Hence [1] is equivalent to

$$\frac{\lambda}{2}\,\phi(m)\,.\sum_{a=1}^{p^u-1} a^{t\phi(p^u)-2} \equiv 0 \pmod{p^u}, \text{ for each } p \mid n. \qquad [3\cdot 2]$$

Now since $t\phi(p^u) - 2 \equiv 0 \pmod{p-1}$, only if $p = 2$ or 3,

[3·2] holds if $\frac{\lambda}{2}\phi(m) \equiv 0 \ (\text{mod. } \{n, 6\})$.

When $n = 2^u$, $u \geqq 1$, then $\phi(m) = 1$ and $\lambda = 4 = \{n^2, 12\}$.

When $n = 2^u m$, $u \geqq 0$, $\{m, 2\} = 1$, if $\phi(m) \equiv 0 \pmod{2}$,

then $\lambda = \{n, 6\} = \{n^2, 6\}$.

When $n = 2^u m$, $u \geqq 0$, $\{m, 2\} = 1$, if $\phi(m) \equiv 0 \pmod{4}$,

then $\lambda = \{n, 3\} = \{n^2, 3\}$.

Hence if $n = 2^u N$, $\{N, 2\} = 1$, $u \geqq 0$;

then $\lambda = \left\{n^2, \dfrac{12}{l}\right\}$, where $\phi(N) = l \pmod{4}$, $1 \leqq l \leqq 4$. (3·4)

REFERENCES.

1. S. Chowla, "A Generalisation of a Theorem of Wolstenholme," *The Mathematics Student*, Vol. 1 (1933), pp. 140-141. See also *Journal of the London Mathematical Society*, Vol. 5 (1930), pp. 158-160.

2. Hansraj Gupta, "Operations with Δ on G-Functions." (*In Press.*)

Note on the Summation of Finite Algebraic Series

By J. A. Macdonald.

In order that an algebraic series of a finite number of terms may be summed in simple form, there must exist a difference equation satisfied by the terms. But, owing to the fact that no general method of finding this equation is given in textbooks, the beginner does not acquire confidence in summation until his experience is wide enough to include all of the usual types of such series; and even then he may come to the conclusion that the finding of the difference equation is no more than a lucky chance. It is therefore proposed in the following notes to show that the ordinary types of summable terminating series can be reduced to one, and to find a single general expression for the sum. It will also be shown that the criterion of summability is the "convergency ratio."

https://doi.org/10.1017/S1757748900002395 Published online by Cambridge University Press

1. *Classification of Ordinary Types of Series.*

The following types of terminating series are of common occurrence.

Type I. Series with the same number of polynomial factors in each term: for example,

$$a_1 a_2 a_3 + a_2 a_3 a_4 + \ldots . + a_n a_{n+1} a_{n+2},$$

where a_n is a polynomial in n.

Type II. Series in which typical terms are reciprocals of those of Type I: for example,

$$\frac{1}{b_1 b_2 b_3} + \frac{1}{b_2 b_3 b_4} + \ldots . + \frac{1}{b_n b_{n+1} b_{n+2}}.$$

Type III. Series with a constant number of factors in the numerators of the terms, as also in the denominators: for example,

$$\frac{a_1 a_2}{b_1 b_2 b_3} + \frac{a_2 a_3}{b_2 b_3 b_4} + \ldots . + \frac{a_n a_{n+1}}{b_n b_{n+1} b_{n+2}}.$$

Type IV. Series in which the number of factors in numerator and denominator increases from term to term: for example,

$$\frac{a_1}{b_1 b_2} + \frac{a_1 a_2}{b_1 b_2 b_3} + \ldots . + \frac{a_1 a_2 \ldots . a_n}{b_1 b_2 \ldots . b_{n+1}}.$$

Overlap. It will be observed that in all the examples given above the successive terms have the greatest possible "overlap." (We exclude the trivial case when all terms are identical). For example,

$$a_1 \overline{a_2 a_3} + \overline{a_2 a_3} a_4 + \ldots .$$

But we may of course have for consecutive terms

$$a_1 a_2 a_3,\ a_3 a_4 a_5,\ \ldots .,\ \text{or even}\quad a_1 a_2 a_3,\ a_4 a_5 a_6 \ldots .$$

