8

Canonical commutation relations

Throughout this chapter $(\mathcal{Y}, \omega)$ is a pre-symplectic space, that is, $\mathcal{Y}$ is a real vector space equipped with an anti-symmetric form ω. From the point of view of classical mechanics, $\mathcal{Y}$ will have the interpretation of the *dual of a phase space*, or, as we will say for brevity, of a *dual phase space*. Note that for quantum mechanics dual phase spaces seem more fundamental that phase spaces.

In this chapter we introduce the concept of a *representation of the canonical commutation relations* (a *CCR representation*). According to a naive definition, a CCR representation is a linear map

$$\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \tag{8.1}$$

with values in self-adjoint operators on a certain Hilbert space satisfying

$$[\phi^{\pi}(y_1), \phi^{\pi}(y_2)] = \mathrm{i} y_1 \cdot \omega y_2 \mathbb{1}. \tag{8.2}$$

We will call (8.2) the *canonical commutation relations in the Heisenberg form*. They are unfortunately problematic, because one needs to supply them with the precise meaning of the commutator of unbounded operators on the left hand side.

Weyl proposed replacing (8.2) with the relations satisfied by the operators $\mathrm{e}^{\mathrm{i}\phi^{\pi}(y)}$. These operators are bounded, and therefore one does not need to discuss domain questions. In our definition of CCR representations we will use the *canonical commutation relations in the Weyl form* (8.4). Under additional regularity assumptions they imply the CCR in the Heisenberg form.

We will introduce two kinds of CCR representations. The usual definition is appropriate to describe *neutral bosons*. In the case of *charged bosons* a somewhat different formalism is used, which we introduce under the name "*charged CCR representations*". Charged CCR representations can be viewed as special cases of (neutral) CCR representations, where the dual phase space $\mathcal{Y}$ is complex and a somewhat different notation is used.

8.1 CCR representations

8.1.1 Definition of a CCR representation

Let $\mathcal{H}$ be a Hilbert space. Recall that $U(\mathcal{H})$ denotes the set of unitary operators on $\mathcal{H}$.

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Definition 8.1 *A* representation of the canonical commutation relations *or a* CCR representation over $(\mathcal{Y}, \omega)$ in $\mathcal{H}$ *is a map*

$$\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H}) \tag{8.3}$$

satisfying

$$W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\cdot\omega y_2}W^\pi(y_1+y_2). \tag{8.4}$$

$W^\pi(y)$ *is then called the* Weyl operator corresponding to $y \in \mathcal{Y}$.

Remark 8.2 *The superscript π is an example of the "name" of a given CCR representation. It is attached to W, which is the generic symbol for "Weyl operators". Later on the same superscript will be attached to other generic symbols, e.g. field operators ϕ.*

Remark 8.3 *Sometimes we will call (8.3)* neutral CCR representations*, to distinguish them from* charged CCR representations *introduced in Def. 8.35.*

Proposition 8.4 *Consider a CCR representation (8.3). Let $y, y_1, y_2 \in \mathcal{Y}$, $t_1, t_2 \in \mathbb{R}$. Then*

$$\begin{aligned} W^{\pi*}(y) = W^\pi(-y), \qquad W^\pi(0) = \mathbb{1}, \\ W^\pi(t_1 y)W^\pi(t_2 y) = W^\pi\big((t_1+t_2)y\big), \\ W^\pi(y_1)W^\pi(y_2) = \mathrm{e}^{-\mathrm{i}y_1\cdot\omega y_2}W^\pi(y_2)W^\pi(y_1). \end{aligned} \tag{8.5}$$

Definition 8.5 *A CCR representation (8.3) is called* regular *if*

$$\mathbb{R} \ni t \mapsto W^\pi(ty) \in U(\mathcal{H}) \quad \textit{is strongly continuous for any } y \in \mathcal{Y}. \tag{8.6}$$

8.1.2 CCR representations over a direct sum

CCR representations can be easily tensored with one another:

Proposition 8.6 *If*

$$\mathcal{Y}_i \ni y_i \mapsto W^i(y) \in U(\mathcal{H}_i), \quad i = 1, 2, \tag{8.7}$$

are two CCR representations, then

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 \ni (y_1, y_2) \mapsto W^1(y_1) \otimes W^2(y_2) \in U(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is also a CCR representation.

8.1.3 Cyclicity and irreducibility

Consider a CCR representation (8.3). The following concepts are parallel to the analogous concepts in the representation theory of groups or C^*-algebras:

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Definition 8.7 *We say that a subset $\mathcal{E} \subset \mathcal{H}$ is* cyclic for (8.3) *if* $\mathrm{Span}\{W^\pi(y)\Psi : \Psi \in \mathcal{E},\ y \in \mathcal{Y}\}$ *is dense in $\mathcal{H}$. We say that $\Psi_0 \in \mathcal{H}$ is* cyclic for (8.3) *if $\{\Psi_0\}$ is cyclic for (8.3).*

Definition 8.8 *We say that the CCR representation (8.3) is* irreducible *if the only closed subspaces of $\mathcal{H}$ invariant under the $W^\pi(y)$ for $y \in \mathcal{Y}$ are $\{0\}$ and $\mathcal{H}$.*

Proposition 8.9 (1) *A CCR representation is irreducible iff $B \in B(\mathcal{H})$ and $[W^\pi(y), B] = 0$ for all $y \in \mathcal{Y}$ implies that B is proportional to identity.*

(2) *In the case of an irreducible representation, all non-zero vectors in $\mathcal{H}$ are cyclic.*

8.1.4 Characteristic functions of CCR representations

Definition 8.10 *We say that $\mathcal{Y} \ni y \mapsto G(y) \in \mathbb{C}$ is a* characteristic function *if for $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, $y_1, \dots, y_n \in \mathcal{Y}$ and $n \in \mathbb{N}$ we have*

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j G(-y_i + y_j) \mathrm{e}^{\frac{\mathrm{i}}{2} y_i \cdot \omega y_j} \geq 0. \tag{8.8}$$

Note that for any CCR representation $y \mapsto W(y) \in U(\mathcal{H})$ and any vector $\Psi \in \mathcal{H}$

$$G(y) := (\Psi | W(y) \Psi) \tag{8.9}$$

is a characteristic function. We will see that every characteristic function comes from a certain CCR representation and a cyclic vector, as in (8.9).

Until the end of this subsection we assume that $y \mapsto G(y)$ is a characteristic function. Set $\mathcal{H}_0 = c_{\mathrm{c}}(\mathcal{Y}, \mathbb{C})$, as in Def. 2.6, that is, $\mathcal{H}_0$ is the vector space of finitely supported functions on $\mathcal{Y}$. Equip it with the sesquilinear form $(\cdot|\cdot)$ defined by

$$(\delta_{y_1} | \delta_{y_2}) := \mathrm{e}^{\frac{\mathrm{i}}{2} y_1 \cdot \omega y_2} G(-y_1 + y_2).$$

It follows from (8.8) that $(\cdot|\cdot)$ is semi-positive definite. Let $\mathcal{N}$ be the space of vectors in $\xi \in \mathcal{H}_0$ such that $(\xi|\xi) = 0$. Set $\mathcal{H} := (\mathcal{H}_0/\mathcal{N})^{\mathrm{cpl}}$.

For any $y \in \mathcal{Y}$ we define a linear operator $W_0(y)$ on $\mathcal{H}_0$ by

$$W_0(y)\delta_{y_1} := \mathrm{e}^{\frac{\mathrm{i}}{2} y \cdot \omega y_1} \delta_{y_1 + y}.$$

The operator $W_0(y)$ preserves the form $(\cdot|\cdot)$, hence it preserves $\mathcal{N}$. Therefore, it defines a linear operator $W(y)$ on $\mathcal{H}_0/\mathcal{N}$ by

$$W(y)\xi := W_0(y)\xi + \mathcal{N}, \quad \xi \in \mathcal{H}_0.$$

$W(y)$ extends to a unitary operator on $\mathcal{H}$. We set $\Psi := \delta_0 + \mathcal{N}$.

Proposition 8.11 *Consider the family of operators*

$$\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H}) \tag{8.10}$$

constructed above from a characteristic function $y \mapsto G(y)$.

(1) *(8.10) is a CCR representation, Ψ is a cyclic vector and $G(y) = (\Psi|W(y)\Psi)$.*
(2) *The following conditions are equivalent:*
   (i) *(8.10) is regular.*
   (ii) *$\mathbb{R} \ni t \mapsto G(y_1 + ty_2)$ is continuous for any $y_1, y_2 \in \mathcal{Y}$.*

8.1.5 Intertwining operators

Let

$$\mathcal{Y} \ni y \mapsto W^1(y) \in U(\mathcal{H}_1), \tag{8.11}$$

$$\mathcal{Y} \ni y \mapsto W^2(y) \in U(\mathcal{H}_2) \tag{8.12}$$

be CCR representations over the same pre-symplectic space $\mathcal{Y}$.

Definition 8.12 *We say that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$* intertwines (8.11) and (8.12) *iff*

$$AW^1(y) = W^2(y)A, \quad y \in \mathcal{Y}.$$

We say that (8.11) and (8.12) are unitarily equivalent *if there exists $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ intertwining (8.11) and (8.12).*

The proof of the following proposition is essentially identical to the proof of Thm. 6.29:

Proposition 8.13 *If the representations (8.11) and (8.12) are irreducible, then the set of operators intertwining them is either $\{0\}$ or $\{\lambda U \ : \lambda \in \mathbb{C}\}$ for some $U \in U(\mathcal{H})$.*

8.1.6 Schrödinger representation

Let $\mathcal{X}$ be a finite-dimensional real vector space. Equip $\mathcal{X}^\# \oplus \mathcal{X}$ with its canonical symplectic form. It follows from Thms. 4.28 and 4.29 that the map

$$\mathcal{X}^\# \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot D)} \in U\big(L^2(\mathcal{X})\big) \tag{8.13}$$

is an irreducible regular CCR representation.

