THE STRUCTURE OF A SPECIAL CLASS OF NEAR-RINGS

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1. Introduction

It is well known that a Boolean ring is isomorphic to a subdirect sum of two-element fields. In [3] a near-ring $(B, +, \cdot)$ is said to be Boolean if there exists a Boolean ring $(B, +, \wedge, 1)$ with identity such that $\cdot$ is defined in terms of $+$, $\wedge$, and 1 and, for any $b \in B$, $b \cdot b = b$. A Boolean near-ring B is called special if $a \cdot b = (a \vee x) \wedge b$, where x is a fixed element of B. It was pointed out that a special Boolean near-ring is a ring if and only if $x = 0$. Furthermore, a special Boolean near-ring does not have a right identity unless $x = 0$. It is natural to ask then whether any Boolean near-ring (which is not a ring) can have a right identity. Also, how are the subdirect structures of a special Boolean near-ring compared to those of a Boolean ring. It is the purpose of this paper to give a negative answer to the first question and to show that the subdirect structures of a special Boolean near-ring are very 'close' to those of a Boolean ring. In fact, we will investigate a class of near-rings that include the special Boolean near-rings and the Boolean semi-rings as defined in [8].

2. Preliminaries

A (*left*) *near-ring* is an algebraic system $(R, +, \cdot)$ such that

(i) $(R, +)$ is a group,

(ii) $(R, \cdot)$ is a semigroup,

(iii) $x(y+z) = xy+xz$ for all $x, y, z \in R$.

In particular, if R contains a multiplicative semigroup S whose elements generate $(R, +)$ and satisfy.

(iv) $(x+y)s = xs+ys$, for all $x, y \in R$ and $s \in S$, we say that R is a distributively generated (d.g.) near-ring.

The most natural example of a near-ring is given by the set R of mappings of an additive group (not necessary abelian) into itself. If the mappings are added by adding images and multiplication is iteration, then the system $(R, +, \cdot)$ is a near-ring. If S is a multiplicative semigroup of endomorphisms of R and R' is the subnear-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [5].

An element r of R is *right* (*anti-right*) *distributive* if

$$(b+c)r = br+cr \qquad ((b+c)r = cr+br)$$

for all $b, c \in R$. It follows at once that an element r is right distributive if and only if $(-r)$ is anti-right distributive. In particular, any element of a d.g. near-ring is a finite sum of right and anti-right distributive elements.

The kernels of near-ring homomorphisms are called *ideals*. Blackett [2] showed that K is an ideal of a near-ring R if and only if K is a normal subgroup of $(R, +)$ that satisfied

(i) $RK \leqq K$ and
(ii) $(m+k)n-mn \in K$, for all $m, n \in R$ and $k \in K$.

3. Subdirect sums of near-rings

The theory of subdirect sum representation for rings carries over almost word for word to near-rings [4]. A nonzero near-ring R is subdirectly irreducible if and only if the intersection of all the nonzero ideals of R is nonzero. The near-ring analogue of Birkhoff's [1] fundamental result for rings can be stated as follows.

THEOREM 3.1. [4] *Every near-ring R is isomorphic to a subdirect sum of subdirectly irreducible near-rings.*

For a more detailed discussion of subdirect sums of near-rings, see [4]. By using the technique of subdirect sum representation, it was shown in [6] that every d.g. near-ring R with the property that $x^2 = x$ for all x in R is a Boolean ring.

4. β-near-rings

DEFINITION 4.1. A near-ring R is called a *β-near-ring* if for each x in R, $x^2 = x$ and $xyz = yxz$ for all $x, y, z \in R$.

EXAMPLE 4.2. Let $(R, +)$ be a nontrivial group. Define multiplication by $a \cdot b = b$ for all $a, b \in R$. Then $(R, +, \cdot)$ is a β-near-ring for which $\cdot$ is not commutative and $(R, +)$ need not be of characteristic two.

