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Nonoscillation of arbitrary order retarded differential equations of non-homogeneous type

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The object of the present paper is to study the delay differential equation of arbitrary order namely

$$y^{(n)}(t) + a(t)y_\tau(t) = f(t) \ , \quad n \geq 2 \quad \text{(an integer)}$$

and prove a nonoscillation theorem under the general situation in which $a(t)$ and $f(t)$ are allowed to oscillate arbitrarily often on some positive half real line. This is accomplished by way of two differential inequalities of nth order.

1.

Recently Onose [2] and Singh [3] studied the oscillation properties of the solutions of the equations

$$y^{(n)}(t) + a(t)y(t) = 0 \ , \tag{1.1}$$

$$y^{(2n)}(t) + a(t)y\big(t-\tau(t)\big) = 0 \tag{1.2}$$

under the restrictive assumption that a be eventually non-negative on some positive half real axis. The purpose here is to study the delay differential equation of arbitrary order namely

$$y^{(n)}(t) + a(t)y_\tau(t) = f(t) \ , \quad n \geq 2 \quad \text{(an integer)}, \tag{1.3}$$

and prove a nonoscillation theorem under the general situation in which a

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and f are allowed to oscillate arbitrarily often on some positive half real line. This is accomplished by way of two differential inequalities of nth order. The following assumptions will hold throughout this paper:

(i) $y_\tau(t) \equiv y\big(t-\tau(t)\big)$,

(ii) $a : (-\infty, \infty) \to (-\infty, \infty)$ is continuous,

(iii) $\tau : [0, \infty) \to [0, \infty)$ is continuous and bounded.

In what follows, it will be shown that if g and h are eventually positive functions such that

$$g^{(n)}(t) + t^{n-1}|a(t)|g(t) \le 0 , \tag{1.4}$$

$$h^{(n)}(t) + t^{n-1}|f(t)|h(t) \le 0 , \tag{1.5}$$

then equation (1.3) has bounded nonoscillatory solutions. It is interesting to note that these differential inequalities are independent of the delay term.

We call a function $F \in C\big[t_0, \infty\big)$ oscillatory if it has arbitrarily large zeros on $\big[t_0, \infty\big)$. Otherwise we call it nonoscillatory.

We shall only consider continuous and extendable solutions of equation (1.3) over some half line $\big[t_0, \infty\big)$, $t_0 > 0$.

2.

THEOREM 1. *Let* g *and* h *be* n *times differentiable functions on some half line* $[T, \infty)$, $T \ge t_0 > 0$ *such that*

$$\liminf_{t\to\infty} g(t) > 0 , \quad \liminf_{t\to\infty} h(t) > 0 , \tag{2.1}$$

$$g^{(n)}(t) + t^{n-1}|a(t)|g(t) \le 0 , \tag{2.2}$$

$$h^{(n)}(t) + t^{n-1}|f(t)|h(t) \le 0 \tag{2.3}$$

eventually. Then equation (1.3) *has bounded nonoscillatory solutions.*

Proof. Let T be large enough so that $g(t) > 0$ in $[T, \infty)$. Then by inequality (2.2), there exists $T_1 > T$ such that

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$$g^{(n)}(t) \le 0 \;, \quad g(t) > 0 \;, \quad t \ge T_1 \;. \tag{2.4}$$

Conclusion (2.4) forces all preceding derivatives to be monotonic. Two cases arise.

Case 1. $g'(t) \ge 0$, $t \ge T_1$.

Conclusion (2.4) also implies that $g^{(n-1)}(t) \ge 0$ because otherwise $g(t)$ will eventually become negative. Dividing (2.2) by $g(t)$ and integrating between $[T_1, t]$, we have

$$\frac{g^{(n-1)}(t)}{g(t)} - \frac{g^{(n-1)}(T_1)}{g(T_1)} + \int_{T_1}^{t} \frac{g^{(n-1)}(s)g'(s)}{g^2(s)}\,ds + \int_{T_1}^{t} s^{n-1}|a(s)|\,ds \le 0 \;. \tag{2.5}$$

Since $g^{(n-1)}(t)$ and $g'(t)$ are nonnegative for $t \ge T_1$, (2.5) implies

$$\lim_{t\to\infty} \int_{T_1}^{t} s^{n-1}|a(s)| < \infty \;. \tag{2.6}$$

Case 2. $g'(t) < 0$, $t \ge T_1$.

