BISECANTS OF FINITE COLLECTIONS OF SETS IN LINEAR SPACES

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1. Introduction. The question posed by Sylvester **(6)** concerning the collinearity of a finite set of points in E^2 having the property that each two together with some third be collinear has been the inspiration for numerous investigations. The original question was answered by the following theorem.

THEOREM 1. *If a finite set of $k \geqslant 2$ points in affine n-space A^n or in projective n-space P^n is not a subset of a line, then there exists a line in that space containing precisely two of the points.*

Generalizations of this result and additional references are contained in **(3; 4; 5)**.

In some papers the analogous problem was considered for the case when points in the above theorem are replaced by disjoint sets. The strongest result, obtained in **(3)**, is the following.

THEOREM 2. *If $\{S_i\}$ is a finite collection of two or more disjoint, non-empty compact sets in E^n with $S = \cup S_i$ infinite, then either S is a subset of a line or there exists a hyperplane intersecting exactly two members of the family.*

Examples show (see §2) that the assumption that S is infinite cannot, in general, be dropped, but the suspicion prevailed that the number of counter-examples was severely limited. In §2 we show that this suspicion is, in a certain sense, justified. Specifically we prove Theorem 2.1, according to which the counter-examples mentioned above must be confined to dimensions 2 and 3.

In §3 we free Theorem 2 from its Euclidean setting, by the use of different and, we believe, substantially simpler methods than those used in **(3)**.

The result of the present paper might be summarized as follows.

THEOREM 3. *If $\{S_i\}$ is a finite collection of two or more non-empty disjoint compact sets in a real normed linear space, then at least one of the following holds:*

1. *There exists a hyperplane intersecting precisely two of the sets.*
2. *$\cup S_i$ spans a space of dimension* 1.
3. *$\cup S_i$ is finite and spans a space of dimension* 2 *or* 3.

We had hoped to be able to characterize those exceptional sets in dimensions 2 and 3, but we have thus far been unsuccessful.

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2. The finite case.

Example 2.1. Consider a regular polygon of $2n$ vertices in E^2 together with the n ideal points defined by the sides of this polygon. Label members of a set of alternate vertices with the number 1, each of the remaining vertices with the number 2, and each of the ideal points with the number 3. S_i is the set of points labelled i. It is easy to verify that a line cutting any two of these sets cuts the third.

Example 2.2. In E^3 consider the vertices and centre of a cube together with the three ideal points defined by the edges of the cube. Label each of the vertices with the number 1 or 2 in such a way that no two adjacent vertices carry the same number. Label the centre and each ideal point with the number 3. Define S_i to be the set of points numbered i. Here again it is easy to see that a line cutting any two of the sets cuts the third set.

In dimension 2 a variety of examples somewhat different from those in Example 1 can be constructed but in dimension 3, Example 2 is the only one known to us. Example 2 is essentially the well-known desmic configuration (cf. N. Altshiller-Court, *Modern pure solid geometry* [New York, 1935]).

THEOREM 2.1. *If $\{S_i\}$ is a finite collection of two or more non-empty disjoint finite sets of a linear vector space over an ordered field such that $\cup S_i$ spans a subspace of at least dimension* 4, *then there exists a line* (*and therefore also a hyperplane*) *cutting precisely two of the sets.*

Proof. Motzkin has observed **(5)** that a pencil of lines in affine 3-space, A^3, not all in the same plane must contain a pair of lines such that the plane defined by these lines contains none of the other lines.

This follows at once if we consider a section of the pencil by a plane and appeal to the original Sylvester theorem in the plane of the section.

We now choose a pair of points p_1 and p_2 of $\cup S_i$ where p_1 and p_2 are from different S_i. The points of $\cup S_i - \{p_1, p_2\}$ define a pencil of planes with line $p_1 p_2$ as axis. A section of this pencil by a properly chosen 3-space defines a pencil of lines in that 3-space not all in a plane and, by the Motzkin observation, two of the lines of this pencil define a plane free of any of the other lines of the pencil. This plane together with the points p_1 and p_2 spans a 3-space Γ such that the points of $\cup S_i$ in this 3-space are on precisely two planes of the original pencil of planes. Each of these planes contains at least one point of $\cup S_i - \{p_1, p_2\}$.

It is now an easy matter to check that if a collection of two or more finite non-empty and disjoint sets in A^3 lie on two planes and not on one, then there is a line intersecting precisely two of the sets.

3. 2-Secants in normed linear spaces.

