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SOME NON-EMBEDDING THEOREMS FOR THE GRASSMANN MANIFOLDS $G_{2,n}$ AND $G_{3,n}$

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In recent years, the problem of embedding the projective spaces in Euclidean spaces was studied very much, by different methods. Usually, the negative results on the embedding problem are proved by using suitable homotopy invariants. The best known example of such homotopy invariants is given by the Stiefel–Whitney classes.

Let $G_{m,n}$ be the Grassman manifold of the non-oriented, m-dimensional vector subspaces (m-planes) of the vector space R^{m+n}. Then $G_{m,n}$ is a compact manifold of dimension mn. The manifold $G_{1,n}$ is just the real projective space P_n.

In this paper we generalise some non-embedding results obtained in the case of the projective space P_n to the case of the Grassman manifolds $G_{2,n}$ and $G_{3,n}$. By the classical results of Whitney

$$G_{2,n} \subset R^{4n}; \quad G_{2,n} \subseteq R^{4n-1}$$
$$G_{3,n} \subset R^{6n}; \quad G_{3,n} \subseteq R^{6n-1}$$

where $\subset$ denotes an embedding and $\subseteq$ denotes an immersion.

Let M be an n-dimensional compact manifold. It is known that if $M \subset R^{n+k}$ then the dual Stiefel–Whitney classes $\bar{w}_i(M)$, $i = k, k+1, \ldots, m$, of M vanish. By finding some non-vanishing dual Stiefel–Whitney classes of $G_{2,n}$ and $G_{3,n}$ we obtain a lower bound for the dimension of the Euclidean space in which these manifolds could be embedded.

Our main results are given by the following two theorems:

Theorem 1. *Let $s = 2^r$ be the least integer exceeding n, i.e. $2^{r-1} \leq n < 2^r$.*
(i) *If $n \neq s-1$ then $G_{2,n} \not\subset R^{2s-2}$ $(G_{2,n} \not\subseteq R^{2s-3})$;*
(ii) *For $n = s-1$, $G_{2,s-1} \not\subset R^{3s-2}$ $(G_{2,s-1} \not\subseteq R^{3s-3})$.*

Theorem 2. *Let $s = 2^r$ be the integer obtained from the condition $2^{r+1} < 3n < 2^{r+2}$, where $n \geq 3$.*
(i) *If $\frac{2}{3}s < n \leq s-3$, then $G_{3,n} \not\subset R^{3s-3}$;*
(ii) *For $n = s-2$; $G_{3,s-2} \not\subset R^{4s-3}$; for $n = s-1$ and $s \geq 8$; $G_{3,s-1} \not\subset R^{5s-3}$; $G_{3,3} \not\subset R^{15}$;*
(iii) *If $s \leq n < \frac{4}{3}s$ then $G_{3,n} \not\subset R^{6s-3}$.*

We should remark that (modulo the multiplication by 2) the non-embedding codimension obtained in Theorem 1 for $G_{2,n}$, in the case $n \neq 2^r - 1$, is similar to the non-embedding codimension for the projective space P_n, obtained using the Stiefel–Whitney classes (see **7**). Thus, our result is as strong as is the corresponding theorem in the case of projective space. In the case $n = 2^r - 1$ our result is stronger since, for the projective space P^{2^r-1}, the use of the Stiefel–Whitney classes gives no information on the non-embedding codimension.

Example. Let $n = 2^r = s$. Then $G_{2,s} \not\subset R^{4s-2}$ and our result is quite strong because $G_{2,s} \subset R^{4s}$ and it would remain to study the possibility to embed $G_{2,s}$ in R^{4s-1}.

The non-embedding codimension, calculated from Theorem 2, is greater than 6 in the case (i), is greater than $\frac{3}{2}(n-2)$ in the case (iii), greater than or equal to $2^r + 6$ for $n = 2^r - 2$ and greater than or equal to 2^{r+1} for $n = 2^r - 1$. Then, in the case (i), the non-embedding codimension for $G_{3,n}$ is similar (modulo the multiplication by 3) to the non-embedding codimension for the projective space P_n. The case (iii) furnishes a quite strong result for $n = s = 2^r$. The non-embedding codimension is, in this case, $3s - 3$. Thus, it would remain to study the possibility to embed $G_{3,s}$ in R^{6s-2} or in R^{6s-1}, since $G_{3,s} \subset R^{6s}$.