Definition 8.14 *(8.13) is called the* Schrödinger representation over $\mathcal{X}^\# \oplus \mathcal{X}$.

Conversely let $(\mathcal{Y}, \omega)$ be a finite-dimensional symplectic space and

$$\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H}) \tag{8.14}$$

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be a regular CCR representation. By Thm. 1.47, there exists a space $\mathcal{X}$ such that $\mathcal{Y}$ can be identified with $\mathcal{X}^{\#} \oplus \mathcal{X}$ as symplectic spaces. Thus we can rewrite (8.14) as

$$\mathcal{X}^{\#} \oplus \mathcal{X} \ni (\eta, q) \mapsto W(\eta, q)$$

satisfying

$$W(\eta_1, q_1)W(\eta_2, q_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}(\eta_1 \cdot q_2 - \eta_2 \cdot q_1)} W(\eta_1 + \eta_2, q_1 + q_2).$$

The maps

$$\begin{aligned} \mathcal{X}^{\#} &\ni \eta \mapsto W(\eta, 0), \\ \mathcal{X} &\ni q \mapsto W(0, q) \end{aligned}$$

are strongly continuous unitary groups satisfying

$$W(\eta, 0)W(0, q) = \mathrm{e}^{-\mathrm{i}\eta \cdot q} W(0, q)W(\eta, 0).$$

The following theorem is a corollary to the Stone–von Neumann theorem:

Theorem 8.15 *Under the above stated assumptions, there exists a Hilbert space $\mathcal{K}$ and a unitary operator $U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$ such that*

$$W(\eta, q)U = U\mathrm{e}^{\mathrm{i}(\eta \cdot x + q \cdot D)} \otimes \mathbb{1}_{\mathcal{K}}.$$

The representation is irreducible iff $\mathcal{K} = \mathbb{C}$.

Proof It suffices to use Thm. 4.34 and the identities

$$W(\eta, q) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta \cdot q} W(\eta, 0)W(0, q), \quad \mathrm{e}^{\mathrm{i}(\eta \cdot x + q \cdot D)} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta \cdot q} \mathrm{e}^{\mathrm{i}\eta \cdot x} \mathrm{e}^{\mathrm{i}q \cdot D}.$$

□

The following corollary follows directly from Thm. 4.29 and Prop. 8.13:

Corollary 8.16 *Suppose that $\mathcal{Y}$ is a finite-dimensional symplectic space. Let $\mathcal{Y} \ni y \mapsto W_i(y) \in U(\mathcal{H})$, $i = 1, 2$, be two regular irreducible CCR representations. Then there exists $U \in U(\mathcal{H}_1, \mathcal{H}_2)$, unique up to a phase factor, such that $UW_1(y) = W_2(y)U$.*

8.1.7 Weighted Schrödinger representations

Suppose that $\mathcal{X}$ is a finite-dimensional vector space with a Lebesgue measure $\mathrm{d}x$. Fix $m \in L^2_{\mathrm{loc}}(\mathcal{X})$ such that $m \neq 0$ a.e.. Define the measure $\mathrm{d}\mu(x) = |m|^2(x)\mathrm{d}x$. Then

$$L^2(\mathcal{X}, \mathrm{d}\mu) \ni \Psi \mapsto U\Psi := m\Psi \in L^2(\mathcal{X}, \mathrm{d}x)$$

is a unitary operator. If in addition $m \in L^2(\mathcal{X})$, then $U1 = m$.

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The following theorem is obvious:

Theorem 8.17

$$\mathcal{X}^* \oplus \mathcal{X} \ni (\eta, q) \mapsto U^* \mathrm{e}^{\mathrm{i}\eta\cdot x + \mathrm{i}q\cdot D} U$$
$$= \mathrm{e}^{\mathrm{i}\eta\cdot x + \mathrm{i}q\cdot D + m^{-1}(x) q\cdot\nabla m(x)} \in U\big(L^2(\mathcal{X}, \mathrm{d}\mu)\big)$$

is a regular irreducible CCR representation.

Remark 8.18 *If $V(x) := \frac{1}{2} m^{-1}(x)\Delta m(x)$ is sufficiently regular, then we can define the Schrödinger operator $H := -\frac{1}{2}\Delta + V(x)$. If $m \in L^2(\mathcal{X})$, then we have $Hm = 0$.*

The operator H in the $L^2(\mathcal{X}, \mathrm{d}\mu)$ representation looks like

$$U^* H U = -\frac{1}{2}\Delta - m^{-1}(x)\nabla m(x)\cdot\nabla.$$

It is called the Dirichlet form corresponding to H. *If $m \in L^2(\mathcal{X})$, then 1 is its eigenstate with the eigenvalue 0.*

8.1.8 Examples of non-regular CCR representations

In most applications to quantum physics, CCR representations are regular. However, non-regular representations are also useful. In this subsection we describe a couple of examples of non-regular CCR representations.

Recall that, for a set I, $l^2(I)$ denotes the Hilbert space of square summable families of complex numbers indexed by I.

Example 8.19 *Consider the Hilbert space $l^2(\mathcal{Y})$ and the following operators:*

$$W^{\mathrm{d}}(y) f(x) := \mathrm{e}^{-\frac{\mathrm{i}}{2} y\cdot\omega x} f(x+y). \tag{8.15}$$

Then

$$\mathcal{Y} \ni y \mapsto W^{\mathrm{d}}(y) \in U\big(l^2(\mathcal{Y})\big) \tag{8.16}$$

is a CCR representation.

Note that $\mathbb{R} \ni t \mapsto W^{\mathrm{d}}(ty)$ is not strongly continuous for non-zero $y \in \mathcal{Y}$. Hence (8.16) is non-regular.

Example 8.20 *Let $\mathcal{X}$ be a real vector space (of any dimension). Recall that $\mathcal{X}^\# \oplus \mathcal{X}$ is naturally a symplectic space. On $l^2(\mathcal{X})$ define the following operators:*

$$V(\eta) f(x) := \mathrm{e}^{\mathrm{i}\eta\cdot x} f(x), \quad \eta \in \mathcal{X}^\#;$$
$$T(q) f(x) := f(x - q), \quad q \in \mathcal{X}.$$

Then

$$\mathcal{X}^\# \oplus \mathcal{X} \ni (\eta, q) \mapsto V(\eta) T(q) \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot q} \in U\big(l^2(\mathcal{X})\big) \tag{8.17}$$

is a CCR representation.

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Note that $\mathbb{R} \ni t \mapsto T(tq)$ *is not strongly continuous for non-zero* $q \in \mathcal{X}$*. Hence (8.17) is non-regular.*

8.1.9 Bogoliubov transformations

Let

$$\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H}) \tag{8.18}$$

be a CCR representation.

Recall that $\mathcal{Y}^{\#}$ denotes the space of linear functionals on $\mathcal{Y}$, and $Sp(\mathcal{Y})$ the group of symplectic transformations of $\mathcal{Y}$. Let $v \in \mathcal{Y}^{\#}$, $r \in Sp(\mathcal{Y})$. Clearly, the map

$$\mathcal{Y} \ni y \mapsto W^{v,r}(y) := \mathrm{e}^{\mathrm{i}v\cdot y} W(ry) \in U(\mathcal{H}) \tag{8.19}$$

is a CCR representation.

Definition 8.21 *(8.19) can be called the* Bogoliubov transformation of (8.18) by (v, r)*. Alternatively, if* $r = \mathbb{1}$*, it can be called the* Bogoliubov translation by v *or, if* $v = 0$*, the* Bogoliubov rotation by r.

The pairs (v, r) that appear in (8.19) are naturally interpreted as elements of the group $\mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$, the semi-direct product of $\mathcal{Y}^{\#}$ and $Sp(\mathcal{Y})$, with the product given by

$$(v_2, r_2)(v_1, r_1) := (r_1^{\#} v_2 + v_1, r_2 r_1).$$

Note that $\mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$ can be viewed as a subgroup of the affine group $\mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y}^{\#}) = ASp(\mathcal{Y}^{\#})$, with the homomorphic embedding

$$\mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y}) \ni (v, r) \mapsto (r^{\# -1} v, r^{\# -1}) \in ASp(\mathcal{Y}^{\#}).$$

Proposition 8.22 (1) *If* $(v_1, r_1), (v_2, r_2) \in \mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$*, then*

$$\left(W^{(v_1, r_1)}\right)^{(v_2, r_2)}(y) = W^{(v_1, r_1)(v_2, r_2)}(y).$$

(2) *The set of* $(v, r) \in \mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$ *such that (8.19) is unitarily equivalent to (8.18) is a subgroup of* $\mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$ *containing* $\omega\mathcal{Y} \rtimes \{\mathbb{1}\} \subset \mathcal{Y}^{\#} \rtimes \{\mathbb{1}\}$.
(3) *(8.19) is regular iff (8.18) is.*
(4) *(8.19) is irreducible iff (8.18) is.*

Proof To see that for $v \in \omega\mathcal{Y}$ (8.19) and (8.18) are equivalent, we note

$$W^{v,\mathbb{1}}(y) = W(\omega^{-1}v) W(y) W(-\omega^{-1}v).$$ □

Proposition 8.23 *Let* $\mathcal{Y}$ *be finite-dimensional and* ω *symplectic. Then*

(1) *(8.18) and (8.19) are unitarily equivalent for any* $(v, r) \in \mathcal{Y}^{\#} \rtimes Sp(\mathcal{Y})$.

