A NOTE ON CENTRAL GROUP EXTENSIONS

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If A, B, H, K are abelian groups and $\phi: A \to H$ and $\psi: B \to K$ are epimorphisms, then a given central group extension G of H by K is not necessarily a homomorphic image of a group extension of A by B. Take for instance $A = Z(2)$, $B = Z \oplus Z$, $H = Z(2)$, $K = V_4$ (Klein's fourgroup). Then the dihedral group D_8 is a central extension of H by K but it is not a homomorphic image of $Z \oplus Z \oplus Z(2)$, the only group extension of A by the free group B.

In [1] is is shown that there exists a certain class of loops L which can be considered to be extensions of an abelian group A by a group B. It turns out that this *loop extension* theory contains the Hölder-Schreier theory as a special case. The purpose of the present note is to show that if we have the above situation in this more general extension theory, we can prove the following:

THEOREM. *Any central group extension G of H by K is a homomorphic image of a central loop extension L of A by B.*

Before we prove this theorem, let us recall some definitions:

If C is any subgroup of A, then a function $f: B \times B \to A$ is called a (B, A, C)-*quasi factor system* if it satisfies the following conditions for all $a, b, c \in B$:

(1) $f(b,0) = f(0,b) = 0$

(2) $f(a+b,c) + f(a,b) - f(a,b+c) - f(b,c) \in C$

(3) $f(-b,b) = f(b,-b)$.

Note that if $C = 0$, then f is a *factor system.*

Given abelian groups A and B, C a subgroup of A and a (B, A, C)-quasi factor system f, then the cartesian product $L = B \times A$ with the operation

$$(4) \qquad (b,a) + (b',a') = (b+b', a+a'+f(b,b'))$$

is a *central loop extension* of A by B.

If L is an arbitrary central loop extension of A by B with associated quasi factor system f and G is any central group extension of H by K with associated factor system g, then it follows directly that the mapping

$$(5) \qquad \theta: L \to G;\ (b,a)\theta = (b\psi, a\phi)$$

is an epimorphism if and only if

(6) $$g(b\psi, b'\psi) = \{f(b,b')\}\phi \text{ for all } b, b' \in B.$$

We now proceed to prove the theorem.

PROOF. Consider an arbitrary central group extension G of H by K with associated factor system g. Let $\operatorname{Ker}\phi = C$ and let Φ be the isomorphism $A/C \cong H$. We now construct a loop extension L of A by B such that G is a homomorphic image of L.

To this end, select representatives $r(h)$ in the cosets modulo C in A corresponding to $h \in H$ under Φ, and select, in particular, 0 in C.

Define $f: B \times B \to A$; $f(b,b') = r\{g(b\psi, b'\psi)\}$.

It is clear that f satisfies (1).

To show that f satisfies (2), we consider arbitrary $b, b', b'' \in B$. Since g is a factor system, we have

$$g(b\psi + b'\psi, b''\psi) + g(b\psi, b'\psi) - g(b\psi, b'\psi + b''\psi) - g(b'\psi, b''\psi) = 0.$$

Going over to A/C under Φ, we have the required result. Finally,

$$f(-b, b) = r\{g(-b\psi, b\psi)\} = r\{g(b\psi, -b\psi)\} = f(b, -b).$$

Thus, f is a (B, A, C)-quasi factor system and so $L = B \times A$, with the operation (4), is a central loop extension of A by B.

By definition, f satisfies (6) and so the mapping θ defined by (5) is the required epimorphism.

Reference

[1] G. J. Hauptfleisch, 'Quasi-group extensions of Abelian Groups' (Thesis, Leiden, 1965.)

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