When the factors are polynomials in n, we can convert a given series into a sum of series in which the terms have the greatest possible overlap : for example,

$$\Sigma\,(an^2 + bn + c)$$

would be considered as

$$\Sigma\, an\,(n+1) + \Sigma\,(b-a)\,n + \Sigma\, c.$$

2. *Reduction to a Single Type.*

It will now be shown that the fundamental type of series is Type IV. For we can express the other types in terms of Type IV as follows:

Type I.

$$a_1 a_2 a_3 + a_2 a_3 a_4 + \ldots = a_1 a_2 a_3 + a_1 a_2 a_3 a_4 \left(\frac{1}{a_1} + \frac{a_5}{a_1 a_2} + \frac{a_5 a_6}{a_1 a_2 a_3} + \ldots\right).$$

Type II.

$$\frac{1}{b_1 b_2 b_3} + \frac{1}{b_2 b_3 b_4} + \ldots . = \frac{1}{b_1 b_2 b_3} + \frac{1}{b_2 b_3}\left(\frac{1}{b_4} + \frac{b_2}{b_4 b_5} + \frac{b_2 b_3}{b_4 b_5 b_6} + \ldots .\right).$$

Type III.

$$\frac{a_1 a_2}{b_1 b_2 b_3} + \frac{a_2 a_3}{b_2 b_3 b_4} + \ldots . = \frac{a_1 a_2}{b_1 b_2 b_3} + \frac{a_1 a_2 a_3}{b_2 b_3}\left(\frac{1}{a_1 b_4} + \frac{a_4 b_2}{a_1 a_2 b_4 b_5} + \ldots .\right).$$

3. *The Standard Summation Formula.*

The standard formula which we shall use is not new (see for example Chrystal's *Algebra*, Vol. II, pp. 392-3), but the following elementary proof has perhaps advantages of simplicity.

Let c, d_1, d_2, $\ldots .$, d_n be any numbers, subject only to the condition that none of them is zero, and that none of the sums

$$c + d_1,\ c + d_2,\ \ldots .,\ c + d_n$$

is zero. Then we have

$$\begin{aligned}
1 &= \frac{c}{c+d_1} + \frac{d_1}{c+d_1} \\
&= \frac{c}{c+d_1} + \frac{d_1}{c+d_1}\left(\frac{c}{c+d_2} + \frac{d_2}{c+d_2}\right) \\
&= \frac{c}{c+d_1} + \frac{cd_1}{(c+d_1)(c+d_2)} + \frac{d_1 d_2}{(c+d_1)(c+d_2)}\left(\frac{c}{c+d_3} + \frac{d_3}{c+d_3}\right) \\
&= \frac{c}{c+d_1} + \frac{cd_1}{(c+d_1)(c+d_2)} + \frac{cd_1 d_2}{(c+d_1)(c+d_2)(c+d_3)} + \frac{d_1 d_2 d_3}{(c+d_1)(c+d_2)(c+d_3)}.
\end{aligned}$$

Hence, proceeding in this way, we have

$$1 = c\left\{\frac{1}{c+d_1} + \frac{d_1}{(c+d_1)(c+d_2)} + \frac{d_1 \ldots d_{n-1}}{(c+d_1)\ldots(c+d_n)}\right\} + \frac{d_1 \ldots d_n}{(c+d_1)\ldots(c+d_n)},$$

and so

$$\frac{1}{c+d_1} + \frac{d_1}{(c+d_1)(c+d_2)} + \ldots + \frac{d_1 \ldots d_{n-1}}{(c+d_1)\ldots(c+d_n)} = \frac{1}{c}\left\{1 - \frac{d_1 \ldots d_n}{(c+d_1)\ldots(c+d_n)}\right\}.$$

The chief characteristics of this series (which is of Type IV) are these:

(*a*) each term has one factor less in the numerator than in the denominator;

(*b*) from the second term onwards there is a constant difference c between the r^{th} factor in the denominator and the r^{th} factor in the numerator of each term;

(*c*) the "convergency ratio" is $d_{r-1}/(c + d_r)$.