Remark. It would be interesting to see what (affirmative) methods used in the embedding problem for the projective spaces can be generalised to the case of the Grassmann manifolds.

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1. The Stiefel–Whitney classes of the manifolds $G_{2,n}$ and $G_{3,n}$

Let γ be the canonical m-vector bundle on the Grassmann manifold $G_{m,n}$ and let γ^* be its dual. Let τ be the tangent bundle of $G_{m,n}$. Then τ can be expressed with the help of the canonical vector bundle γ by the formula (see (**4, 6, 8**))

$$\tau \oplus (\gamma \otimes \gamma^*) = (m+n)\gamma. \tag{1}$$

Thus follows the corresponding formula satisfied by the total Stiefel–Whitney class $w(G_{m,n})$ of $G_{m,n}$

$$w(G_{m,n})w(\gamma \otimes \gamma^*) = w^{m+n} \tag{2}$$

where

$$w = 1 + w_1 + w_2 + \cdots + w_m; \quad w_i \in H^i(G_{m,n}; Z_2); \quad (i = 1, \ldots, m),$$

is the total Stiefel–Whitney class of the canonical bundle γ.

In the cases $m=2$, $m=3$, using the formula for the Stiefel–Whitney class of a tensor product of two vector bundles (1, 5, 9), it follows that

$$w(\gamma\otimes\gamma^*)=(1+w_1)^2=1+w_1^2;\quad m=2$$

$$w(\gamma\otimes\gamma^*)=(1+w_1^2+w_2+w_1w_2+w_3)^2=[(1+w_1)(1+w_1+w_2)+w_3]^2;\quad m=3$$

Hence

$$w(G_{2,n})(1+w_1)^2=(1+w_1+w_2)^{n+2} \tag{3}$$

$$w(G_{3,n})[(1+w_1)(1+w_1+w_2)+w_3]^2=(1+w_1+w_2+w_3)^{n+3}. \tag{4}$$

Remark. From these formulae it follows that $w_1(G_{2,n})=(n+2)w_1$; $w(G_{3,n})=(n+3)w_1$, so that $G_{2,n}(G_{3,n})$ is orientable if and only if n is an even (odd) number. In the general case of $G_{m,n}$ one can see easily, using the results of (9), that $w(\gamma\otimes\gamma^*)$ is the square of a certain expression, so that $w_1(\gamma\otimes\gamma^*)=0$ and, then $w_1(G_{m,n})=(m+n)w_1$. Thus, it follows that $G_{m,n}$ is orientable if and only if $m+n$ is an even number.

We are interested in obtaining the total dual Stiefel–Whitney classes $\bar{w}(G_{2,n})$, $\bar{w}(G_{3,n})$ of the manifolds $G_{2,n}$, $G_{3,n}$ i.e. the classes characterised by the relations:

$$w(G_{2,n})\bar{w}(G_{2,n})=1;\quad w(G_{3,n})\bar{w}(G_{3,n})=1.$$

Let r be the unique integer such that the dimension $2n$ of $G_{2,n}$ satisfies the relation

$$2^r\leq 2n<2^{r+1}.$$

One excludes $G_{2,1}$ which is diffeomorphic to the projective plane.

