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# REMARKS ON OP AND TOWBER RINGS

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**1. Introduction.** In this paper all rings considered have identity and are commutative, and all modules are finitely generated. We shall make liberal use of the definitions and notation established in [**6**; **7**].

Towber observed in [**9**] that a local Outer Product ring (OP-ring) must have v-dimension $\leqq 2$, and so a local OP-ring is either regular of global dimension $\leqq 2$ or it has infinite global dimension. Since the global dimension of a noetherian ring is the supremum of the global dimensions of its localizations, we immediately obtain the following result.

THEOREM 1.1. *The global dimension of a noetherian OP-ring is either* $\infty$ *or* $\leqq 2$.

All of these cases do in fact occur, for if $K$ is a field, then $K$, $K[x]$, and $K[x, y]$ are OP-rings of global dimensions 0, 1, and 2, respectively, and $Z_4$ is an OP-ring of infinite global dimension (see [**4**, Chapter VI, Exercise 1] and Theorem 2.1 of this paper).

In all the cases we have considered, the Towber rings are precisely the OP-rings of finite global dimension; we conjecture that this is always the case for noetherian rings. Of course, a noetherian Towber ring is an OP-ring by definition and has finite global dimension [**7**, Theorem 4.7]; thus our conjecture amounts to the converse. In this paper we shall give several instances which support this conjecture.

A noetherian ring of global dimension $\leqq 1$ is a finite direct sum of Dedekind domains [**1**, Proposition 4.13]; Dedekind domains are Towber rings [**9**, Theorem 1.2]; and the finite direct sum of Towber rings is a Towber ring [**7**, Theorem 5.4]; hence we have the following result.

THEOREM 1.2. *A noetherian ring of global dimension* $\leqq 1$ *is a Towber ring.*

This, of course, provides a partial answer to our conjecture, and reduces the question to rings of global dimension 2.

In § 2 we investigate the OP and Towber properties in several special classes of rings (principal ideal rings (PIR), rings with descending chain condition (DCC), local rings, and non-semi-simple integral domains, i.e. integral domains with non-zero Jacobson radical) and verify the conjecture in each of these cases. We also state a conjecture equivalent to the original one, which may be a little more accessible.

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In § 3 we study the structure of $u^\perp$ under various conditions, and verify the conjecture formulated in § 2, and hence also the original conjecture, for unique factorization domains (UFD). Along the way we show that a unique factorization domain (which need not be noetherian) of global dimension $\leqq 2$ is a Towber ring if and only if its finitely generated projective modules are free; this generalizes slightly the equivalence of [**7**, parts (1) and (3) of Theorem 7.5]. We also show, as a consequence of these considerations, that in a noetherian integral domain of global dimension $\leqq 2$ the inverse of any fractional ideal is an invertible ideal.

We close with some general remarks about the relation between the structure of $u^\perp$ and the decomposability of $u$; roughly speaking, these are not as closely related as some of the results in [**7**; **9**] would seem to imply.

## 2. Examples.

*Definition* 2.1 [**11**, Chapter IV, § 15, p. 245]. A principal ideal ring (PIR) is called *special* if it has a unique proper prime ideal and that prime ideal is nilpotent.

LEMMA 2.1. *A special PIR is an OP-ring.*

*Proof.* Every vector over such a ring is a constant multiple of a unimodular vector; thus it will be sufficient to show that all unimodular vectors are outer products, i.e., that these rings are H-rings. But they are surely local rings, and local rings are H-rings by [**7**, Corollary 2.12].

THEOREM 2.1. *Every PIR is an OP-ring.*

*Proof.* A PIR is a finite direct sum of principal ideal domains (PID) and special PIRs [**11**, Chapter IV, Theorem 33] and the OP-property is invariant under direct sums [**7**, Theorem 5.4], hence this follows from Lemma 2.1 and [**6**, Theorem 2.2].

As in [**7**], we shall use $D(R)$ to denote the global dimension of the ring $R$. See also [**7**, § 4] for the definition of $\mathrm{T}_p{}^n$.

THEOREM 2.2. *If $R$ is a PIR, then the following statements are equivalent:*
(1) *$R$ is a Towber ring;*
(2) *$R$ has property $\mathrm{T}_2{}^4$;*
(3) *$R$ is a finite direct sum of PIDs;*
(4) $D(R) \leqq 1$;
(5) $D(R) < \infty$.