(2) *Let* $\mathrm{Op}(\cdot)$ *and* $\mathrm{Op}^{(v,r)}(\cdot)$ *denote the Weyl quantization w.r.t. (8.18) and (8.19) respectively. (See (8.42) later on for the definition of the Weyl quantization.) For* $b \in \mathcal{S}'(\mathcal{Y}^{\#})$*, set*

$$b^{v,r}(w) = b(r^{\#}w + v), \qquad w \in \mathcal{Y}^{\#}.$$

Then $\mathrm{Op}^{(v,r)}(b) = \mathrm{Op}\left(b^{(v,r)}\right)$.

8.2 Field operators

Throughout the section, $(\mathcal{Y}, \omega)$ is a pre-symplectic space and we are given a regular CCR representation

$$\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H}). \tag{8.20}$$

8.2.1 Definition of field operators

By regularity and (8.6), $\mathbb{R} \ni t \mapsto W^{\pi}(ty)$ is a strongly continuous unitary group. By Stone's theorem, for any $y \in \mathcal{Y}$, we can define its self-adjoint generator

$$\phi^{\pi}(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W^{\pi}(ty)\big|_{t=0}.$$

In other words, $\mathrm{e}^{\mathrm{i}\phi^{\pi}(y)} = W^{\pi}(y)$.

Definition 8.24 $\phi^{\pi}(y)$ *will be called the* field operator corresponding to $y \in \mathcal{Y}$. *(Sometimes the name* Segal field operator *is used.)*

Theorem 8.25 *Let* $y, y_1, y_2 \in \mathcal{Y}$.

(1) $W^{\pi}(y)$ *leaves invariant* $\mathrm{Dom}\,\phi^{\pi}(y_1)$ *and*

$$[\phi^{\pi}(y), W^{\pi}(y_1)] = y_1 \cdot \omega y\, W^{\pi}(y_1). \tag{8.21}$$

(2) $\phi^{\pi}(ty) = t\phi^{\pi}(y)$, $t \in \mathbb{R}$.

(3) *One has* $\mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2) \subset \mathrm{Dom}\,\phi^{\pi}(y_1 + y_2)$ *and*

$$\phi^{\pi}(y_1 + y_2) = \phi^{\pi}(y_1) + \phi^{\pi}(y_2), \ \text{ on } \mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2). \tag{8.22}$$

(4) *In the sense of quadratic forms on* $\mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2)$*, we have*

$$[\phi^{\pi}(y_1), \phi^{\pi}(y_2)] = \mathrm{i}y_1 \cdot \omega y_2 \mathbb{1}. \tag{8.23}$$

Proof (8.21) follows immediately from differentiating in t the identity

$$W^{\pi}(ty)W^{\pi}(y_1) = W^{\pi}(y_1)W^{\pi}(ty)\mathrm{e}^{-\mathrm{i}ty\cdot\omega y_1}.$$

To obtain (8.22), we note that, for $\Psi \in \mathrm{Dom}\,\phi^\pi(y_1) \cap \mathrm{Dom}\,\phi^\pi(y_2)$,

$$\begin{aligned} t^{-1}\big(W^\pi(t(y_1+y_2)) - 1\!\!1\big)\Psi &= \mathrm{e}^{-\frac{\mathrm{i}}{2}t^2 y_1\cdot\omega y_2} W^\pi(ty_1)t^{-1}\big(W^\pi(ty_2) - 1\!\!1\big)\Psi \\ &\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}t^2 y_1\cdot\omega y_2} t^{-1}\big(W^\pi(ty_1) - 1\!\!1\big)\Psi \\ &\quad + t^{-1}\big(\mathrm{e}^{-\frac{\mathrm{i}}{2}t^2 y_1\cdot\omega y_2} - 1\!\!1\big)\Psi \\ &\underset{t\to 0}{\longrightarrow} \mathrm{i}\phi(y_2)\Psi + \mathrm{i}\phi(y_1)\Psi. \end{aligned}$$

By differentiating the identity

$$\big(W^\pi(t_1y_1)\Psi_1 | W^\pi(t_2y_2)\Psi_2\big) = \mathrm{e}^{-\mathrm{i}t_1t_2y_1\cdot\omega y_2}\big(W^\pi(t_2y_2)\Psi_1 | W^\pi(t_1y_1)\Psi_2\big)$$

w.r.t. t_1 and t_2, and setting $t_1 = t_2 = 0$, we obtain (8.23). □

Sometimes it is convenient to introduce CCR representations with help of field operators, as described in the following proposition. We recall that $Cl_{\mathrm{h}}(\mathcal{H})$ denotes the set of self-adjoint operators on $\mathcal{H}$.

Proposition 8.26 *Let $\mathcal{Y} \ni y \mapsto \phi^\pi(y) \in Cl_{\mathrm{h}}(\mathcal{H})$ be a map such that*

(1) $\phi^\pi(ty) = t\phi^\pi(y)$, $t \in \mathbb{R}$;
(2) $\mathrm{e}^{\mathrm{i}\phi^\pi(y_1)}\mathrm{e}^{\mathrm{i}\phi^\pi(y_2)} = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\cdot\omega y_2}\mathrm{e}^{\mathrm{i}\phi^\pi(y_1+y_2)}$, $y_1, y_2 \in \mathcal{Y}$.

Then $\mathcal{Y} \ni y \mapsto W^\pi(y) := \mathrm{e}^{\mathrm{i}\phi^\pi(y)}$ is a regular CCR representation, and $\phi^\pi(y)$ are the corresponding Segal field operators.

Remark 8.27 *Let $\mathcal{X} \subset \mathcal{Y}$ be an isotropic subspace. Then the field operators $\phi^\pi(q)$ with $q \in \mathcal{X}$ commute with one another. Hence*

$$\phi^\pi(q),\ q \in \mathcal{X},$$

is an $\mathcal{X}^\#$-vector of commuting self-adjoint operators (see Def. 2.77). If f is a cylindrical Borel function on $\mathcal{X}^\#$, then the operator $f(\phi^\pi)$ is well defined by the functional calculus.

8.2.2 Common domain of field operators

Definition 8.28 *The* Schwartz space for the CCR representation (8.20) *is defined as the intersection of $\mathrm{Dom}\,\phi^\pi(y_1)\cdots\phi^\pi(y_n)$ for $y_1, \dots, y_n \in \mathcal{Y}$. It is denoted $\mathcal{H}^{\infty,\pi}$ and has the structure of a topological vector space with semi-norms $\|\phi^\pi(y_1)\cdots\phi^\pi(y_n)\Psi\|$.*

Clearly, polynomials in $\phi^\pi(y)$ act as operators on $\mathcal{H}^{\infty,\pi}$.

Theorem 8.29 *Let $\mathcal{Y}$ be finite-dimensional. Then*

(1) *$\mathcal{H}^{\infty,\pi}$ is dense in $\mathcal{H}$.*
(2) *If $\omega = 0$, then $\mathcal{H}^{\infty,\pi}$ coincides with the space of C^∞ vectors for the vector of commuting self-adjoint operators ϕ^π.*
(3) *If ω is non-degenerate, then $\Psi \in \mathcal{H}^{\infty,\pi}$ iff the function $\mathcal{Y} \ni y \mapsto (\Psi|W^\pi(y)\Psi)$ belongs to $\mathcal{S}(\mathcal{Y})$.*

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(4) *If* $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$ *and (8.20) is the Schrödinger representation in* $L^2(\mathcal{X})$*, then* $\mathcal{H}^{\infty,\pi}$ *equals* $\mathcal{S}(\mathcal{X})$*.*

Proof (2) is obvious. (3) follows from Thms. 8.15 and 4.30. (4) follows from Thm. 4.15.

Let us prove (1). Set $\mathcal{Y}_0 = \mathrm{Ker}\,\omega$. Let $\mathcal{Y}_1 \subset \mathcal{Y}$ be a complementary space to $\mathcal{Y}_0$. $\mathcal{Y}_1$ is symplectic, hence we can assume that, for some space $\mathcal{X}$, $\mathcal{Y}_1 = \mathcal{X}^{\#} \oplus \mathcal{X}$ with the canonical symplectic form. By Thm. 8.15, there exists a unitary map $U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$ such that

$$W^{\pi}(y_1) = UW(y_1) \otimes 1\!\!1_{\mathcal{K}} U^*, \quad y_1 \in \mathcal{Y}_1,$$

where $W(y)$ denote the Weyl operators in the Schrödinger representation. Now we know from (3) that $U\ \mathcal{S}(\mathcal{X}) \overset{\mathrm{al}}{\otimes} \mathcal{K}$ is contained in the Schwartz space for $\mathcal{Y}_1 \ni y_1 \mapsto W^{\pi}(y_1)$.

Using that $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are orthogonal for ω and Thm. 4.29, we obtain that $U^* W^{\pi}(y_0) U = 1\!\!1 \otimes W^{\pi_0}(y_0)$ for $y_0 \in \mathcal{Y}_0$, where $\mathcal{Y}_0 \ni y_0 \mapsto W^{\pi_0}(y_0) \in U(\mathcal{K})$ is a CCR representation. By (2), the corresponding Schwartz space $\mathcal{K}^{\infty,\pi_0}$ is dense in $\mathcal{K}$. Thus $U\ \mathcal{S}(\mathcal{X}) \overset{\mathrm{al}}{\otimes} \mathcal{K}^{\infty,\pi_0} \subset \mathcal{H}^{\infty,\pi}$ is dense in $\mathcal{H}$. □

If $\mathcal{Y}$ has an arbitrary dimension, then Thm. 8.29 is still useful, because it can be applied to finite-dimensional subspaces of $\mathcal{Y}$. In particular, Thm. 8.29 implies that for an arbitrary symplectic space $\mathcal{Y}$, the spaces $\mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2)$ considered in Thm. 8.25 are dense in $\mathcal{H}$.

8.2.3 Non-self-adjoint fields

As in Subsect. 1.3.5, we can equip $\mathbb{C}\mathcal{Y}$ with the anti-symmetric form $\omega_{\mathbb{C}}$.