4. *Value of the Summation Constant* c *in Types I, II and III, and Expressions for the Sum of the Series.*

Type I.

$$\sum_{r=1}^{n} a_r a_{r+1} \ldots a_{r+s-1} = a_1 a_2 \ldots a_s + a_1 a_2 \ldots a_{s+1}\left(\frac{1}{a_1} + \frac{a_{s+2}}{a_1 a_2} + \ldots + \frac{a_{s+2} \ldots a_{n+s-1}}{a_1 \ldots a_{n-1}}\right)$$

The condition of summability by means of the standard identity is

$$a_1 - a_{s+2} = a_2 - a_{s+3} = \ldots\ldots = a_{n-2} - a_{n+s-1} = c,$$

and the sum to n terms of the series is

$$a_1 a_2 \ldots\ldots a_s + \frac{a_1 \ldots\ldots a_{s+1}}{(a_1 - a_{s+2})}\left(1 - \frac{a_{s+2} \ldots\ldots a_{n-1} a_n \ldots\ldots a_{n+s}}{a_1 \ldots\ldots a_{s+1} a_{s+2} \ldots\ldots a_{n-1}}\right)$$

$$= a_1 a_2 \ldots\ldots a_s + \frac{1}{(a_1 - a_{s+2})}(a_1 \ldots\ldots a_{s+1} - a_n \ldots\ldots a_{n+s}).$$

In this case there are $s+1$ independent quantities, namely

$$a_1, a_2, \ldots\ldots, a_{s+1}.$$

If two of these are identical, they can be expressed in terms of factorial polynomials, in the usual way. For example,

$$\Sigma r(r+1)(r+2)^2 = \Sigma r(r+1)(r+2)(r+3) - \Sigma r(r+1)(r+2).$$

Type II.

$$\Sigma \frac{1}{b_r b_{r+1} \ldots\ldots b_{r+s-1}}.$$

Here it is unnecessary to segregate the first term, but we shall do so for the sake of uniformity, writing the series as

$$\frac{1}{b_1 \ldots\ldots b_s} + \frac{1}{b_2 \ldots\ldots b_s}\left(\frac{1}{b_{s+1}} + \frac{b_2}{b_{s+1} b_{s+2}} + \ldots\ldots + \frac{b_2 \ldots\ldots b_{n-1}}{b_{s+1} \ldots\ldots b_{s+n-1}}\right).$$

The condition for summability by the standard method is

$$b_{s+1} - b_2 = \dots\dots = b_{s+n-2} - b_{n-1} = c,$$

and in such a case the sum to n terms is

$$\frac{1}{b_1 \dots\dots b_s} + \frac{1}{b_{s+1} - b_2}\left(\frac{1}{b_2 \dots\dots b_s} - \frac{1}{b_{n+1} \dots\dots b_{s+n-1}}\right).$$

Type III.

In the same way as above,

$$\Sigma \frac{a_r \dots a_{r+s-1}}{b_r \dots b_{r+t-1}} = \frac{a_1 \dots a_s}{b_1 \dots b_t} + \frac{1}{a_1 b_{t+1} - a_{s+2} b_2}\left(\frac{a_1 \dots a_{s+1}}{b_2 \dots b_t} - \frac{a_n \dots a_{n+s}}{b_{n+1} \dots b_{n+t-1}}\right),$$

with the condition

$$a_1 b_{t+1} - a_{s+2} b_2 = a_2 b_{t+2} - a_{s+3} b_3 = \dots\dots = c.$$

Corollary. If a_r and b_r are polynomials linear in r, then

$$\Sigma \frac{a_r \dots\dots a_{r+s}}{b_r \dots\dots b_{r+t}}$$

can be summed by the formula, if, and only if $t = s + 2$.

For, putting $a_r = p_1 r + q_1$, and $b_r = p_2 r + q_2$, we have

$$c = \{p_1 r + q_1\}\{p_2 (r + t) + q_2\} - \{p_1 (r + s + 1) + q_1\}\{p_2 (r + 1) + q_2\}.$$

It is found on simplification that the coefficient of r^2 is zero, and that the coefficient of r is $p_1 p_2 (t - s - 2)$. Hence c cannot be independent of r, unless $t = s + 2$. If this condition is fulfilled,

$$c = (s + 1)(p_2 q_1 - p_1 q_2 - p_1 p_2).$$

5. *Determination of the Constant* c *by means of the Convergency Ratio.*

It will be observed that the transformation of types I, II, III into the standard type IV does not alter the convergency ratio. In type I, for example, the ratio is a_{r+s-1}/a_{r-1}, and this is unchanged when the series is expressed in the type IV form.