EXAMPLE 4.3. The Boolean semirings as defined in [8] are β-near-rings for which addition is commutative.

EXAMPLE 4.4. The special Boolean near-rings as defined in [3] are β-near-rings.

It is easily seen that if a β-near-ring R has a right identity, then R is a Boolean ring. In fact, we have the following much stronger result.

THEOREM 4.5. *Let R be a near-ring with the property that $x^2 = x$ for all x in R and has a right identity e. Then R is a Boolean ring.*

PROOF. Since e is right distributive, the equation

$$(e+e)^2 = e+e$$

tells that $e+e = 0$. If x is in R, then

$$x+x = x(e+e) = 0.$$

Hence every element of $(R, +)$ is of order two and consequently $(R, +)$ is commutative.

Let w be an arbitrary element in R. Then

$$(e+w)^2 = e+w$$

yields that

$$(e+w)e+(e+w)w = e+w.$$

It follows that $(e+w)w = 0$ for all $w \in R$. Moreover, $0w+0ww = 0$ implies that $0w(e+w) = 0$. Thus $0w(e+w)w = 0w$ implies that $0w0 = 0w$ and hence $0 = 0w$ for all $w \in R$.

To complete the proof we now show that $(R, \cdot)$ is commutative. This would mean that each element in R is right distributive and hence R is a (commutative) Boolean ring.

Let a and b be arbitrary elements of R. Then

$$\begin{aligned} (ab+ba)(ab+ba) &= ab+ba, \\ (ab+ba)ab+(ab+ba)ba &= ab+ba, \\ (ab+ba)ab &= (ab+ba)+(ab+ba)ba, \\ (ab+ba)ab &= (ab+ba)(e+ba). \end{aligned}$$

Thus we have that

$$\begin{aligned} (ab+ba)abba &= (ab+ba)(e+ba)ba \\ &= (ab+ba)0 \\ &= 0. \end{aligned}$$

It follows that

$$(ab+ba)abab = 0b = 0.$$

Similarly, expand $(ab+ba)(ba+ab) = ab+ba$ as above, we obtain that $(ab+ba)ba = 0$. Consequently $ab+ba = 0$. This completes the proof since every element of $(R, +)$ is of order two.

Note that Theorem 4.5 furnishes a negative answer to the first question mentioned in the introduction.

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5. Subdirect structure of β-near-rings

DEFINITION 5.1. A near-ring $(R, +, \cdot)$ is said to be *small* if there is an element e of R such that e is a left multiplicative identity and for all $x \neq e$ in R, either x is a left identity or else $xy = 0y$ for all y in R.

It is clear that any two-element field is a small near-ring but certainly not conversely. Now we are ready to state our result which compares the subdirect structures of a β-near-ring to those of a Boolean ring.

THEOREM 5.2. *Every β-near-ring R is isomorphic to a subdirect sum of subdirectly irreducible near-rings R_i where each R_i is either a two-element field or a small near-ring.*

To facilitate the discussion on the proof of Theorem 5.2, we first prove a few lemmas which are of interest in their own right.

LEMMA 5.3. *If R is a subdirectly irreducible β-near-ring then R has a left identity.*

PROOF. For each x in R, let

$$A_x = \{y \in R : xy = 0\}. \tag{1}$$

By straight forward calculations, keeping in mind that $xyz = yxz$ for all $x, y, z \in R$, one can easily verify that A_x is an ideal of R. If $A_x = 0$, then x is a left identity since $x(xy-y) = 0$ for all $y \in R$.

Now let

$$N = \{x \in R : A_x \neq 0\}. \tag{2}$$

Suppose $N = R$. Let

$$A = \bigcap_{x \in N} A_x.$$

Then A is not zero since R is subdirectly irreducible. But if $w \neq 0$ is in A, then $w^2 = w = 0$. Thus there exists an element e in R such that $A_e = 0$ and hence e is a left identity.