DEFINITION. *A hyperplane π in a linear vector space E is called a k-secant of a collection $\{S_\alpha\}$ of subsets of E if it intersects exactly k members of the collection.*

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THEOREM 3.1. *Let $\{S_i\}$ be a finite collection of two or more non-empty disjoint compact sets in a real normed linear space X with $\cup S_i$ infinite and suppose no straight line in X contains $\cup S_i$. Then a 2-secant of $\{S_i\}$ exists.*

We contend that it is only necessary to establish Theorem 3.1 in the setting of a strictly convex normed linear space. In the first place, since each S_i is compact, $\cup S_i$ is compact and the subspace spanned by $\cup S_i$ is separable. Thus if we can produce a hyperplane in this subspace of the desired character, the extension to the whole space of the defining linear functional is assured and the resulting hyperplane will satisfy the demands of the theorem. Now the property of the theorem is invariant under a topological isomorphism and by a theorem of Clarkson (**1**; also **2**, p. 518), any separable normed linear space is topologically isomorphic to a strictly convex one.

It is convenient to base the proof of Theorem 3.1 on a series of lemmas, the proofs of which are quite straightforward and hence will be sketched only briefly.

LEMMA 3.1. *If C is a compact set, containing at least two points, in a strictly convex normed linear space, then there exists a pair of distinct parallel hyperplanes each supporting C at precisely one point.*

Proof. Let a, b be a pair of diametral points of C and consider the sphere centred at a and passing through b. Since the sphere is strictly convex, there is a hyperplane σ supporting this sphere at the single point b. This hyperplane also supports C at b. It is an easy matter to show that the hyperplane σ', parallel to σ through a, supports the sphere centred at b and passing through a at the single point a. The hyperplanes σ, σ' are as required in the lemma.

LEMMA 3.2. *Let $\{S_i\}$, $i = 1, 2, \ldots, n$, $n \geqslant 2$, be a finite collection of disjoint non-empty compact sets in a normed linear space, not all on a single line, with S_1 infinite and $\cup S_i$ finite. Then $\{S_i\}$ has a 2-secant.*

Proof. It clearly suffices to show that a line L exists which cuts S_1 and exactly one more S_i, $i \geqslant 2$. We distinguish between the case (i) when a line M exists such that

$$\bigcup_2^n S_i \subset M$$

and (ii) when no such line exists. In the first case, there exists a point s of S_1 not on M. L can then be any line through s and an arbitrary point of $\bigcup_2^n S_i$. In the second case, let a,b,c be a triple of non-collinear points of $\bigcup_2^n S_i$ and consider the collection Λ of all lines joining these points with those of S_1. It is easily seen that this collection is infinite. Thus Λ must contain an L as desired.

LEMMA 3.3. *Let $\{S_i\}$, $i = 1, 2, \ldots, n$, $n \geqslant 2$, be a finite collection of non-empty disjoint compact sets in a normed linear space, not all on a single line,*

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with S_1 infinite. Suppose all but a finite number of points of $\bigcup_2^n S_i$ lie on a line M which contains an accumulation point p of S_1. Then $\{S_i\}$ has a 2-secant.

Proof. As in the proof of Lemma 3.2, we show that a line L exists which cuts exactly two members of the collection. We may clearly assume that

$$\bigcup_2^n S_i \cap M \neq \emptyset.$$

If

$$\bigcup_2^n S_i \subset M,$$

then there is a point q of S_1 not on M. The line through q and any point of $\bigcup_2^n S_i$ may serve as L. If not let

$$a \in \bigcup_2^n S_i \cap M \quad \text{and} \quad b \in \bigcup_2^n S_i - M.$$

The collection of lines Λ joining $\{a, b\}$ with S_1 is obviously infinite and, clearly, at most finitely many of them cut three or more members of $\{S_i\}$.

LEMMA 3.4. *Let $\{S_i\}$, $i = 1, 2, \ldots, n$ $(n \geqslant 2)$, be a finite collection of non-empty disjoint compact sets in a normed linear space X. Let α be a hyperplane, α^+ and α^- the two open half-spaces determined by α, and suppose that*

$$\bigcup_3^n S_i \cap \alpha^-$$

is finite (or empty). Let $\beta \subset \alpha$ be a hyperplane relative to α and β^+ and β^- the two open half-spaces, relative to α, determined by β, and suppose that $\beta \cap S_2 \neq \emptyset$ and

$$\bigcup_3^n S_i \cap \beta = \emptyset.$$

Suppose further that a sequence $\{p_i\}$ of points in $\alpha^+ \cap S_1$ $(\alpha^- \cap S_1)$ exists which converges to $p \in S_1 \cap \beta^-$, and

$$\bigcup_3^n S'_i \cap \beta^- = \emptyset \qquad \left(\bigcup_3^n S'_i \cap \beta^+ = \emptyset\right)$$

where S'_i is the set of accumulation points of S_i. Then there exists a hyperplane in X intersecting S_1 and S_2 and no other S_i.