Since the ratio of the r^{th} term to the $(r-1)^{\text{th}}$ term in the type IV form is always $d_{r-1}/(c + d_r)$, we have for c an equation of the form

$$f(r-1)/\{f(r) + c\} = \phi(r),$$

which in many cases is easily solved.

One or two concrete examples may make this point clear.

(i) In the series $1 . 2 . 3 + 2 . 3 . 4 + \dots.$

the convergency ratio is $\dfrac{r+2}{r-1} = \dfrac{(r-1)+3}{r-1}$.

Here $$f(r-1) = (r-1)+3,$$
and so $$f(r) = r+3, \quad c+f(r) = r-1,$$
so that $$c = -4.$$

(ii) In the series $\Sigma \{r!\,(r+6)!\}/\{(r+4)!\}^2$

the convergency ratio is $\dfrac{r(r+6)}{(r+4)^2} = \dfrac{\{(r-1)+1\}\{(r-1)+7\}}{(r+4)^2}$.

We have therefore $$f(r) = (r+1)(r+7), \quad c+f(r) = (r+4)^2,$$
and so $$c = 9.$$

(iii) In the series $\sum\limits_{r=2}^{n} r(1-a)(1-2a)\ldots\{1-(r-1)a\}$

the ratio is $$\frac{r}{r-1}\{1-(r-1)a\} = \left(\frac{1}{r-1} - a\right)\Big/\left(\frac{1}{r}\right).$$

Hence we have $$f(r) = \frac{1}{r} - a, \quad c+f(r) = \frac{1}{r},$$

and so $$c = a.$$

Here is a case in which $f(r)$ is not a polynomial in r.

(iv) As an example of a series in which the signs of the terms alternate we may take $\Sigma(-)^r\, {}^mC_r$.

Here the ratio is $$-\frac{m-r+1}{r} = \frac{r-1-m}{r}.$$

Hence we have $$f(r) = r-m, \quad c+f(r) = r,$$
and so $$c = m.$$

6. *Standard Forms of Series, and their Sums to* n *terms.*

As examples we shall take the series (i), (ii), (iii), (iv) above, for which the values of c have been determined. Except in the case of (iii) the first term will be segregated.

(i) $$1\,.\,2\,.\,3 + \sum_{r=2}^{n} r(r+1)(r+2).$$

The lowest value of the ratio (when $r=3$) is $\frac{5}{2}$, and since $c=-4$

the series is $$1\,.\,2\,.\,3 + 1\,.\,2\,.\,3\,.\,4\left(\frac{1}{1} + \frac{5}{1\,.\,2} + \ldots\right).$$

The sum of the series is
$$1\,.\,2\,.\,3 - \tfrac{1}{4}\{1\,.\,2\,.\,3\,.\,4 - n(n+1)(n+2)(n+3)\}.$$

(ii) $$(1!\,7!)/(5!)^2 + \sum_{r=2}^{n} \{r!\,(r+6)!\}/\{(r+4)!\}^2.$$

The lowest value of the ratio (for $r = 3$) is $3 \,.\, 9/7^2$, and since $c = 9$, the third term is $3 \,.\, 9/\{(9 + 3 \,.\, 9)\, 7^2\}$.

The series in standard form is therefore

$$\frac{1!\,7!}{(5!)^2} + \frac{2!\,8!}{(5!)^2}\left\{\frac{1}{6^2} + \frac{3 \,.\, 9}{6^2 \,.\, 7^2} + \frac{(3 \,.\, 9)(4 \,.\, 10)}{6^2 \,.\, 7^2 \,.\, 8^2} + \ldots.\right\}.$$

The sum is $$\frac{1}{9}\left\{\frac{5 \,.\, 6 \,.\, 7}{2 \,.\, 3 \,.\, 4} - \frac{(n+5)(n+6)(n+7)}{(n+2)(n+3)(n+4)}\right\}.$$

(iii) The summation being from $r = 2$ to $r = n$ the series will be taken as it stands.

The lowest value of the ratio is $(\frac{1}{2} - a)/\frac{1}{3}$, and since $c = a$, the second term is $(\frac{1}{2} - a)/(\frac{1}{2} \,.\, \frac{1}{3})$.