LEMMA 5.4. *If R is a subdirectly irreducible β-near ring and if $z \neq 0$ such that $A_z \neq 0$, then $zy = 0y$ for all $y \in R$.*

PROOF. Since $A_z \neq 0$, it follows that $z \in N$ as defined in (2). Since $A \neq 0$, let $w \neq 0$ be an element in A. Thus $xw = 0$ for all $x \in N$. If $wy = 0$ for some $y \neq 0$ in R, then $w \in N$ and $w^2 = w = 0$, which is a contradiction. It follows that $wy \neq 0$ for any $y \neq 0$ in R. This means that $A_w = 0$ and hence w is a left identity. Thus

$$zy = zwy = 0y \text{ for all } y \in R. \tag{3}$$

LEMMA 5.5. *If R is a subdirectly irreducible β-near-ring with the property that $0y = 0$ for all y in R, then each $x \neq 0$ in R is a left identity.*

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PROOF. Let N be the set as defined in (2). Then Lemma 5.4 implies that if there exists an element $z \neq 0$ in R such that $A_z \neq 0$, then $zy = 0y$ for all y in R. In particular, $zz = 0z = 0$. This contradiction implies that $N = 0$. Hence each *nonzero element in R is a left identity.*

LEMMA 5.6. *If R is a subdirectly irreducible β-near-ring with a nonzero right distributive element r, then R is the two element field.*

PROOF. Since r is right distributive, it follows that $0r = 0$. From Lemma 5.4 and (3) with $z = r$ and $y = r$, we see that $A_r = 0$. Now let

$$L_r = \{y \in R : yr = 0\}.$$

Since r is right distributive and $xyz = yxz$ for all $x, y, z \in R$, it is easily verified that L_r is an ideal of R. Suppose that $L_r \neq 0$. Let

$$L = L_r \cap A, \text{ where } A = \bigcap_{x \in N} A_x.$$

There exists a $w \neq 0$ in L such that $xw = 0$ for each $x \in N$ and $wr = 0$. This is a contradiction since w is a left identity. Thus $L_r = 0$ and we conclude that r is a right identity as well as a left identity. Thus $(R, \cdot)$ is commutative and $0x = 0$ for all x in R. By Lemma 5.5 each $x \neq 0$ in R is a left identity and it follows that $x = xr = r$. Consequently R is the two-element field.

We may now complete the proof of Theorem 5.2.

PROOF OF THEOREM 5.2. Let R be a β-near-ring. By Theorem 3.1, R is isomorphic to a subdirect sum of subdirectly irreducible near-rings R_i. Now each R_i is a homomorphic image of R and therefore a β-near-ring. If R_i has a nonzero right distributive element then it is a two-element field by Lemma 5.6. If R_i does not have a nonzero right distributive element, then R_i is a small near-ring by Lemmas 5.3 and 5.4.

Since special Boolean near-rings and Boolean semi-rings as defined in [3] and [8] respectively are β-near-rings, Theorem 5.2 furnishes the subdirect structures of those near-rings as well.

An immediate corollary of Lemma 5.6 is the following characterization of Boolean rings.

COROLLARY 5.7. *A near-ring R is a Boolean ring if and only if R is a β-near-ring and every nonzero homomorphic image of R has a nonzero right distributive element.*

Since a homomorphic image of a d.g. near-ring is again a d.g. near-ring [5], we have

COROLLARY 5.8. *Every d.g. β-near-ring is a Boolean ring.*

6. Remarks

In view of Theorem 4.5, one naturally asks that if R is a near-ring with a right identity, for what positive integers n such that $x^n = x$ for all x in R would imply that $(R, \cdot)$ is commutative. Of course it is well known that if R is a ring, then $(R, \cdot)$ is commutative for all n. By a result in [7, Cor. 3.7] an affirmative answer for $n = n_0$ would imply that if R is a near-ring with a right identity and $x^{n_0} = x$ for all x in R, then R is a commutative ring with identity. Thus it is of interest to known the answers to the questions just mentioned above.

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