Definition 8.30 *For* $w = y_1 + \mathrm{i}y_2$*,* $y_1, y_2 \in \mathcal{Y}$*, we define the* field operator

$$\phi^{\pi}(w) := \phi^{\pi}(y_1) + \mathrm{i}\phi^{\pi}(y_2) \ \textit{with domain}\ \mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2).$$

Proposition 8.31 (1) *For* $w = y_1 + \mathrm{i}y_2$*,* $y_1, y_2 \in \mathcal{Y}$*,*

$$\phi^{\pi}(w) \ \textit{is closed on}\ \mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2).$$

(2) *For* $w_1, w_2 \in \mathbb{C}\mathcal{Y}$*,* $\lambda_1, \lambda_2 \in \mathbb{C}$*,*

$$\phi^{\pi}(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 \phi^{\pi}(w_1) + \lambda_2 \phi^{\pi}(w_2) \ \textit{on}\ \mathrm{Dom}\,\phi^{\pi}(w_1) \cap \mathrm{Dom}\,\phi^{\pi}(w_2).$$

(3) *For* $w_1, w_2 \in \mathbb{C}\mathcal{Y}$*,*

$$[\phi^{\pi}(w_1), \phi^{\pi}(w_2)] = \mathrm{i}w_1 \cdot \omega_{\mathbb{C}} w_2 1\!\!1 \ \textit{as a quadratic form on}\ \mathrm{Dom}\,\phi^{\pi}(w_1) \cap \mathrm{Dom}\,\phi^{\pi}(w_2).$$

Proof By Thm. 8.25, we have, for $\Psi \in \mathrm{Dom}\,\phi^{\pi}(y_1) \cap \mathrm{Dom}\,\phi^{\pi}(y_2)$,

$$\|\phi^{\pi}(y_1 + \mathrm{i}y_2)\Psi\|^2 = \|\phi^{\pi}(y_1)\Psi\|^2 + \|\phi^{\pi}(y_2)\Psi\|^2 - y_1 \cdot \omega y_2 \|\Psi\|^2. \tag{8.24}$$

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We know that $\phi^\pi(y_1)$ and $\phi^\pi(y_2)$ are self-adjoint, hence closed. Therefore, $\mathrm{Dom}\,\phi^\pi(y_1)$ and $\mathrm{Dom}\,\phi^\pi(y_2)$ are complete in the graph norms. Hence so is $\mathrm{Dom}\,\phi^\pi(y_1) \cap \mathrm{Dom}\,\phi^\pi(y_2)$ in the intersection norm. This proves (1). (2) is immediate and (3) follows immediately from Thm. 8.25 (4). □

8.2.4 CCR over a Kähler space

In this subsection we assume that ω is symplectic. We fix a CCR representation (8.20). We use the notation and results of Subsects. 1.3.6, 1.3.8 and 1.3.9.

The following proposition shows that choosing a sufficiently large subspace of commuting field operators that annihilate a certain vector is equivalent to fixing a Kähler structure in $(\mathcal{Y}, \omega)$.

Proposition 8.32 *Suppose that $\mathcal{Z}$ is a complex subspace of $\mathbb{C}\mathcal{Y}$ such that*

(1) $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$,
(2) $\overline{z}_1, \overline{z}_2 \in \overline{\mathcal{Z}}$ *implies* $\phi^\pi(\overline{z}_1)\phi^\pi(\overline{z}_2) = \phi^\pi(\overline{z}_2)\phi^\pi(\overline{z}_1)$ *(or, equivalently, $\overline{\mathcal{Z}}$ is isotropic for $\omega_\mathbb{C}$).*

Then there exists a unique pseudo-Kähler anti-involution j *on* $(\mathcal{Y}, \omega)$ *such that*

$$\mathcal{Z} = \{y - \mathrm{i}\mathrm{j}y \ : \ y \in \mathcal{Y}\}. \tag{8.25}$$

If in addition
(3) *there exists a non-zero $\Omega \in \mathcal{H}$ such that $\Omega \in \mathrm{Dom}\,\phi^\pi(\overline{z})$ and $\phi^\pi(\overline{z})\Omega = 0$, $z \in \mathcal{Z}$, then* j *is Kähler.*

Proof By (1), each $y \in \mathcal{Y}$ can be written uniquely as $y = z_y + \overline{z}_y$. Clearly, z_y depends linearly on y. We have $\overline{\mathrm{i}(2z_y - y)} = \mathrm{i}(2z_y - y)$. Hence $\mathrm{j}y := \mathrm{i}(2z_y - y)$ defines $\mathrm{j} \in L(\mathcal{Y})$, and (8.25) is true.

(2) implies

$$\begin{aligned} 0 &= (y_1 + \mathrm{i}\mathrm{j}y_1)\cdot\omega_\mathbb{C}(y_2 + \mathrm{i}\mathrm{j}y_2) \\ &= y_1\cdot\omega y_2 - (\mathrm{j}y_1)\cdot\omega(\mathrm{j}y_2) + \mathrm{i}\big((\mathrm{j}y_1)\cdot\omega y_2 + y_1\cdot\omega\mathrm{j}y_2\big). \end{aligned}$$

Hence

$$y_1\cdot\omega y_2 - (\mathrm{j}y_1)\cdot\omega(\mathrm{j}y_2) = (\mathrm{j}y_1)\cdot\omega y_2 + y_1\cdot\omega\mathrm{j}y_2 = 0,$$

which shows that j is symplectic and infinitesimally symplectic, hence pseudo-Kähler.

Then we compute using (3):

$$\begin{aligned} 0 &= \|\phi(y + \mathrm{i}\mathrm{j}y)\Omega\|^2 \\ &= \big(\Omega|\phi^\pi(y)^2\Omega\big) + \big(\Omega|\phi^\pi(\mathrm{j}y)^2\Omega\big) - \mathrm{i}\big(\Omega|[\phi^\pi(\mathrm{j}y), \phi^\pi(y)]\Omega\big) \geq -y\cdot\omega\mathrm{j}y. \end{aligned}$$

□

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Motivated in part by the above proposition, let us fix j, a pseudo-Kähler anti-involution on $(\mathcal{Y}, \omega)$. Recall that the space $\mathcal{Z}$ given by (8.25) is called the holomorphic subspace of $\mathbb{C}\mathcal{Y}$ (see Subsect. 1.3.6).

Definition 8.33 *We define the* (abstract) creation *and* annihilation operators associated with j *by*

$$a^{\pi*}(z) := \phi^{\pi}(z), \quad a^{\pi}(z) := \phi^{\pi}(\overline{z}), \quad z \in \mathcal{Z}.$$

By Prop. 8.31, if $z = y - \mathrm{ij}y \in \mathcal{Z}$, then $a^{\pi}(z) = \phi^{\pi}(y) + \mathrm{i}\phi^{\pi}(\mathrm{j}y)$, $a^{\pi*}(z) = \phi^{\pi}(y) - \mathrm{i}\phi^{\pi}(\mathrm{j}y)$ are closed operators on $\mathrm{Dom}\,\phi^{\pi}(y) \cap \mathrm{Dom}\,\phi^{\pi}(\mathrm{j}y)$.

Proposition 8.34 (1) *One has $\phi^{\pi}(z, \overline{z}) = a^{\pi*}(z) + a^{\pi}(z)$, $z \in \mathcal{Z}$.*
(2)

$$[a^{\pi*}(z_1), a^{\pi*}(z_2)] = 0, \quad [a^{\pi}(z_1), a^{\pi}(z_2)] = 0,$$
$$[a^{\pi}(z_1), a^{\pi*}(z_2)] = \overline{z}_1 \cdot z_2 \mathbb{1}, \quad z_1, z_2 \in \mathcal{Z}.$$

Proof (1) is immediate, since $(z, \overline{z}) = (z, 0) + (0, \overline{z})$. The first line of (2) follows from the fact that $\mathcal{Z}$, $\overline{\mathcal{Z}}$ are isotropic for $\omega_{\mathbb{C}}$ (see Subsect. 1.3.9). To prove the second line we write

$$\begin{aligned}[a^{\pi}(z_1), a^{\pi*}(z_2)] &= [\phi^{\pi}(\overline{z_1}), \phi^{\pi}(z_2)] = \mathrm{i}\overline{z_1} \cdot \omega_{\mathbb{C}} z_2 \mathbb{1} \\ &= -\mathrm{i}\overline{z_1} \cdot \mathrm{j}_{\mathbb{C}} z_2 \mathbb{1} = \overline{z_1} \cdot z_2 \mathbb{1},\end{aligned}$$

using Subsect. 1.3.9 and the fact that $\mathrm{j}_{\mathbb{C}} z_2 = \mathrm{i} z_2$, since $z_2 \in \mathcal{Z}$. □

Note that in the case of a Fock representation, considered in Chap. 9, the space $\mathcal{Y}$ has a natural Kähler structure. The abstract creation and annihilation operators defined in Def. 8.33 coincide then with the usual creation and annihilation operators.

If the space $\mathcal{Y}$ is equipped with a charge 1 symmetry, then we have a natural pseudo-Kähler structure (see Subsect. 1.3.11). The corresponding creation and annihilation operators are then called *charged field operators*. However, in this case we prefer to use a slightly different formalism, which is described in the next subsection.

8.2.5 Charged CCR representations

CCR representations, as defined in Def. 8.1, are used mainly to describe neutral bosons. Therefore, sometimes we will call them *neutral CCR representations*. In the context of charged bosons one uses another formalism described in the following definition.

