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CENTRAL COMMUTATORS

A. CARANTI AND C.M. SCOPPOLA

We give examples of finite groups of odd prime power order in which the commutators lying in the centre do not generate the intersection of the centre and the commutator subgroup.

The problem of finding a finite p-group G with commutator subgroup G_2 and centre $Z(G)$ so that $G_2 \cap Z(G)$ is not generated by commutators has been considered by R. Oliver, and communicated to the authors by J. Brandt.

To our knowledge, no group having this property has been explicitly recorded in the literature. The example we propose hereafter is meant to display the ideas involved in the construction of it; examples of smaller order, however, can be found, as remarked at the end of this paper.

Notation follows [1]. We use freely elementary commutator calculus as in III.1 of [1]. Matrices act on vectors from the right; that is, vectors are regarded as row vectors.

Let p be an odd prime number. Let E be an elementary abelian group of order p^3 , viewed as a vector space over $GF(p)$, and let

$$e_1, e_2, e_3$$

be a base of E. Let y be the automorphism of E given, with respect to

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the given base, by the matrix

$$\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} .$$

Since $p > 2$, y has order p. Then the semidirect product $H = E\langle y\rangle$ is a group of order p^4, and of nilpotency class 3. Indeed, if we set $x = e_1$, we get

$$(1) \qquad \begin{cases} [x,y] = e_2, \ [x,y,y] = e_3, \\ H_2 = [H,H] = \langle e_2,e_3\rangle, \ H_3 = [H_2,H] = \langle e_3\rangle = Z(H) . \end{cases}$$

H is group 12 of Satz III.12.6 of [1] (note that H has exponent p for $p > 3$, while H has exponent 9 for $p = 3$, check [1, Aufgabe 29, p. 349].

Let now A be an elementary abelian p-group of order p^{10}, viewed as a vector space over $GF(p)$, and let

$$a_1, a_2, a_3, c_2, c_3, b_1, b_2, b_3, d_2, d_3$$

be a base of A. Consider the elements $\xi, \eta \in Aut(A)$ given, with respect to the given base, in matrix form by

$$\xi = \begin{bmatrix} 1 & 1 & & & & & & & & \\ & 1 & 1 & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & 1 & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & 1 & \\ & & 1 & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & 1 \\ & & & & & & & & & 1 \end{bmatrix}, \quad \eta = \begin{bmatrix} 1 & & & 1 & & & & & & \\ & 1 & & & & & & 1 & & \\ & & 1 & & & & & & & \\ & & & 1 & 1 & & & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & 1 & & & \\ & & & & & & 1 & 1 & & \\ & & & & & & & 1 & & \\ & & 1 & & & & & & 1 & \\ & & & & & & & & & 1 \end{bmatrix} .$$

A straightforward computation gives

$$\xi\eta = \eta\xi ;$$

since ξ and η are unipotent, their Jordan forms have blocks of size at

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most 3, and $p > 2$, we have

$$\xi^p = \eta^p = 1;$$

since $\langle a_1, a_2, a_3 \rangle$ is invariant under ξ but not invariant under η, η is not a power of ξ. Therefore $\langle \xi, \eta \rangle$ is an elementary abelian subgroup of $Aut(A)$ of order p^2. Then setting $\rho(x) = \xi$, $\rho(y) = \eta$, a homomorphism $\rho : H \longrightarrow Aut(A)$ is defined, with

(2) $$ker(\rho) = H_2.$$

Let G be the semidirect product of A and H relative to ρ, so that $|G| = p^{14}$. Set $a_1 = a$, $b_1 = b$; then in G

(3) $$\begin{cases} a_2 = [a,x] \\ a_3 = [a,x,x] = [b,y,x] = [b,x,y] \\ c_2 = [a,y] \\ c_3 = [a,y,y], \end{cases}$$

(4) $$\begin{cases} b_2 = [b,y] \\ b_3 = [b,y,y] = [a,x,y] = [a,y,x] \\ d_2 = [b,x] \\ d_3 = [b,x,x] . \end{cases}$$