For typographical reasons, we replace 2^r by s from now on. Then

$$(1+w_1+w_2)^{2s}=[(1+w_1+w_2)^2]^s=(1+w_1^2+w_2^2)^s=\cdots=1+w_1^{2s}+w_2^{2s}=1,$$

since the dimensions of w_1^{2s}, w_2^{2s} exceed the dimension $2n$ of $G_{2,n}$. Thus, multiplying (3) by $(1+w_1+w_2)^{2s-n-2}$, one obtains

$$\bar{w}(G_{2,n})=(1+w_1)^2(1+w_1+w_2)^{2s-n-2};\quad 2^{r-1}\leq n<2^r=s. \tag{5}$$

In the same way, considering the unique integer r defined by the conditions

$$2^{r+1}<3n<2^{r+2}$$

one obtains

$$\bar{w}(G_{3,n})=[(1+w_1)(1+w_1+w_2)+w_3]^2(1+w_1+w_2+w_3)^{4s-n-3}. \tag{6}$$

One excludes the case of $G_{3,1}$ which is diffeomorphic to the three-dimensional projective space.

The cohomology ring, modulo 2, of the Grassmann manifold $G_{m,n}$ was determined by S. S. Chern who gave, in (2, 3), the following multiplication formula

$$(a_1, a_2, \ldots, a_m)(0, 0, \ldots, h) = \sum (b_1, b_2, \ldots, b_m)$$

where $(a_1, \ldots, a_m)$; $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_m \leqslant n$ is a Schubert cocycle modulo 2, of dimension $a_1 + a_2 + \cdots + a_m$, $(0, 0, \ldots, 0, h) = \bar{w}_h$ is the dual Stiefel–Whitney class of dimension h of γ and the summation is extended over all combinations $(b_1, \ldots, b_m)$ such that

$$0 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_m \leqslant n; \quad a_i \leqslant b_i \leqslant a_{i+1}; \quad i = 1, \ldots, m; \; a_{m+1} = n,$$
$$a_1 + \cdots + a_m + h = b_1 + \cdots + b_m.$$

Remark. Let $\bar{w}_0 = 1$ and $\bar{w}_h = 0$ for $h < 0$ or $h > n$. Then, the given multiplication formula implies the following formula

$$(a_1, \ldots, a_m) = \det(\bar{w}_{a_i + i - j})$$

i.e. every Schubert cocycle can be expressed in terms of $\bar{w}_h$.

The Stiefel–Whitney classes of γ are

$$w_i = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{i})$$

The following lemmas can be verified by an easy induction on h.

Lemma 1. *The following formula is valid in the cohomology ring* $H^*(G_{m,n}; Z_2)$

$$w_m^h = (1, \ldots, 1)^h = (h, \ldots, h); \quad h \leqslant n. \tag{7}$$

Lemma 2. *Let* $w = 1 + w_1 + w_2$ *be the total Stiefel–Whitney class of the canonical bundle* γ *on* $G_{2,n}$. *Then*

$$w_1^h = (0, 1)^h = \sum_{0 \leqslant i \leqslant h/2} \binom{h+1}{i} (i, h-i) \tag{8}$$

where $\binom{h+1}{i}$ is the binomial coefficient (reduced modulo 2).

Corollary. $w_1^{s-1} = (0, s-1)$; $\quad w_1^s = (0, s) + (1, s-1)$.

Lemma 3. *The dual Stiefel–Whitney classes* $\bar{w}_h$ *of* γ *on* $G_{2,n}$ *are given by the formula*

$$\bar{w}_h = \sum_{0 \leqslant k \leqslant h/2} \binom{h-k}{k} w_2^k w_1^{h-2k}; \quad h = 1, \ldots, n. \tag{9}$$

Corollary. $\bar{w}_{s-1} = w_1^{s-1}$.

Lemma 4. *Let $w = 1 + w_1 + w_2 + w_3$ be the total Stiefel–Whitney class of γ on $G_{3,n}$. Then*

$$w_1^h = (0,0,1)^h$$
$$= \sum \left[\binom{k+l-1}{k}\binom{h}{k+l-2} + \binom{k+l+1}{k+1}\binom{h+1}{k+l} \right](l,k,h-k-l) \qquad (10)$$

$$w_2^h = (0,1,1)^h$$
$$= \sum \left[\binom{l+1}{k+l-h}\binom{h+1}{l} + \binom{l-1}{k+l-h-1}\binom{h}{l-2} \right](l,k,2h-k-l) \qquad (11)$$

where, as usual, $\binom{l}{k} = 0$ for $k < 0$ or $l < k$ and the summations are extended over all admissible Schubert cocycles.