Proof. Let γ_i denote the hyperplane spanned by β and p_i. Since

$$\bigcup_3^n S_i \cap \beta = \emptyset \quad \text{and} \quad \bigcup_3^n S_i \cap \alpha^- \text{ is finite,}$$

an i_0 exists such that $i \geqslant i_0$ implies

$$\left(\bigcup_3^n S_j \cap \alpha^-\right) \cap \gamma_i = \emptyset.$$

Let $\{p_{i_k}\}$ be a subsequence of $\{p_i\}$ with the property that all γ_{i_k} are distinct, $i_k \geqslant i_0$, $k = 1, 2, \ldots$, and write $\alpha_k = \gamma_{i_k}$. Let $\alpha_k{}^+$ be the open half-space determined by α_k that contains β^+ (β^-). The family $\{\alpha_k{}^+\}$ is clearly an open cover of $\bigcup_3^n S'_i$. Thus

$$\bigcup_3^n S'_i \subset \alpha_{k_1}{}^+ \cup \alpha_{k_2}{}^+ \cup \ldots \cup \alpha_{k_m}{}^+ \cap \overline{\alpha^+}.$$

Since this set is convex and disjoint from an open ball $B(p, \epsilon)$ centred at p and of sufficiently small radius $\epsilon > 0$, all α_k such that $p_{i_k} \in B(p, \epsilon)$ will be disjoint from

$$\bigcup_3^n S'_i$$

and intersect both S_1 and S_2. Clearly only finitely many of these hyperplanes can intersect

$$\bigcup_3^n S_i.$$

Thus $\{\alpha_k\}$ contains at least one member which satisfies the demands of the lemma.

Proof of the theorem. The $\bigcup_1^n S'_i$ is a non-empty compact set. Hence, by Lemma 3.1, there is a hyperplane π supporting this set at a single point p, which we may suppose to be a point of S_1. Let π^+ be the open half-space defined by π that contains

$$\bigcup_1^n S'_i - \{p\}$$

and π^- the complementary open half-space defined by π. Obviously π^- contains at most finitely many points of $\bigcup_2^n S_i$. Suppose $\pi_1 \subset \pi^+$ is a hyperplane parallel to π and Z the central projection of

$$\pi^+ \cap \bigcup_2^n S_i$$

on π_1 with p the centre of projection. Z is clearly compact. If Z is empty, then the theorem follows by Lemma 3.2, and if Z consists of a single point, the theorem follows by Lemma 3.3.

Suppose then that Z contains at least two points with δ_1 and δ_2 a pair of parallel support hyperplanes relative to π_1 supporting Z at the single points d_1 and d_2 respectively, as guaranteed by Lemma 3.1. Let ρ_i be the hyperplane through p and δ_i. These hyperplanes clearly support

$$\bigcup_2^n S_i \cap \pi^+$$

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at sets of points lying entirely on the rays pd_i. We distinguish between two cases:

1. There exists a sequence $\{p_i\}$ of points in $S_1 - \{p\}$ approaching p with no point in at least one of the two hyperplanes, say ρ_1.

2. There is no such sequence.

In Case 1, let r_1 be the point of $\bigcup_2^n S_i$ on the ray pd_1 nearest p and r^1 the point furthest from p. Let σ_1 and σ^1 be the hyperplanes relative to ρ_1 through r_1 and r^1, respectively, parallel to $\pi \cap \rho_1$. There must be a subsequence of $\{p_i\}$ approaching p from either ρ_1^+ or ρ_1^-, where ρ_1^+ and ρ_1^- are the two open half-spaces defined by ρ_1. The theorem follows in either case by an application of Lemma 3.4. If ρ_1^+ is the half-space containing Z, σ_1 serves as the β in the lemma in the first instance and σ^1 in the second; ρ_1 serves as α in both cases.

In Case 2, there are infinitely many points of $S_1 - \{p\}$ in $\rho_1 \cap \rho_2 \cap U$ for each neighbourhood U of p. Obviously, then, a line L through r_1 and

$$q \in S_1 - \{p\}$$

in ρ_1 exists such that

$$L \cap \bigcup_3^n S_i = \emptyset.$$

By an argument similar to the one used in the proof of Lemma 3.4 it is readily seen that a hyperplane through L, which is disjoint from $\bigcup_3^n S_i$, exists. This completes the proof of the theorem.

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