The series in standard form is therefore

$$(1 - a)\left\{\frac{1}{\frac{1}{2}} + \frac{(\frac{1}{2} - a)}{\frac{1}{2} \,.\, \frac{1}{3}} + \frac{(\frac{1}{2} - a)(\frac{1}{3} - a)}{\frac{1}{2} \,.\, \frac{1}{3} \,.\, \frac{1}{4}} + \ldots.\right\}.$$

The sum is

$$\frac{1 - a}{a}\{1 - (1 - 2a)(1 - 3a) \ldots. (1 - na)\}, \text{ for } n \geqq 2.$$

(iv) In order that the standard identity may be applied, this series must be written in the form

$$\frac{(-m)}{1!} + \frac{(-m)(-m+1)}{2!} + \ldots. + \frac{(-m)(-m+1)\ldots.(-m+n-1)}{n!}.$$

Segregating the first term, we have

$$\frac{(-m)}{1!} + \sum_{r=2}^{n} \frac{(-m)(-m+1)\ldots.(-m+r-1)}{r!}.$$

The lowest value of the ratio $(r=3)$ is $\dfrac{-m+2}{3}$; and since $c = m$ the third term of the series in standard form is $\dfrac{-m+2}{2 \,.\, 3}$.

The series in standard form is therefore

$$-\frac{m}{1!} + (-m)(-m+1)\left\{\frac{1}{2!} + \frac{-m+2}{3!} + \ldots.\right\}.$$

The sum is

$$-\frac{m}{1!} + \frac{-m(-m+1)}{m}\left\{1 - \frac{(-m+2)\ldots(-m+n)}{n!}\right\} = -1 + \frac{(-m+1)\ldots(-m+n)}{n!}.$$

(v) and (vi) The reader may find it interesting to segregate first terms and apply the method to the series

$$\{\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots(n+\tfrac{1}{2})\}^{-1}-\{\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots(n-\tfrac{1}{2})1!\}^{-1}+\{\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots(n-\tfrac{3}{2})2!\}^{-1}-\ldots+(-)^n(\tfrac{1}{2}\cdot n!)^{\cdot}$$

and $$1+\sum_{r=2}^{n} r\,\frac{3^{r-1}}{5\,.\,6\,.\,7\,\ldots\,(r+3)}\,.$$

It will be found that $c = n + \frac{1}{2}$ and 1 respectively, the sums of the series being $(-)^n\{(n+\frac{1}{2})\,n!\}^{-1}$ and $4 - \dfrac{3^n\,4!}{(n+3)!}$.

(vii) $$\frac{1}{v_1 w_1}+\sum_{r=2}^{n}\frac{u_1 u_2 \ldots u_{r-1}}{(v_1 v_2 \ldots v_r)(w_1 w_2 \ldots w_r)}\,.$$

This series has $r-1$ factors in the numerator of the r^{th} term, $2r$ factors in the denominator. The convergency ratio is $u_{r-1}/(v_r w_r)$, and the series can be summed by the standard identity if $v_r w_r - u_r$ is constant.

This condition will be fulfilled if

$$v_r = a + d_r,\ \ w_r = b + e_r,\ \text{ and }\ u_r = ae_r + bd_r + d_r e_r,$$

where a and b are constants, and where d_r, e_r are any functions of r.

The summation constant c is here ab, and the sum of the series to n terms is

$$\frac{1}{ab}\left\{1-\frac{u_1 \ldots u_n}{(v_1 \ldots v_n)(w_1 \ldots w_n)}\right\}.$$

The series (vi) above is the particular case of this general series for which $a = 1$, $b = 1$, $u_r = \dfrac{3}{r}$, $v_r = r + 3$, $w_r = \dfrac{1}{r}$.

General Conclusions.

The outcome of the above discussion seems to be that in cases where the possibility of summation is in question the convergency ratio should be examined, the terms having first been treated in such a way as to give the greatest possible overlap. If a summation constant c exists, it can be found in this way, and the standard form of the series can thus be determined. If there is no such summation constant, the series cannot be summed to a finite number of terms by algebraic methods. Since the nature of a series is intimately bound up with its characteristic convergency ratio, it is suggested that familiarity with the convergency ratios of the more important types of series is not only desirable but almost essential.