Definition 8.35 *Let $(\mathcal{Y}, \omega)$ be a charged pre-symplectic space, that is, a complex vector space equipped with an anti-Hermitian form denoted $(y_1|\omega y_2)$, $y_1, y_2 \in \mathcal{Y}$*

(see Subsect. 1.2.11). Let $\mathcal{H}$ be a Hilbert space. We say that a map

$$\mathcal{Y} \ni y \mapsto \psi^\pi(y) \in Cl(\mathcal{H})$$

is a charged CCR representation *if there exists a regular CCR representation of* $(\mathcal{Y}_\mathbb{R}, \mathrm{Re}\,(\cdot|\omega\cdot))$

$$\mathcal{Y} \ni y \mapsto W^\pi(y) = \mathrm{e}^{\mathrm{i}\phi^\pi(y)} \in U(\mathcal{H}) \tag{8.26}$$

such that

$$\psi^\pi(y) = \frac{1}{\sqrt{2}}(\phi^\pi(y) + \mathrm{i}\phi^\pi(\mathrm{i}y)), \quad y \in \mathcal{Y}.$$

Proposition 8.36 *Suppose that $\mathcal{Y} \ni y \mapsto \psi^\pi(y)$ is a charged CCR representation. Let $y, y_1, y_2 \in \mathcal{Y}$. We have:*

(1) $\psi^\pi(\lambda y) = \overline{\lambda}\psi^\pi(y)$, $\lambda \in \mathbb{C}$.
(2) *On* $\mathrm{Dom}\,\psi^\pi(y_1) \cap \mathrm{Dom}\,\psi^\pi(y_2)$ *we have* $\psi^\pi(y_1 + y_2) = \psi^\pi(y_1) + \psi^\pi(y_2)$.
(3) *In the sense of quadratic forms, we have the identities*

$$\begin{aligned}
[\psi^{\pi*}(y_1), \psi^{\pi*}(y_2)] &= [\psi^\pi(y_1), \psi^\pi(y_2)] = 0, \\
[\psi^\pi(y_1), \psi^{\pi*}(y_2)] &= \mathrm{i}(y_1|\omega y_2)\mathbb{1}.
\end{aligned}$$

By definition, a charged CCR representation determines the neutral CCR representation (8.26) on the symplectic space $(\mathcal{Y}_\mathbb{R}, \mathrm{Re}(\cdot|\omega\cdot))$ with the fields given by

$$\phi^\pi(y) := \frac{1}{\sqrt{2}}\left(\psi^\pi(y) + \psi^{\pi*}(y)\right), \quad y \in \mathcal{Y}. \tag{8.27}$$

In addition, $(\mathcal{Y}_\mathbb{R}, \mathrm{Re}(\cdot|\omega\cdot))$ is equipped with a charge 1 symmetry

$$U(1) \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta} \in Sp(\mathcal{Y}_\mathbb{R}).$$

Conversely, charged CCR representations arise when the underlying symplectic space of a (neutral) CCR representation is equipped with a charge 1 symmetry. Let us make this precise. Suppose that $(\mathcal{Y}, \omega)$ is a symplectic space and

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\mathcal{H})$$

is a regular neutral CCR representation. Suppose that

$$U(1) \ni \theta \mapsto u_\theta = \cos\theta\mathbb{1} + \sin\theta\mathrm{j}_{\mathrm{ch}} \in Sp(\mathcal{Y})$$

is a charge 1 symmetry. We know from Prop. 1.94 (2) that j_{ch} is a pseudo-Kähler anti-involution. Set

$$\psi^\pi(y) = \frac{1}{\sqrt{2}}(\phi^\pi(y) + \mathrm{i}\phi^\pi(\mathrm{j}_{\mathrm{ch}}y)), \quad \psi^{\pi*}(y) = \frac{1}{\sqrt{2}}(\phi^\pi(y) - \mathrm{i}\phi^\pi(\mathrm{j}_{\mathrm{ch}}y)), \quad y \in \mathcal{Y}.$$

Then we obtain a charged CCR representation over $\mathcal{Y}^{\mathbb{C}}$ with the complex structure given by j_{ch} and the anti-Hermitian form

$$(y_1|\omega y_2) := y_1 \cdot \omega y_2 - \mathrm{i} y_1 \cdot \omega \mathrm{j}_{\mathrm{ch}} y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

We can look at this construction as follows. By the standard procedure described in the previous subsection, we introduce the holomorphic subspace for j_{ch}, that is,

$$\mathcal{Z}_{\mathrm{ch}} := \{y - \mathrm{i}\mathrm{j}_{\mathrm{ch}} y \ : \ y \in \mathcal{Y}\} \subset \mathbb{C}\mathcal{Y}.$$

Introduce the creation and annihilation operators associated with j_{ch}:

$$a_{\mathrm{ch}}^{\pi}(z) := \phi^{\pi}(\overline{z}), \quad a_{\mathrm{ch}}^{\pi *}(z) := \phi^{\pi *}(z), \quad z \in \mathcal{Z}_{\mathrm{ch}}.$$

We have a natural identification of the space $\mathcal{Z}_{\mathrm{ch}}$ with $\mathcal{Y}$:

$$\mathcal{Y} \ni y \mapsto z = \frac{1}{\sqrt{2}}(\mathbb{1} - \mathrm{i}\mathrm{j}_{\mathrm{ch}})y \in \mathcal{Z}_{\mathrm{ch}}. \tag{8.28}$$

Then

$$\psi^{\pi}(y) := a_{\mathrm{ch}}^{\pi}(z), \quad \psi^{\pi *}(y) := a_{\mathrm{ch}}^{\pi *}(z).$$

8.2.6 CCR over a symplectic space with conjugation

Let $\mathcal{X}$ be a real vector space. Let $\mathcal{V}$ be a subspace of $\mathcal{X}^{\#}$. Consider the space $\mathcal{V} \oplus \mathcal{X}$ equipped with its canonical pre-symplectic form ω. Clearly, it is also equipped with a conjugation

$$\tau(\eta, q) = (\eta, -q), \quad (\eta, q) \in \mathcal{V} \oplus \mathcal{X}.$$

Let

$$\mathcal{V} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}\phi^{\pi}(\eta, q)} \in U(\mathcal{H})$$

be a regular CCR representation.

Definition 8.37 *The* (abstract) position *and* momentum operators *are* $\mathcal{X}$- *and* $\mathcal{X}^{\#}$-*vectors of commuting self-adjoint operators defined by*

$$\begin{aligned} \eta \cdot x^{\pi} &:= \phi^{\pi}(\eta, 0), \quad \eta \in \mathcal{V}; \\ q \cdot D^{\pi} &:= \phi^{\pi}(0, q), \quad q \in \mathcal{X}. \end{aligned}$$

A natural conjugation on the symplectic space $\mathcal{Y}$ is available in the case of the Schrödinger representation. In this case the operators defined in Def. 8.37 are the usual momentum and position operators.

Recall that for the Schrödinger representation the symplectic space is finite-dimensional. One often considers a conjugation on an infinite-dimensional symplectic space. This is the case for the real-wave representation (see Sect. 9.3), which to some extent can be viewed as a generalization of the Schrödinger representation to infinite dimensions. However, besides a conjugation, the real-wave representation requires an additional structure: $\mathcal{Y}$ needs to be a Kähler space. CCR relations over a Kähler space with a conjugation are discussed in the following subsection.

8.2.7 CCR over a Kähler space with conjugation

Suppose that $\mathcal{X}$ is a real Hilbert space and $c > 0$ is an operator on $\mathcal{X}$. Set

$$\mathcal{Y} := (2c)^{-\frac{1}{2}}\mathcal{X} \oplus (2c)^{\frac{1}{2}}\mathcal{X},$$

which is a symplectic space with a conjugation. (Note that $(2c)^{-\frac{1}{2}}\mathcal{X}$ can be viewed as the space dual to $(2c)^{\frac{1}{2}}\mathcal{X}$). Consider a regular CCR representation

$$(2c)^{-\frac{1}{2}}\mathcal{X} \oplus (2c)^{\frac{1}{2}}\mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}\phi^{\pi}(\eta,q)} \in U(\mathcal{H}).$$

Let x^{π} and D^{π} be the position and momentum operators introduced in Def. 8.37.

We introduce the following definition:

Definition 8.38 *For $w \in \mathbb{C}c^{-\frac{1}{2}}\mathcal{X}$ we define* Schrödinger-type creation *and* annihilation operators

$$a_{\mathrm{sch}}^{\pi *}(w) := \frac{1}{2}w \cdot x^{\pi} - \mathrm{i}cw \cdot D^{\pi}, \quad a_{\mathrm{sch}}^{\pi}(w) := \frac{1}{2}\overline{w} \cdot x^{\pi} + \mathrm{i}c\overline{w} \cdot D^{\pi}.$$

By Subsect. 8.2.3, $a_{\mathrm{sch}}^{\pi}(w)$ and $a_{\mathrm{sch}}^{\pi *}(w)$ are closed and the adjoints of each other on their natural domains.