Corollary.
$$w_1^s = (0,0,s) + (0,1,s-1) + (1,1,s-2),$$
$$w_1^{s+1} = (0,0,s+1) + (0,2,s-1) + (1,2,s-2),$$
$$w_2^s = (0,s,s) + (1,s-1,s) + (2,s-1,s-1),$$
$$w_2^{s+1} = (0,s+1,s+1) + (2,s-1,s+1) + (3,s-1,s).$$

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. We have seen that the total dual Stiefel–Whitney class of the Grassmann manifold $G_{2,n}$ is given by the formula (5). Let us write $n = s - p$ where $s = 2^r$ and $1 \leqslant p \leqslant 2^{r-1}$. Restricting ourselves to the case $2^{r-1} \leqslant n < 2^r - 1$, i.e. $1 < p \leqslant 2^{r-1}$, we have

$$\bar{w}(G_{2,n}) = (1+w_1)^2(1+w_1+w_2)^{p-2}(1+w_1+w_2)^s$$

According to the Corollary to Lemma 2 we obtain

$$(1+w_1+w_2)^s = 1 + w_1^s + w_2^s = 1 + (0,s) + (1,s-1) + (s,s) = 1,$$

since $n < s-1$. Hence

$$\bar{w}(G_{2,n}) = (1+w_1)^2(1+w_1+w_2)^{p-2}.$$

Then

$$\bar{w}_{2p-2}(G_{2,n}) = w_1^2 w_2^{p-2} = (0,1)^2(p-2,p-2) = (p-1,p-1) + (p-2,p) \neq 0,$$

since $2 \leqslant p \leqslant 2^{r-1}$. This proves (i).

According to the fact that $\binom{s-1}{k} \equiv 1$ modulo 2, it follows that

$$\bar{w}(G_{2,s-1}) = (1+w_1)^2(1+w_1+w_2)^{s-1} = (1+w_1)^2 \sum_{h=0}^{s-1} (1+w_1)^h w_2^{s-h-1}$$
$$= \sum_{h=0}^{s-1} (1+w_1)^{h+2} w_2^{s-h-1}.$$

Then

$$\bar{w}_k(G_{2,s-1}) = \sum_{0 \leq 2i \leq k} \binom{s+1-i}{k-2i} w_1^{k-2i} w_2^i.$$

We have

$$\bar{w}_s(G_{2,s-1}) = \sum_{i=0}^{s/2} \binom{s+1-i}{i+1} w_1^{s-2i} w_2^i = \binom{s+1}{1} w_1^s + \binom{s}{2} w_1^{s-2} w_2 + \cdots$$
$$= w_1^s + w_1^{s-4} w_2^2 + \cdots = (1, s-1) + \cdots \neq 0.$$

The unwritten terms in the last sum are of the form $(i, s-1)$, with $i > 1$ because $w_2^2 = (2,2)$, etc., and hence do not affect the term $(1, s-1)$. This proves (ii).

Remark. Let us suppose that $k > 2^r = s$, i.e. $k = s+p+1$, and $0 \leq p$, $p \leq s-3$. Then

$$\bar{w}_{s+p+1}(G_{2,s-1}) = \sum_{0 \leq 2i \leq k} \binom{s+1-i}{s+p+1-2i} w_1^{k-2i} w_2^i$$
$$= \sum_{p-1 \leq 2i \leq s+p} \binom{s+1-i}{i-p} w_1^{s+p+1-2i} w_2^i$$
$$= \sum_{0 \leq 2i \leq s+1-p} \binom{s+1-p-i}{i} w_1^{s+1-p-2i} w_2^{p+i}$$
$$= w_2^p \sum_{0 \leq 2i \leq s+1-p} \binom{s+1-p-i}{i} w_1^{s+1-p-i} w_2^i$$
$$= w_2^p \bar{w}_{s+1-p} = (p,p)(0, s+1-p) = (p, s+1) = 0,$$

since $n = s-1 < s+1$.