*Proof.* (1) $\Leftrightarrow$ (2). This is [**7**, Theorem 7.4].

(2) $\Rightarrow$ (3). This is a consequence of [**7**, Proposition 4.4 and Theorem 5.4], and the structure theorem for PIRs mentioned above.

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(3) $\Rightarrow$ (4). If $R$ is a PID, then $D(R) \leqq 1$, and $D(\sum_i \oplus R_i) = \sup_i D(R_i)$.

(4) $\Rightarrow$ (5). This is clear.

(5) $\Rightarrow$ (1). If $D(R) < \infty$, then no direct summand of $R$ is a special PIR, since a non-trivial special PIR is an irregular local ring and so has infinite global dimension. Hence $R$ is a direct sum of PIDs and therefore a Towber ring [**7**, Theorem 5.4; **6**, Theorem 2.2].

Thus the conjecture mentioned in the introduction is valid for PIRs: these are all OP-rings, and they are Towber rings if and only if their global dimension is finite. The next theorem establishes the conjecture for (commutative) rings with DCC.

THEOREM 2.3. *For a ring $R$ with DCC the following statements are equivalent:*

(1) *$R$ is a Towber ring;*

(2) *$R$ has property* $T_2^4$*;*

(3) *$R$ is a finite direct sum of fields;*

(4) $D(R) = 0$*;*

(5) $D(R) < \infty$.

*Proof.* (1) $\Leftrightarrow$ (2). This is [**7**, Theorem 7.4].

(2) $\Rightarrow$ (3). Every ring with DCC is a direct sum of noetherian primary rings [**11**, Chapter IV, Theorem 3], i.e., noetherian rings with exactly one proper prime ideal. Since property $T_2^4$ is inherited by every summand of $R$, we need only show that such a ring with property $T_2^4$ must be a field.

Let $R$ be a noetherian primary ring with maximal ideal $M$; then $\bigcap_{k=0}^{\infty} M^k = (0)$ by Krull's Theorem, and $R$ has the DCC, and so $M$i s nilpotent. This contradicts [**7**, Proposition 4.4] unless $M = (0)$, i.e., unless $R$ is a field.

(3) $\Rightarrow$ (4). Fields have global dimension 0, and $D(\sum_i \oplus R_i) = \sup_i D(R_i)$.

(4) $\Rightarrow$ (5). This is obvious.

(5) $\Rightarrow$ (1). If $A$ is a primary ring with maximal ideal not (0), then Krull dimension $A = 0$ and v-dim $A > 0$, and so $D(A) = \infty$. Thus $D(R) < \infty$ implies that all summands of $R$ are primary rings whose sole proper prime ideal is (0), i.e., fields. Of course, fields are Towber rings and the Towber property is preserved by finite direct sums.

Towber has shown that a local ring is an OP-ring if and only if it has v-dimension $\leqq 2$ [**10**] and a Towber ring if it has global dimension $\leqq 2$ [**9**, Theorem 1.1]. In [**7**, Theorem 4.7] we showed that a local ring is a Towber ring only if it has global dimension $\leqq 2$; thus our conjecture also holds for local rings.

Since a noetherian ring of finite global dimension is a (finite) direct sum of integral domains [**1**, Corollary 4.4] and since the OP and Towber properties are invariant under direct sums, it would be sufficient to verify the conjecture for integral domains. For the rest of this paper we shall restrict ourselves to this case.

Thus, let $R$ be an integral domain, $\Omega$ its maximal spectrum, $\Omega_p$ its prime spectrum, and $J$ its Jacobson radical. For any closed subset $F$ of $\Omega$ and any ideal $I$ in $R$ let

$$I(F) = \bigcap_{x \in F} x \qquad \text{and} \qquad W(I) = \{p \in \Omega_p | \, p \supseteq I\};$$

then $I(F)$ is an ideal in $R$ and $W(I)$ is a closed subset of $\Omega_p$.