Proposition 8.39 (1) *For $\eta \in \mathbb{C}(2c)^{-\frac{1}{2}}\mathcal{X}$, $q \in \mathbb{C}(2c)^{\frac{1}{2}}\mathcal{X}$, we have*

$$\eta \cdot x^{\pi} = a_{\mathrm{sch}}^{\pi *}(\eta) + a_{\mathrm{sch}}^{\pi}(\overline{\eta}), \quad q \cdot D^{\pi} = \frac{1}{2\mathrm{i}}\big(a_{\mathrm{sch}}^{\pi}(c^{-1}\overline{q}) - a_{\mathrm{sch}}^{\pi *}(c^{-1}q)\big). \tag{8.29}$$

(2) *For $w_1, w_2 \in \mathbb{C}\mathcal{X}$,*

$$\begin{aligned} \big[a_{\mathrm{sch}}^{\pi}(w_1), a_{\mathrm{sch}}^{\pi *}(w_2)\big] &= \overline{w_1} \cdot cw_2 \mathbb{1}, \\ \big[a_{\mathrm{sch}}^{\pi}(w_1), a_{\mathrm{sch}}^{\pi}(w_2)\big] &= \big[a_{\mathrm{sch}}^{\pi *}(w_1), a_{\mathrm{sch}}^{\pi *}(w_2)\big] = 0. \end{aligned} \tag{8.30}$$

It is easy to interpret Schrödinger-type creation and annihilation operators in terms of an appropriate Kähler structure on $\mathcal{Y}$ with a conjugation, following the terminology of Subsect. 1.3.10. Let us equip $\mathcal{Y} = (2c)^{-\frac{1}{2}}\mathcal{X} \oplus (2c)^{\frac{1}{2}}\mathcal{X}$ with the anti-involution

$$\mathrm{j} := \begin{bmatrix} 0 & -(2c)^{-1} \\ 2c & 0 \end{bmatrix}.$$

Clearly, the pair j, ω is Kähler. The corresponding scalar product of $(\eta_i, q_i) \in \mathcal{Y}$, $i = 1, 2$, is

$$(\eta_1, q_1) \cdot (\eta_2, q_2) = \eta_1 \cdot 2c\eta_2 + q_1 \cdot (2c)^{-1} q_2. \tag{8.31}$$

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Let us consider the map

$$\mathbb{C}c^{-\frac{1}{2}}\mathcal{X} \ni w \mapsto z := \frac{1\!\!1 - \mathrm{ij}}{2}(w, 0) = \Big(\frac{1}{2}w, -\mathrm{i}cw\Big) \in \mathbb{C}\mathcal{Y}. \tag{8.32}$$

(8.32) is unitary onto $\mathcal{Z}$, the holomorphic subspace of $\mathbb{C}\mathcal{Y}$ associated with j. Then we have

$$a^{\pi}_{\mathrm{sch}}(w) = a^{\pi}(z), \quad a^{\pi*}_{\mathrm{sch}}(w) = a^{\pi*}(z),$$

where $a^{\pi*}(z)$, resp. $a^{\pi}(z)$, are the creation, resp. annihilation operators associated with the anti-involution j, as in Subsect. 8.2.4.

In what follows we drop the superscript π. A standard choice of c is $c = 1\!\!1$, for which

$$\mathrm{j} = \begin{bmatrix} 0 & -\frac{1}{2}1\!\!1 \\ 21\!\!1 & 0 \end{bmatrix},$$

and leads to the formulas

$$\begin{aligned} &a^*_{\mathrm{sch}}(w) = \frac{1}{2}w \cdot x - \mathrm{i}w \cdot D, \, a_{\mathrm{sch}}(w) = \frac{1}{2}\overline{w} \cdot x + \mathrm{i}\overline{w} \cdot D, \\ &w \cdot x = a^*_{\mathrm{sch}}(w) + a_{\mathrm{sch}}(\overline{w}), \, w \cdot D = \frac{1}{2\mathrm{i}}(-a^*_{\mathrm{sch}}(w) + a_{\mathrm{sch}}(\overline{w})), \qquad w \in \mathbb{C}\mathcal{X}. \end{aligned}$$

This choice is the most convenient in the context of the *real-wave representation*, which will be described later.

In another choice, which is often found in the literature, one takes $c = \frac{1}{2}1\!\!1$ and multiplies $a_{\mathrm{sch}}(w)$ and $a^*_{\mathrm{sch}}(w)$ by $\sqrt{2}$ to keep the commutation relation $[a_{\mathrm{sch}}(w_1), a^*_{\mathrm{sch}}(w_2)] = \overline{w_1} \cdot w_2$, which leads to the formulas

$$\begin{aligned} a^*_{\mathrm{sch}}(w) &= \frac{1}{\sqrt{2}}w \cdot x - \frac{\mathrm{i}}{\sqrt{2}}w \cdot D, \quad a_{\mathrm{sch}}(w) = \frac{1}{\sqrt{2}}\overline{w} \cdot x + \frac{\mathrm{i}}{\sqrt{2}}\overline{w} \cdot D, \\ w \cdot x &= \frac{1}{\sqrt{2}}\big(a^*_{\mathrm{sch}}(w) + a_{\mathrm{sch}}(\overline{w})\big), \quad w \cdot D = \frac{1}{\mathrm{i}\sqrt{2}}\big(-a^*_{\mathrm{sch}}(w) + a_{\mathrm{sch}}(\overline{w})\big), \quad w \in \mathbb{C}\mathcal{X}. \end{aligned}$$

This choice is more symmetric, but leads to the appearance of ugly square roots of 2; therefore we will not use it.

8.3 CCR algebras

In some approaches to quantum physics the initial step consists in choosing a $*$-algebra, usually a C^*- or W^*-algebra, which is supposed to describe observables of a system. Only after choosing a state (or a family of states) and making the corresponding GNS construction, we obtain a representation of this $*$-algebra in a Hilbert space. This philosophy allows us to study a quantum system in a representation-independent fashion.

Many authors try to apply this approach to bosonic systems. This raises the question whether one can associate with a given pre-symplectic space $(\mathcal{Y}, \omega)$

a natural and useful $*$-algebra describing the canonical commutation relations over $\mathcal{Y}$.

The analogous question has a rather satisfactory answer in the fermionic case. In particular, there exists an obvious choice of a C^*-algebra describing the CAR over a given Euclidean space. It turns out, however, that in the bosonic case the situation is much more complicated, since for a given pre-symplectic space several natural choices of CCR algebra are possible.

This question is discussed in this section. Throughout this section, $(\mathcal{Y}, \omega)$ is a pre-symplectic space and we discuss various $*$-algebras associated with $\mathcal{Y}$. We will see that each choice has its drawbacks. In the literature, the most popular choice seems to be the Weyl CCR algebra, which we discuss in Subsect. 8.3.5. One can, however, argue that, at least in the case of regular representations, it is more natural to use what we call the regular CCR algebra discussed in Subsect. 8.3.4. Some authors prefer to use the polynomial CCR algebra, discussed in Subsect. 8.3.1, which is purely algebraic and is not a C^*-algebra.

Unfortunately, the C^*-algebraic approach to bosonic systems has some serious problems. Many authors apply it in the case of free dynamics (given by Bogoliubov automorphisms). In the case of physically interesting interacting dynamics, the C^*-algebraic approach is not easy to apply. In fact, in the case of bosonic systems with infinite-dimensional phase spaces it is usually difficult to find a natural C^*-algebra preserved by a non-trivial dynamics. Sometimes, in such a case one can apply W^*-algebras, which we do not discuss here.

In the approach to canonical commutation relations discussed in this book, the central role is played by CCR representations, as defined in Def. 8.1. We view various CCR algebras introduced in this section more as academic curiosities than as basic tools. Therefore, the reader in a hurry may skip this section on the first reading.

8.3.1 Polynomial CCR $*$-algebras

In this subsection we discuss the polynomial CCR $*$-algebra over $\mathcal{Y}$. Note that for non-zero ω we cannot represent $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ as an algebra of bounded operators on a Hilbert space. The usefulness of this $*$-algebra for rigorous mathematical physics is rather limited.

Definition 8.40 *The* polynomial CCR $*$-algebra over $\mathcal{Y}$, *denoted by* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$, *is defined to be the unital complex $*$-algebra generated by elements $\phi(y)$, $y \in \mathcal{Y}$, with relations*

$$\phi(\lambda y) = \lambda\phi(y),\ \lambda \in \mathbb{R}, \quad \phi(y_1 + y_2) = \phi(y_1) + \phi(y_2),$$
$$\phi^*(y) = \phi(y), \quad \phi(y_1)\phi(y_2) - \phi(y_2)\phi(y_1) = \mathrm{i}y_1 \cdot \omega y_2 \mathbb{1}.$$

Let us describe basic properties of $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$.

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Proposition 8.41 (1) *Let* $r \in ASp(\mathcal{Y})$. *Then there exists a unique* $*$-*isomorphism* $\hat{r} : \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}) \to \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ *such that* $\hat{r}(\phi(y)) = \phi(ry)$, $y \in \mathcal{Y}$.

(2) *Let* $\mathcal{Y}_1$ *be a subspace of* $\mathcal{Y}$. *Then* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}_1)$ *is naturally embedded in* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$, *such that, for* $y \in \mathcal{Y}_1$, $\phi(y)$ *in the sense of* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}_1)$ *coincide with* $\phi(y)$ *in the sense of* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$. *If moreover* $\mathcal{Y}_1 \neq \mathcal{Y}$, *then* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}_1) \neq \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$.

Definition 8.42 $\hat{r}$ *defined in Prop. 8.41 is called the* Bogoliubov automorphism of $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ corresponding to r.

Proposition 8.43 *Let* $\mathcal{H}$ *be a Hilbert space and let* $\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi^{\pi}(y)} \in U(\mathcal{H})$ *be a regular CCR representation. Recall that* $\mathcal{H}^{\infty,\pi}$ *denotes the Schwartz space for a given regular CCR representation, and was defined in Def. 8.28. Then there exists a unique* $*$-*representation* $\pi : \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}) \to L(\mathcal{H}^{\infty,\pi})$ *such that* $\pi(\phi(y)) = \phi^{\pi}(y)$.

8.3.2 Stone–von Neumann CCR algebras

In this subsection we always assume that $(\mathcal{Y}, \omega)$ is a finite-dimensional pre-symplectic space. We set $\mathcal{Y}_0 := \mathrm{Ker}\,\omega \subset \mathcal{Y}$. In this case there exists a natural candidate for a CCR algebra suggested by the Stone–von Neumann theorem (Thm. 8.15), which implies the following proposition:

Proposition 8.44 (1) *Let* $\mathfrak{M}_i \subset B(\mathcal{H}_i)$, $i = 1, 2$, *be von Neumann algebras with distinguished unitary elements* $W_i(y)$ *depending* σ-*weakly continuously on* $y \in \mathcal{Y}$. *Let* $\mathfrak{Z}_i$ *be the centers of* $\mathfrak{M}_i$. *Assume that*

(i) $W_i(y_1)W_i(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \cdot \omega y_2} W_i(y_1 + y_2)$, $y_1, y_2 \in \mathcal{Y}$;

(ii) $\mathrm{Span}\{W_i(y) \ : \ y \in \mathcal{Y}\}$ *is* σ-*weakly dense in* $\mathfrak{M}_i$;

(iii) $\mathfrak{Z}_i$ *are* $*$-*isomorphic to* $L^{\infty}(\mathcal{Y}_0^{\#})$;

(iv) $\mathfrak{M}_i' = \mathfrak{Z}_i$.