Hence the result (ii) obtained by using the Stiefel–Whitney classes is the best possible.

Proof of Theorem 2. The condition $2^{r+1} < 3n < 2^{r+2}$ implies $n < 2^{r+1}$ so that we can write $n = 2^{r+1} - p$, where $2^{r+1} < 3p < 2^{r+2}$. Then

$$\bar{w}(G_{3,n}) = [(1+w_1)(1+w_1+w_2)+w_3]^2(1+w_1+w_2+w_3)^{2s+p-3},$$

where $s = 2^r$. Excluding the cases $r = 0$, $r = 1$ when $n = 1$, $n = 2$ where one obtains manifolds diffeomorphic to the projective space P_3 or to the Grassmann manifold $G_{2,3}$ which have been studied, it follows that $p \geq 3$. Then

$$\begin{aligned}(1 + w_1 + w_2 + w_3)^{2s} = 1 + w_1^{2s} + w_2^{2s} + w_3^{2s} = 1 + (0, 0, 2s) + (0, 1, 2s - 1)\\ + (1, 1, 2s - 2) + (0, 2s, 2s) + (1, 2s - 1, 2s)\\ + (2, 2s - 1, 2s - 1) + (2s, 2s, 2s).\end{aligned}$$

Since $n < \frac{1}{3}2^{r+2} = \frac{4}{3}s$ and $r > 1$, we have $n < 2s - 2$, hence

$$(1 + w_1 + w_2 + w_3)^{2s} = 1.$$

Thus

$$\bar{w}(G_{3,n}) = [(1 + w_1)(1 + w_1 + w_2) + w_3]^2(1 + w_1 + w_2 + w_3)^{p-3}.$$

The Stiefel–Whitney class of maximal dimension, with non-vanishing coefficient is

$$\begin{aligned}\bar{w}_{3p-3}(G_{3,n}) = (w_1^2w_2^2 + w_3^2)w_3^{p-3} = (p - 3, p - 3, p - 3)(0, 1, 1)^2(0, 0, 1)^2\\ + (p - 1, p - 1, p - 1) = (p - 3, p - 1, p + 1) + (p - 3, p, p)\\ + (p - 2, p - 2, p + 1) + (p - 1, p - 1, p - 1).\end{aligned}$$

If $p - 1 \leq n$, i.e. $\frac{1}{3}2^{r+1} < p \leq 2^r$, or equivalently $2^r \leq n < \frac{1}{3}2^{r+2}$, then $(p - 1, p - 1, p - 1) \neq 0$ and hence $\bar{w}_{3p-3}(G_{3,n}) \neq 0$. This proves (iii).

If $p - 1 > n$, i.e. $2^r < p < \frac{1}{3}2^{r+2}$, or, equivalently $\frac{1}{3}2^{r+1} < n < 2^r$, we shall write $n = 2^r - q = s - q$, where $0 < q < \frac{1}{3}s$. Then

$$\bar{w}(G_{3,n}) = [(1 + w_1)(1 + w_1 + w_2) + w_3]^2(1 + w_1 + w_2 + w_3)^{2s+s+q-3}.$$

Excluding the cases $q = 1$, $q = 2$, i.e. $n = s - 1$, $n = s - 2$, which will be treated separately, we have $w_1^s = 0$, $w_2^s = 0$, $w_3^s = 0$. Hence

$$\bar{w}(G_{3,n}) = [(1 + w_1)(1 + w_1 + w_2) + w_3]^2(1 + w_1 + w_2 + w_3)^{q-3}.$$

Thus we obtain as above

$$\bar{w}_{3q-3}(G_{3,n}) \neq 0$$

and this proves (i).