LEMMA 2.2. *If $R$ is noetherian and $F$ is irreducible, then $I(F)$ is a prime ideal in $R$ and $F = W(I(F)) \cap \Omega$.*

*Proof.* $F = W(I) \cap \Omega$ for some ideal $I$, since $F$ is closed. Then certainly $I \subseteq I(F)$, and so $W(I(F)) \subseteq W(I)$ and $W(I(F)) \cap \Omega \subseteq F$. The opposite inclusion is clear, thus $W(I(F)) \cap \Omega = F$, and it remains only to show that $I(F)$ is prime. Let $p_1, \ldots, p_k$ be the prime ideals belonging to $I(F)$; then $W(I(F)) = W(p_1) \cup \ldots \cup W(p_k)$ and

$$F = [W(p_1) \cap \Omega] \cup \ldots \cup [W(p_k) \cap \Omega].$$

Since $F$ is irreducible, it follows that $F = W(p_i) \cap \Omega$ for some $i$. Then $p_i \subseteq \bigcap_{x \in F} x = I(F)$, and so $p_i = I(F)$ and $I(F)$ is prime.

LEMMA 2.3. *If $R$ is a non-semi-simple noetherian integral domain, then* $\dim \Omega \leqq D(R) - 1$.

*Proof.* If $F_0 \subset F_1 \subset \ldots \subset F_n$ is a proper chain of irreducible closed sets in $\Omega$, then $I(F_0) \supseteq I(F_1) \supseteq \ldots \supseteq I(F_n)$ is a chain of prime ideals in $R$, and the inclusions are proper, since, by Lemma 2.2 $I(F_\alpha) = I(F_{\alpha+1})$ would imply $F_\alpha = F_{\alpha+1}$. Of course

$$I(F_n) = \bigcap_{x \in F_n} x \supseteq \bigcap_{x \in \Omega} x = J \qquad \text{and} \qquad J \neq (0);$$

thus $I(F_n) \neq (0)$ and $I(F_0) \supset I(F_1) \supset \ldots \supset I(F_n) \supset (0)$ is a proper chain of prime ideals in $R$ of length $n + 1$. Thus the Krull dimension of $R$ must be $\geqq \dim \Omega + 1$ and of course $D(R) \geqq$ Krull dimension of $R$, and hence $D(R) \geqq \dim \Omega + 1$ also.

THEOREM 2.4. *For non-semi-simple noetherian integral domains the Towber property is equivalent to global dimension* $\leqq 2$.

*Proof.* The Towber property implies global dimension $\leqq 2$ [**7**, Theorem 4.7], and so we need only show the converse. Of course $D(R) \leqq 2$ implies $\dim \Omega \leqq 1$; thus by [**8**, Theorem 1], every projective $R$-module is the direct sum of a free module and an ideal. Then $R$ is a Towber ring [**7**, Theorem 3.5].

Again, our conjecture holds for the given class of rings. The next theorem will enable us to restate the conjecture in a possibly more accessible form.

THEOREM 2.5. *Let $R$ be a noetherian integral domain with property* $T_2^3$. *Then* $D(R) \leqq 2$ *if and only if $R$ is integrally closed.*

*Proof. Necessity.* A noetherian integral domain of finite global dimension is integrally closed [**1**, Proposition 4.2].

*Sufficiency.* All the hypotheses are retained by localization, and so we may assume that $R$ is a local ring. Then $R$ has v-dimension $\leqq 2$ [**7**, Proposition 4.6], and so has Krull dimension $\leqq 2$ also. If the Krull dimension is 0, then $R$ is a field and $D(R) = 0$, while if the Krull dimension is 2, then $R$ is regular and $D(R) = 2$. If the Krull dimension of $R$ is 1, then the maximal ideal of $R$ is the only non-trivial prime ideal; in this case $R$ is noetherian, integrally closed, and its only prime is maximal; thus $R$ is a Dedekind domain [**11**, Chapter V, Theorem 13] and $D(R) = 1$. Thus $D(R) \leqq 2$ in any case.

In view of the above theorem our conjecture amounts to the following statement.

For integrally closed noetherian domains, the OP and Towber properties are equivalent.

In the next section (Theorem 3.4) we shall show that this is, at any rate, the case for UFDs.