Here again conclusion (2.4) implies that for $t \ge T_1$, $g^{(n)}(t) \le 0$, $g^{(n-1)}(t) \ge 0$, $g'(t) < 0$, $g(t) > 0$. Again, we will show that (2.6) holds. Suppose to the contrary

$$\int_{T_1}^{\infty} t^{n-1}|a(t)|\,dt = +\infty \;. \tag{2.7}$$

Then from (2.5) and (2.7), it follows

$$\lim_{t\to\infty} \int_{T_1}^{t} \frac{g^{(n-1)}(s)g'(s)}{g^2(s)}\,ds = -\infty \;. \tag{2.8}$$

But

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$$\lim_{t\to\infty}\int_{T_1}^{t}\frac{g^{(n-1)}(s)g'(s)}{g^2(s)}\,ds \ge \lim_{t\to\infty}\left[g^{(n-1)}(T_1)\int_{T_1}^{t}\frac{g'(s)}{g^2(s)}\,ds\right]$$

$$= \lim_{t\to\infty}\left\{g^{(n-1)}(T_1)\left[-\frac{1}{g(t)}+\frac{1}{g(T_1)}\right]\right\} > -\infty$$

by condition (2.1). This contradiction shows that (2.6) holds. Similarly (2.3) leads to

$$\int^{\infty} t^{n-1}|f(t)|dt < \infty .$$

To complete the proof, we set up the following integral equation

$$(2.9)\quad y(t) = \frac{1}{2} - K\int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,a(t)y\big(t-\tau(t)\big) + K\int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,f(t)dt ,$$

where $K = 1$ when n is even and $K = -1$ when n is odd. It is obvious that a solution of (2.9) is also a solution of equation (1.3).

We now set up a sequence of estimates:

$$(2.10)\quad y_0(t) \equiv 1 ,$$

$$(2.11)\quad y_j(t) = \frac{1}{2} - K\int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,a(t)y_{j-1}\big(t-\tau(t)\big)dt + K\int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,f(t)dt ,$$

and choose t large enough so that

$$(2.12)\qquad \int_t^{\infty} t^{n-1}|a(t)|dt < 1/4$$

and

$$(2.13)\qquad \int_t^{\infty} t^{(n-1)}|f(t)|dt < 1/4 .$$

Due to the boundedness of τ all estimates in (2.11) are well defined to the right of some large $T > 0$.

From (2.10), (2.11), (2.12) and (2.13),

$$|y_1(t)| \le 1/2 + \int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,|a(t)|dt + \int_t^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}\,|f(t)|dt$$

$$\le 1/2 + 1/4 + 1/4 = 1 .$$

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Similarly for each j ,

$$|y_j(t)| \le 1 .$$

Now in the manner of Theorem 3 of [1], $|y_{j+1}-y_j| \le 1$ for all j , and $\{y_j\}$ converges uniformly to a solution of (2.9); and the proof is complete.

3.

EXAMPLE 1. Consider the equation

$$y^{(5)}(t) + \frac{\cos t}{1+t^{12}}\, y(t-\pi) = \frac{\sin t}{1+t^{12}} . \tag{3.1}$$

Let $g(t) = t^{3/2}$. Now

$$g^{(5)}(t) + t^4|a(t)|g(t) = -\frac{45}{32}\, t^{-7/2} + \frac{t^4 \cdot t^{3/2}|\cos t|}{1+t^{12}}$$

$$= -\frac{45}{32}\, t^{-7/2}\left[1 - \frac{32}{45}\,\frac{t^9}{1+t^{12}}\,|\cos t|\right] < 0 \quad \text{for large } t .$$

Similarly when $h(t) = t^{3/2}$, $f(t) = \frac{\sin t}{1+t^{12}}$, then

$$h^{(5)}(t) + t^4|f(t)|h(t) < 0 \quad \text{for large } t .$$

Hence equation (3.1) has a bounded nonoscillatory solution.

EXAMPLE 2. Consider the equation

$$y^{(6)}(t) + \frac{\cos t}{1+t^{12}}\, y(t-\pi) = \frac{\sin t}{1+t^{12}} . \tag{3.2}$$

Here we take $g(t) = t^{5/2}$,

$$g^{(6)}(t) + t^5|a(t)|g(t) = -\frac{225}{64}\, t^{-7/2} + \frac{t^5 \cdot t^{5/2}|\cos t|}{1+t^{12}}$$

$$= -\frac{225}{64}\, t^{-7/2}\left[1 - \frac{64}{225}\,\frac{t^{11}}{1+t^{12}}\,|\cos t|\right]$$

$$< 0 \quad \text{for large } t ,$$

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and

$$h(t) = t^{5/2} , \quad f(t) = \frac{\sin t}{1+t^{12}} ,$$

$$h^{(6)}(t) + t^5|f(t)|h(t) = -\frac{225}{64} t^{-7/2}\left[1 - \frac{64}{225}\frac{t^{11}}{1+t^{12}}|\sin t|\right]$$
$$< 0 \quad \text{for large } t .$$

Thus (3.2) has a bounded nonoscillatory solution.

If we take $n = 3$, $f(t) = 0$ in (1.3), then we arrive at known results (see [4]).

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