In the cases $n = s - 1$, $n = s - 2$ we have

$$\begin{aligned}\bar{w}(G_{3,s-2}) = [(1 + w_1)(1 + w_1 + w_2) + w_3]^2(1 + w_1 + w_2 + w_3)^{s-1},\\ \bar{w}(G_{3,s-1}) = [(1 + w_1)(1 + w_1 + w_2) + w_3]^2(1 + w_1 + w_2 + w_3)^{s-2}.\end{aligned}$$

Recalling that $\binom{s-1}{k} \equiv 1$ modulo 2, and according to the Corollary to

Lemma 4 we have $w_1^s w_3 = 0$, then

$$\begin{aligned}\bar{w}(G_{3,s-2}) &= (1+w_1)^2 \sum_{k=0}^{s-1} (1+w_1+w_2)^{k+2} w_3^{s-k-1} + \sum_{k=0}^{s-1} (1+w_1+w_2)^k w_3^{s-k+1} \\ &= (1+w_1)^2(1+w_1+w_2)^{s+1} + (1+w_1)^2(1+w_1+w_2)^s w_3 \\ &\quad + w_1^2 \sum_{k=2}^{s-1} (1+w_1+w_2)^k w_3^{s-k+1} \\ &= 1 + w_1 + w_1^2 + w_2 + w_1^3 + w_3 + w_1^2 w_2 + w_1^2 w_3 \\ &\quad + w_1^s + w_1^{s+1} + w_1^{s+2} + w_1^s w_2 + w_1^{s+3} + w_1^{s+2} w_2 \\ &\quad + w_1^2[w_3^2(1+w_1+w_2)^{s-1} + w_3^3(1+w_1+w_2)^{s-2} + \cdots].\end{aligned}$$

Using the formula (10) it follows that $w_1^{s+3} = (1, 4, s-2) \neq 0$. Since in the last sum the $(s+3)$-dimensional Schubert cocycles of this kind (with 1 in the first place) do not appear (because $w_3^2 = (2, 2, 2)$, etc.) it follows that

$$\bar{w}_{s+3}(G_{3,s-2}) \neq 0.$$

Example. By direct calculations one obtains $\bar{w}_{11}(G_{3,6}) \neq 0$ and $\bar{w}_k(G_{3,6}) = 0$ for $12 \leqslant k \leqslant 18$.

In the case $n = s-1$ one obtains

$$\begin{aligned}\bar{w}(G_{3,s-1}) &= (1+w_1)^2 \sum_{0 \leqslant 2k \leqslant s-2} (1+w_1+w_2)^{2k+2} w_3^{s-2k-2} \\ &\quad + \sum_{0 \leqslant 2k \leqslant s-2} (1+w_1+w_2)^{2k} w_3^{s-2k} \\ &= 1 + w_1^s + w_2^s + w_1^2 \sum_{k=1}^{s/2} (1+w_1+w_2)^{2k} w_3^{s-2k}.\end{aligned}$$

Then

$$w_2^s = (2, s-1, s-1)$$

and, in the last sum, the terms which could furnish 2s-dimensional Schubert cocycles with 2 in the first place are

$$w_3^2(w_2^{s-4} w_1^2 + w_2^{s-6} w_1^6) = (2, s-1, s-1) + (2, s-1, s-1) + \cdots$$

where the unwritten terms do not contain 2 in the first place. Hence

$$\bar{w}_{2s}(G_{3,s-1}) = (2, s-1, s-1) + \cdots \neq 0.$$

Remark. The situation described above is valid for $r \geqslant 3$. If $r = 2$ one obtains directly $\bar{w}_6(G_{3,3}) \neq 0$ and this completes the proof of (ii).

Remark. If $r = 3$ one obtains by direct calculations that $\bar{w}_{16}(G_{3,7}) \neq 0$ and $\bar{w}_k(G_{3,7}) = 0$ for $k = 17, 18, 19, 20, 21$. The same result is valid in the case $r = 4$.

Remark. Since $\bar{w}(G_{2,n}) \neq 1$ and $\bar{w}(G_{3,n}) \neq 1$ for $n > 1$ it follows that $G_{2,n}$ and $G_{3,n}$ are not parallelizable.

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