THEOREM 2.6. *Let $R$ be a noetherian integral domain with global dimension $\leqq 2$ and trivial class group. Then $R$ is a Towber ring if and only if all finitely generated projective $R$-modules are free.*

*Proof.* If projective $R$-modules are free, then $R$ is a Towber ring [**7**, Theorem 3.5]; thus suppose that $R$ is a Towber ring, and let $P$ be a (finitely generated) projective $R$-module. By [**7**, Theorem 7.1], there is an $R$-module $I$ of rank 1 for which $P \oplus I$ is free. $I$ is then projective, hence isomorphic to an invertible ideal, and hence free, since by hypothesis invertible ideals are principal. Then $P$ is also free, since Towber rings are certainly H-rings.

Of course, UFDs always have trivial class group, and so it follows in particular that a noetherian UFD of global dimension $\leqq 2$ is a Towber ring if and only if its projective modules are free. In the next section (Theorem 3.2) we show that this is in fact true without the noetherian hypothesis.

**3. The structure of $u^\perp$.** Let $R$ be an integral domain with field of quotients $K$ and let $u \in \bigwedge^s K^n$. As in [**7**], we shall consider $R^n$ to be contained in $K^n$ and $\bigwedge R^n$ in $\bigwedge K^n$, and shall use $u_K{}^\perp$ to denote $\{v \in K^n | v \wedge u = 0\}$ and $u_R{}^\perp$ (or sometimes simply $u^\perp$ if $u \in \bigwedge^s R^n$) for $u_K{}^\perp \cap R^n$.

PROPOSITION 3.1. *Let $R$ be an integral domain, let $P \otimes Q = R^n$ and rank $P = s$. Then $P = u^\perp$ for some decomposable vector $u \in \bigwedge^s R^n$.*

*Proof.* As usual we use $P_K$ to denote the $K$-space $P \oplus_R K$, and consider this to be contained in $K^n$; then $K^n = P_K \oplus Q_K$. Let $\{v_1, \ldots, v_s\}$ be a basis for $P_K$ and $\{v_{s+1}, \ldots, v_n\}$ a basis for $Q_K$, and let $u = v_1 \wedge \ldots \wedge v_s \in \bigwedge^s K^n$. Then $P_K = u_K{}^\perp$ and $P = P_K \cap R^n$, and so $P = u_R{}^\perp$. By clearing denominators if necessary we may assume that $v_1, \ldots, v_s \in P$; then $u$ is a decomposable vector in $\bigwedge^s R^n$ and $P = u^\perp$, as desired.

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Thus every projective $R$-module actually occurs as the annihilator of some decomposable vector.

PROPOSITION 3.2 (Towber). *Let $R$ be a UFD and $u$ a Plücker vector in $\bigwedge^k R^n$ whose natural coordinates have greatest common divisor* 1. *Then $u$ is decomposable if and only if $u^\perp$ is free.*

*Proof.* This is [**9**, Theorem 2.3]. (Actually, Towber states his theorem without the gcd condition, but his proof assumes this and the theorem would be false without it. For if it were true, then by our Proposition 3.1 every projective module over a UFD would be free, and this is not the case.)

THEOREM 3.1. *If $R$ is a (not necessarily noetherian) UFD with property $T_k^n$ and $u$ is a Plücker vector in $\bigwedge^k R^n$, then $u^\perp$ is free.*

*Proof.* If the gcd of the natural coordinates of $u$ is 1, the theorem follows from Proposition 3.2. If the gcd is not 1, we factor it out; the resulting vector is also a Plücker vector, and hence decomposable. Thus $u = cv$, where $c \in R$ and $v^\perp$ is free by Proposition 3.2. Of course $u^\perp = v^\perp$, hence this proves the theorem.

COROLLARY 3.1. *If $R$ is a (not necessarily noetherian) Towber UFD, then finitely generated projective $R$-modules are free.*

*Proof.* By Proposition 3.1, every finitely generated projective module occurs as a $u^\perp$ for some decomposable exterior vector $u$. Decomposable vectors are certainly Plücker vectors; thus the corollary then follows from Theorem 3.1.

By [**7**, Proposition 3.1; **9**, Proposition 1.3], any integral domain of global dimension $\leqq 2$ whose projective modules are free is a Towber ring. This, together with the preceding corollary, yields the following result.

THEOREM 3.2. *A (not necessarily noetherian) UFD of global dimension $\leqq 2$ is a Towber ring if and only if its finitely generated projective modules are all free.*

An argument similar to that of Theorem 3.1 yields a simple proof of the following well-known result, which we used in § 2.