Furthermore, it is easy to see that a_3, b_3, c_3, d_3 are left fixed by ξ and η, and then

(5) $$a_3, b_3, c_3, d_3 \in Z(G) .$$

Since $A \triangleleft G$, $[A,H] \leq A$; by (2),

(6) $$[A,H_2] = 1 ,$$

and then $AH_2 = A \times H_2$; therefore, by easy commutator identities

$$G_2 = [G,G] = [AH,AH] = [A,H]H_2 = [A,H] \times H_2$$

and then, by (1), G_2 is the 10-dimensional direct sum of

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$$[A,H] = \langle a_2,\ a_3,\ c_2,\ c_3,\ b_2,\ b_3,\ d_2,\ d_3\rangle$$

and

$$H_2 = \langle [x,y]\ ,\ [x,y,y]\rangle\ .$$

By (6) and the above $[G_2,A] = 1$; then

$$G_3 = [G_2,G] = [G_2,AH] = [G_2,H] =$$

$$= [[A,H]H_2,H] = [A,H,H]H_3 = [A,H,H] \times H_3\ ,$$

and thus, by (1),

(7) G_3 is the 5-dimensional direct sum of $[A,H,H]$

$= \langle a_3,\ b_3,\ c_3,\ d_3\rangle$ and $H_3 = \langle [x,y,y]\rangle$.

By (5), (1) and (6) $G_3 \leq Z(G)$ and G has class 3. Consider now the element $g = [a,x][b,y]^{-1} \in G_2$; (3) and (4) yield

$$[g,x] = [a,x,x,][b,y,x]^{-1} = 1,\ [g,y] = [a,x,y][b,y,y]^{-1} = 1\ .$$

Since A is abelian, and $G = \langle A,x,y\rangle$, we obtain $g \in Z(G) \cap G_2$; by (7), however, $g \not\in G_3$. We now show that all central commutators of G, i.e. the commutators of G lying in $Z(G)$, are in fact in G_3. This will imply that g is not a product of central commutators. Suppose

(8) $$w = [g_1,g_2] \in Z(G),\ g_1,g_2 \in G.$$

It is enough to show $w \equiv 1 (mod\ G_3)$. For a suitable 2×4 matrix $t = [t_{ij}]$, with entries in $\mathbb{Z}$ we have, mod G_3,

(9) $$w \equiv [a^{t_{11}}b^{t_{12}}x^{t_{13}}y^{t_{14}},\ a^{t_{21}}b^{t_{22}}x^{t_{23}}y^{t_{24}}]$$

$$\equiv [a,x]^{m_{13}}[a,y]^{m_{14}}[b,x]^{m_{23}}[b,y]^{m_{24}}[x,y]^{m_{34}},$$

where $m_{ij} = t_{1i}t_{2j} - t_{1j}t_{2i}$. Since congruences are mod G_3 , and $G_3 \leq Z(G)$ we get, by (8), (3) and (4),

$$1 = [w,y] = b_3^{m_{13}+m_{24}}c_3^{m_{14}}a_3^{m_{23}}[x,y,y]^{m_{34}}\ .$$

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Therefore by (7) we have, mod p, $m_{13} + m_{24} \equiv 0$, $m_{14} \equiv m_{23} \equiv m_{34} \equiv 0$.

Now the well known relation

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0$$

holds among the 2 × 2 minors of t (see for instance [1], Beispiel III.1.12); we obtain $m_{ij} \equiv 0$ for $\{i,j\} \neq \{1,2\}$, and thus, by (9), $w \equiv 1 (\mathrm{mod}\, G_3)$, as required.

Note that an example of smaller order can be obtained by adding to a presentation of G as above the relation $[b,x,x] = 1$. It is conceivable that examples of even smaller order exist.

Reference

[1] B. Huppert, *Endliche Gruppen I* (Springer, Berlin, 1967).

Department of Mathematics,
University of Trento,
I-38050 Povo (Trento),
ITALY.

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