Then there exists a unique σ-*weakly continuous* $*$-*isomorphism* $\rho : \mathfrak{M}_1 \to \mathfrak{M}_2$ *such that*

$$\rho\big(W_1(y)\big) = W_2(y), \quad y \in \mathcal{Y}.$$

Moreover, there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $\rho(\cdot) = U \cdot U^*$. *If* U_i, $i = 1, 2$, *are two such operators, then* $U_1^* U_2 \in \mathfrak{Z}_1$ *and* $U_1 U_2^* \in \mathfrak{Z}_2$.

(2) *Identify* $\mathcal{Y}$ *with* $\mathcal{Y}_0 \oplus \mathcal{X}^{\#} \oplus \mathcal{X}$ *and* ω *with the canonical symplectic form on* $\mathcal{X}^{\#} \oplus \mathcal{X}$ *extended by zero on* $\mathcal{Y}_0$. *Let* v *denote the generic variable in* $\mathcal{Y}_0^{\#}$ *and the corresponding multiplication operator. Then the von Neumann algebra*

$$L^{\infty}(\mathcal{Y}_0^{\#}) \otimes B\big(L^2(\mathcal{X})\big) \subset B\big(L^2(\mathcal{Y}_0^{\#} \oplus \mathcal{X})\big) \tag{8.33}$$

and the family of its elements $W(y) := \mathrm{e}^{\mathrm{i}(y_0 v + \eta x + qD)}$, $y = (y_0, \eta, q) \in \mathcal{Y}_0 \oplus \mathcal{X}^{\#} \oplus \mathcal{X}$, *satisfy the requirements of (1). (*$\otimes$ *used in (8.33) is the tensor multiplication in the category of* W^*-*algebras; see Subsect. 6.3.2.)*

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Prop. 8.44 suggests the following definition:

Definition 8.45 *A* Stone–von Neumann CCR algebra over $\mathcal{Y}$ *is defined as a von Neumann algebra* $\mathfrak{M}$ *with distinguished unitary elements* $W(y)$, $y \in \mathcal{Y}$, *satisfying the conditions of Prop. 8.44. It is denoted* $\mathrm{CCR}(\mathcal{Y})$ *and the Hilbert space it acts on is denoted* $\mathcal{H}_{\mathcal{Y}}$.

Prop. 8.44 shows that $\mathrm{CCR}(\mathcal{Y})$ is defined uniquely up to a spatially implementable $*$-isomorphism. Clearly, if $\omega = 0$, then $\mathrm{CCR}(\mathcal{Y}) \simeq L^\infty(\mathcal{Y}^\#)$. If ω is symplectic, then $\mathrm{CCR}(\mathcal{Y}) = B(\mathcal{H}_{\mathcal{Y}})$.

Definition 8.46 *Let* $y \in \mathcal{Y}$. *The corresponding* abstract field operator $\phi(y)$ *is defined as the self-adjoint operator on* $\mathcal{H}_{\mathcal{Y}}$ *such that* $W(y) = \mathrm{e}^{\mathrm{i}\phi(y)}$.

Note that the operators $\phi(y)$ are affiliated to $\mathrm{CCR}(\mathcal{Y})$.

Note also that the definition of the Stone–von Neumann CCR algebra is simpler if ω is symplectic – we can then drop (iii) and (iv) from Prop. 8.44.

The following proposition is an analog of Prop. 8.41 about polynomial CCR $*$-algebras. But whereas Prop. 8.41 was a trivial algebraic fact, Prop. 8.47 is somewhat deeper.

Proposition 8.47 (1) *Let* $r \in ASp(\mathcal{Y})$. *Then there exists a unique spatially implementable* $*$*-isomorphism* $\hat{r} : \mathrm{CCR}(\mathcal{Y}) \to \mathrm{CCR}(\mathcal{Y})$ *such that* $\hat{r}(W(y)) = W(ry)$, $y \in \mathcal{Y}$.

(2) *Let* $\mathcal{Y}_1 \subset \mathcal{Y}$. *Then there is a unique embedding of* $\mathrm{CCR}(\mathcal{Y}_1)$ *in* $\mathrm{CCR}(\mathcal{Y})$, *such that, for* $y \in \mathcal{Y}_1$, $W(y)$ *in the sense of* $\mathrm{CCR}(\mathcal{Y}_1)$ *coincide with* $W(y)$ *in the sense of* $\mathrm{CCR}(\mathcal{Y})$. *If moreover* $\mathcal{Y}_1 \neq \mathcal{Y}$, *then* $\mathrm{CCR}(\mathcal{Y}_1) \neq \mathrm{CCR}(\mathcal{Y})$.

Definition 8.48 $\hat{r}$ *defined in Prop. 8.47 is called the* Bogoliubov automorphism of $\mathrm{CCR}(\mathcal{Y})$ corresponding to r.

Here is yet another reformulation of the Stone–von Neumann theorem (see Thm. 8.15):

Theorem 8.49 *Let* $(\mathcal{Y}, \omega)$ *be symplectic. Let* $\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H})$ *be a regular CCR representation. Then there exists a unique* σ*-weakly continuous* $*$*-representation* $\pi : \mathrm{CCR}(\mathcal{Y}) \to B(\mathcal{H})$ *such that* $\pi(W(y)) = W^\pi(y)$, $y \in \mathcal{Y}$. *Moreover,* π *is isometric and*

$$\pi(\mathrm{CCR}(\mathcal{Y})) = \left\{W^\pi(y) \ : \ y \in \mathcal{Y}\right\}''.$$

If in addition the representation is irreducible, then there also exists a unitary operator $U : \mathcal{H} \to \mathcal{H}_{\mathcal{Y}}$, *unique up to a phase factor, such that* $\pi(\cdot) = U \cdot U^*$.

8.3.3 $\mathcal{S}$- and $\mathcal{S}'$-type operators

In this subsection we fix a finite-dimensional symplectic space $(\mathcal{Y}, \omega)$ and consider the von Neumann algebra $\mathrm{CCR}(\mathcal{Y}) = B(\mathcal{H}_{\mathcal{Y}})$. We will describe an abstract version of the constructions described in Subsect. 4.1.11.

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Definition 8.50 *$\Psi \in \mathcal{H}_{\mathcal{Y}}$ is called an $\mathcal{S}$-*type vector *if the function*

$$\mathcal{Y} \ni y \mapsto (\Psi|W(y)\Psi)$$

belongs to $\mathcal{S}(\mathcal{Y})$. The abstract Schwartz space for $\mathcal{Y}$ *is defined as the set of $\mathcal{S}$-type vectors. It is denoted $\mathcal{H}^\infty$.*

Clearly, $\phi(y)$, $y \in \mathcal{Y}$, leaves $\mathcal{H}^\infty$ invariant. Thus we can define a family of semi-norms

$$\mathcal{H}^\infty \ni \Psi \mapsto \|\phi(y_1)\cdots\phi(y_n)\Psi\|, \quad y_1, \dots, y_n \in \mathcal{Y},$$

which equip $\mathcal{H}^\infty$ with the structure of a Fréchet space.

Definition 8.51 *$\mathcal{H}^{-\infty}$ is defined as the topological dual to $\mathcal{H}^\infty$. It is called the* abstract $\mathcal{S}'$ space for $\mathcal{Y}$.

Note that $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ can be represented as an algebra of linear operators on $\mathcal{H}^\infty$, as well as on $\mathcal{H}^{-\infty}$.

Definition 8.52 *$A \in \mathrm{CCR}(\mathcal{Y})$ is called an $\mathcal{S}$-*type operator *iff it is trace-class and the function*

$$\mathcal{Y} \ni y \mapsto \mathrm{Tr}\, AW(y)$$

belongs to $\mathcal{S}(\mathcal{Y})$. The set of $\mathcal{S}$-type operators is denoted $\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y})$.

Clearly, $\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y})$ is a $*$-algebra. It is equipped with a topology by the family of semi-norms

$$\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y}) \ni A \mapsto |\mathrm{Tr}\,\phi(y_1)\cdots\phi(y_n)A|, \quad y_1, \dots, y_n \in \mathcal{Y}.$$

Definition 8.53 *Continuous linear functionals on $\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y})$ are called $\mathcal{S}'$-*type forms over $\mathcal{Y}$. *Their space is denoted by $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$.*

Let

$$\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y}) \ni A \mapsto B(A) \in \mathbb{C} \tag{8.34}$$

be an $\mathcal{S}'$-type form. Clearly, for any $\Psi_1, \Psi_2 \in \mathcal{H}^\infty$, the operator $|\Psi_2)(\Psi_1|$ belongs to $\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y})$. Thus, (8.34) defines a continuous sesquilinear form on $\mathcal{H}^\infty$:

$$\mathcal{H}^\infty \times \mathcal{H}^\infty \ni (\Psi_1, \Psi_2) \mapsto B\left(|\Psi_2)(\Psi_1|\right) \in \mathbb{C}.$$

In what follows we will use the "operator notation", writing $(\Psi_1|B\Psi_2)$ instead of $B\left(|\Psi_2)(\Psi_1|\right)$. Thus bounded operators can be viewed as elements of $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$, so that we have

$$\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y}) \subset \mathrm{CCR}(\mathcal{Y}) \subset \mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y}).$$

As in Subsect. 4.1.11, we define the adjoint form B^* by $(\Psi_1|B^*\Psi_2) = \overline{(\Psi_2|B\Psi_1)}$. If B_1 or B_2^* extend as continuous operators on $\mathcal{H}^\infty$, then we can

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define $B_2 \circ B_1$ as an element of $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$ by

$$(\Psi_1|B_2 \circ B_1 \Psi_2) := (\Psi_1|B_2(B_1\Psi)), \text{ or } (\Psi_1|B_2 \circ B_1\Psi_2) := (B_2^*\Psi|B_1\Psi).$$

In particular this is possible if B_1 or $B_2 \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$.