THEOREM 3.3. *UFDs have trivial class group.*

*Proof.* Let $I$ be a projective ideal. By Proposition 3.1, $I = u^\perp$ for some $u \in \bigwedge^1 R^n$, and as before we can write $u = cv$ with $v$ a vector whose coordinates have gcd 1. Then $u^\perp = v^\perp$ and $v$ is trivially decomposable; thus $I$ is free by Proposition 3.2, and therefore principal.

THEOREM 3.4. *For noetherian UFDs the OP and Towber properties are equivalent.*

*Proof.* Towber rings are certainly OP-rings; thus we must show the converse. If $R$ is a noetherian UFD with the OP property, then $D(R) \leqq 2$ by Theorem

2.5, and by Theorem 2.6 it will be sufficient to show that projective $R$-modules are free.

In view of Serre's theorem [**8**, Theorem 1], we need only show that projective modules of rank 2 are free, so let $P$ be projective of rank 2 and choose $Q$ so that $P \oplus Q = R^m$ for some $m$. By applying Serre's theorem to $Q$ and Bass' cancellation theorem [**2**, Theorem 9.3] to $P \oplus Q$ we may further assume that rank $Q = 2$. Then $P \oplus Q = R^4$ and there is a decomposable vector $u \in \bigwedge^2 R^4$ such that $P = u^\perp$; let $u = v_1 \wedge v_2$. Then $v_1 \wedge v_2 \neq 0$ (since $P \neq R^4$) and we can find $v_3 \in R^4$ such that $v_1 \wedge v_2 \wedge v_3 \neq 0$ also. Let

$$S = (v_1 \wedge v_2 \wedge v_3)^\perp.$$

Then $S \approx R^3$ by Theorem 3.1, and of course $P \subseteq S$. In particular, $v_1$ and $v_2$ are in $S$, and $P = (v_1 \wedge v_2)^\perp$; thus $P$ is free by another application of Theorem 3.1.

LEMMA 3.1. *Let $R$ be a noetherian integral domain and $K_1, \ldots, K_p$ non-zero fractional ideals of $R$. Let $I_j = R\colon K_j$ for each $j$, and $M = I_1 \oplus \ldots \oplus I_p$. Then $M \approx (v_1 \wedge \ldots \wedge v_p)^\perp$ for some $n$ and some $v_1, \ldots, v_p \in R^n$.*

*Proof.* Since $R$ is noetherian, each $K_j$ is finitely generated; say $K_j$ is generated by $\{a_1^{(j)}, \ldots, a_{s_j}^{(j)}\}$ for each $j$. We may, of course, assume that the $K_j$s are all integral ideals, in which case $a_i^{(j)} \in R$ for all $i$ and $j$. Let

$$n = s_1 + \ldots + s_p$$

and define vectors $v_1, \ldots, v_p \in R^n$ as follows:

$$\begin{aligned} v_1 &= (a_1^{(1)}, \ldots, a_{s_1}^{(1)}, 0, \ldots, 0), \\ v_2 &= (0, \ldots, 0, a_1^{(2)}, \ldots, a_{s_2}^{(2)}, 0, \ldots, 0), \\ &\cdot \\ &\cdot \\ &\cdot \\ v_p &= (0, \ldots, 0, a_1^{(p)}, \ldots, a_{s_p}^{(p)}). \end{aligned}$$

Let $F$ be the field of quotients of $R$. For each $j$, some $a_i^{(j)} \neq 0$ (since $K_j \neq (0)$), and so the $v_j$s are linearly independent over $F$, and

$$(v_1 \wedge \ldots \wedge v_p)_F{}^\perp = Fv_1 \oplus \ldots \oplus Fv_p.$$

Of course $(v_1 \wedge \ldots \wedge v_p)_R{}^\perp = (v_1 \wedge \ldots \wedge v_p)_F{}^\perp \cap R^n$, and so

$$w \in (v_1 \wedge \ldots \wedge v_p)_R{}^\perp$$

if and only if $w \in R^n$ and $w = f_1 v_1 + \ldots + f_p v_p$ for some $f_1, \ldots, f_p \in F$. Clearly $f_1 v_1 + \ldots + f_p v_p \in R^n$ if and only if $f_j a_i^{(j)} \in R$ for all $j$ and all relevant $i$, i.e., if and only if $f_j \in I_j$ for all $j$. Thus $(v_1 \wedge \ldots \wedge v_p)_R{}^\perp = I_1 v_1 \oplus \ldots \oplus I_p v_p \approx I_1 \oplus \ldots \oplus I_p$, as desired.

If $I$ is a non-zero fractional ideal in an integral domain $R$, the ideal $R\colon I$ in general need not be invertible, as examples in [**3**; **5**] show. By Lemma 3.1,

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however, if $R$ is noetherian we do have $R{:}I \approx v^\perp$ for some $v \in \bigwedge^1 R^n$, and hence $\dim_R(R{:}I) \leqq \max(D(R) - 2, 0)$ [**7**, Proposition 3.1]. If $D(R) \leqq 2$, $R{:}I$ is therefore projective and hence invertible; this proves the following result.

THEOREM 3.5. *If $R$ is a noetherian integral domain of global dimension $\leqq 2$, then $R{:}I$ is invertible for every non-zero fractional ideal $I$.*

Again let $R$ be a noetherian integral domain and $u$ a Plücker vector in $\bigwedge^p R^n$. In [**7**, Theorem 3.4; **9**, Theorem 2.2], it is stated, roughly, that if $u^\perp$ is fairly simple, then $u$ is decomposable. The converse, however, is not true; $u^\perp$ can be very complicated even when $u$ is decomposable. One indication of this is given by Proposition 3.1, which states that every projective $R$-module can be realized as a $u^\perp$ for some decomposable $u$. Another indication is [**7**, Proposition 3.2], according to which $u^\perp$ need not even be projective. Specifically, if $R$ is a noetherian integral domain of global dimension $n$ ($\geqq 2$) this proposition yields a Plücker vector $u$ for which $\text{h-dim}(u^\perp) = n - 2$. $u$ is then decomposable over the quotient field of $R$, and so some scalar multiple is decomposable over $R$, and of course has the same annihilator; thus this proposition yields decomposable vectors whose annihilators have arbitrarily large homological dimension (or at any rate the largest homological dimension allowed by [**7**, Proposition 3.1]). If $R$ is not regular (that is, if some localization of $R$ has infinite global dimension), the construction yields a decomposable vector whose annihilator has infinite homological dimension.

The next proposition indicates the universality of this sort of behaviour.

PROPOSITION 3.3. *Let $R$ be a regular noetherian domain of global dimension $\geqq 3$. Then for every pair $(q, n)$ such that $2 \leqq q < n$ there is a decomposable vector $u \in \bigwedge^q R^n$ for which $u^\perp$ is not projective.*

LEMMA 3.2. *If $R$ is a local ring with property $\mathrm{T}_q^{\,n}$, then* $\text{v-dim}\, R \leqq 2$.

The proof of the lemma is a simple adaptation of the argument used in the proof of [**9**, Theorem 2.4]; we will not repeat that argument here.

*Proof of Proposition* 3.3. Since $R$ is noetherian and $D(R) \geqq 3$, there is a maximal ideal $M$ for which $D(R_M) \geqq 3$. By Lemma 3.2, $R_M$ does not have property $\mathrm{T}_q^{\,n}$; thus there is a Plücker vector $u \in \bigwedge^q R_M^{\,n}$ which is not decomposable. By [**9**, Proposition 1.3], $(u^\perp)_{R_M}$ is then not free, and hence not projective, since $R_M$ is a local ring. But $(u^\perp)_{R_M}$ is (isomorphic to) the localization of $u_R{}^\perp$ at $M$ [**7**, Proposition 2.15], and so it follows that $u_R{}^\perp$ is not projective either.

Since $u$ is a Plücker vector, it is decomposable over the quotient field of $R_M$, which is of course the same as the quotient field of $R$; a suitable scalar multiple $cu$ will then be decomposable over $R$, and of course $(cu)_R{}^\perp = u_R{}^\perp$, which is not projective.

Perhaps the real point here is that for any Plücker vector $u$ over an integral domain $R$ one can find a non-zero $r \in R$ such that $ru$ is decomposable. Then $(ru)^{\perp} = u^{\perp}$, $ru$ is decomposable, and $u$ in general will not be; hence the structure of $u^{\perp}$ is not as closely related to the decomposability of $u$ as some of our earlier results might suggest.

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