If $\mathcal{Y} \simeq \mathcal{X}^{\#} \oplus \mathcal{X}$ and we consider the Schrödinger representation on $L^2(\mathcal{X})$, then $\mathrm{CCR}^{\mathcal{S}}(\mathcal{Y})$ coincides with the set of operators whose integral kernel is in $\mathcal{S}(\mathcal{X} \times \mathcal{X})$. $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$ consists then of forms whose distributional kernel is in $\mathcal{S}'(\mathcal{X} \times \mathcal{X})$, which were considered already in Subsect. 4.1.11.

8.3.4 Regular CCR algebras

Until the end of this section, $(\mathcal{Y}, \omega)$ is a pre-symplectic space of arbitrary dimension. Recall that $\mathrm{Fin}(\mathcal{Y})$ denotes the set of finite-dimensional subspaces of $\mathcal{Y}$.

In this subsection we introduce the notion of the regular CCR C^*-algebra over $\mathcal{Y}$. In the literature, it is rarely used. Weyl CCR C^*-algebras are more common. Nevertheless, it is a natural construction. Its use was advocated by I. E. Segal.

Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{Fin}(\mathcal{Y})$ and $\mathcal{Y}_1 \subset \mathcal{Y}_2$. We can define their Stone–von Neumann CCR algebras, as in Def. 8.45. By Prop. 8.47, we have a natural embedding,

$$\mathrm{CCR}(\mathcal{Y}_1) \subset \mathrm{CCR}(\mathcal{Y}_2).$$

We can define the algebraic regular CCR $*$-algebra as the inductive limit of Stone–von Neumann CCR algebras:

Definition 8.54 *We set*

$$\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{alg}}(\mathcal{Y}) := \bigcup_{\mathcal{Y}_1 \in \mathrm{Fin}(\mathcal{Y})} \mathrm{CCR}(\mathcal{Y}_1). \tag{8.35}$$

Clearly, $\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{alg}}(\mathcal{Y})$ is a $*$-algebra equipped with a C^*-norm.

Definition 8.55 *We define the* regular CCR C^*-algebra over $\mathcal{Y}$ *as*

$$\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}) := \left(\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{alg}}(\mathcal{Y})\right)^{\mathrm{cl}}.$$

Clearly, $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$ is a generalization of the Stone–von Neumann algebra $\mathrm{CCR}(\mathcal{Y})$ from Def. 8.45.

We have an obvious extension of Prop. 8.47:

Proposition 8.56 (1) *Let $r \in ASp(\mathcal{Y})$. Then there exists a unique $*$-isomorphism $\hat{r} : \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}) \to \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$ such that $\hat{r}(W(y)) = W(ry)$, $y \in \mathcal{Y}$.*

(2) *Let $\mathcal{Y}_1 \subset \mathcal{Y}$. Then $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_1)$ is naturally embedded in $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$. If moreover $\mathcal{Y}_1 \neq \mathcal{Y}$, then $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_1) \neq \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$.*

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Proof Let us give a proof of (2). Working in the Schrödinger representation we see that $\|W(y_1) - W(y_2)\| = 2$ if $y_1 \neq y_2$. Hence, if $y \in \mathcal{Y} \backslash \mathcal{Y}_1$, then $W(y) \notin \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}_1)$. □

Definition 8.57 *$\hat{r}$ defined in Prop. 8.56 is called the* Bogoliubov automorphism *of* $\mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$ *corresponding to r.*

The following proposition is an extension of Thm. 8.49:

Proposition 8.58 *Suppose that ω is symplectic. Let $\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H})$ be a regular CCR representation. Then there exists a unique $*$-representation $\pi : \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y}) \to B(\mathcal{H})$ such that $\pi(W(y)) = W^{\pi}(y)$, $y \in \mathcal{Y}$, and which, for $\mathcal{Y}_1 \in \mathrm{Fin}(\mathcal{Y})$, is σ-weakly continuous on the sub-algebras $\mathrm{CCR}(\mathcal{Y}_1) \subset \mathrm{CCR}^{\mathrm{reg}}(\mathcal{Y})$. Moreover, π is isometric.*

Proof We use the fact that if ω is symplectic then we can restrict the union in (8.35) to run over finite-dimensional symplectic subspaces of $\mathcal{Y}$. □

8.3.5 Weyl CCR algebra

In this subsection we introduce the notion of the Weyl CCR C^*-algebra over $\mathcal{Y}$. This is the C^*-algebra generated by elements satisfying the Weyl CCR relations over $\mathcal{Y}$. Mathematical physicists use Weyl CCR algebras often in their description of bosonic systems.

Note that Weyl CCR algebras can be viewed as non-commutative generalizations of algebras of almost periodic functions. Indeed, $\mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y})$ consists of almost periodic functions on $\mathcal{Y}$ if $\omega = 0$.

Let us start with the definition of algebraic Weyl CCR algebras.

Definition 8.59 $\mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y})$ *is defined as the $*$-algebra generated by the elements $W(y)$, $y \in \mathcal{Y}$, with relations*

$$W(y)^* = W(-y), \;\; W(y_1)W(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \cdot \omega y_2} W(y_1 + y_2), \;\; y, y_1, y_2 \in \mathcal{Y}.$$

Let $\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H}^{\pi})$ be a CCR representation. Clearly, there exists a unique unital $*$-isomorphism $\pi : \mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y}) \to B(\mathcal{H}^{\pi})$ such that $\pi(W(y)) = W^{\pi}(y)$.

Let $\mathcal{R}(\mathcal{Y})$ be the class of CCR representations over $\mathcal{Y}$. $\mathcal{R}(\mathcal{Y})$ is non-empty. In fact, we always have the (non-regular) CCR representation $\mathcal{Y} \ni y \mapsto W^{\mathrm{d}}(y) \in U\big(l^2(\mathcal{Y})\big)$ defined in (8.15). It yields a corresponding faithful representation $\pi^{\mathrm{d}} : \mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y}) \to B\big(l^2(\mathcal{Y})\big)$.

Definition 8.60 *For $A \in \mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y})$ we set*

$$\|A\| := \sup\big\{ \|\pi(A)\| \; : \; \pi \in \mathcal{R}(\mathcal{Y}) \big\}. \tag{8.36}$$

The Weyl CCR C^*-algebra *is defined as*

$$\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}) := \left(\mathrm{CCR}^{\mathrm{Weyl}}_{\mathrm{alg}}(\mathcal{Y})\right)^{\mathrm{cpl}}.$$

Clearly, $\|\cdot\|$ defined in (8.36) is a C^*-norm and $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ is a C^*-algebra.

Proposition 8.61 (1) *Let $r \in ASp(\mathcal{Y})$. Then there exists a unique $*$-isomorphism $\hat{r} : \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}) \to \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ such that $\hat{r}(W(y)) = W(ry)$, $y \in \mathcal{Y}$.*
(2) *Let $\mathcal{Y}_1 \subset \mathcal{Y}$. Then $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_1)$ is naturally embedded in $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$. If moreover $\mathcal{Y}_1 \neq \mathcal{Y}$, then $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_1) \neq \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$.*
(3) *If $\mathcal{Y} \neq \{0\}$, then $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ is non-separable.*

Proof (1) and (2) are obvious analogs of Prop. 8.41. (3) follows from the fact that $y_1 \neq y_2$ implies $\|W(y_1) - W(y_2)\| = 2$. □

Definition 8.62 *$\hat{r}$ defined in Prop. 8.61 is called the* Bogoliubov automorphism of $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ corresponding to r.

Let us give an analog of Prop. 8.43:

Proposition 8.63 *Let $\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H})$ be a CCR representation. Then there exists a unique $*$-homomorphism*

$$\pi : \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}) \to B(\mathcal{H})$$

such that $\pi(W(y)) = W^\pi(y)$.

If ω is symplectic, the algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ enjoys especially good properties. In particular, there is no need to consider the norms given by all possible representations, since all of them are equal.

Theorem 8.64 *Let ω be symplectic. Then*

(1) *For $A \in \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, $\|A\| = \|\pi^{\mathrm{d}}(A)\|$.*
(2) *Every representation π described in Thm. 8.63 is isometric.*
(3) *$\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ is simple.*

Proof Let us prove (2).

For $y \in \mathcal{Y}$ we define $R(y) \in U\big(l^2(\mathcal{Y})\big)$ by setting

$$(R(y)f)(x) := f(x+y), \quad f \in l^2(\mathcal{Y}).$$

Let $\hat{\mathcal{Y}}$ be the Pontryagin dual of $\mathcal{Y}$ (the space of characters on the group $\mathcal{Y}$ with values in $\{z \in \mathbb{C} : \ |z| = 1\}$). Let $\mathcal{F} : l^2(\mathcal{Y}) \to L^2(\hat{\mathcal{Y}})$ be the (unitary) Fourier transformation. Then $\mathcal{F}R(y)\mathcal{F}^* = \hat{R}(y)$, where $\hat{R}(y) \in U\big(L^2(\hat{\mathcal{Y}})\big)$ is defined by

$$(\hat{R}(y)g)(\chi) := \chi(y)g(\chi), \quad g \in L^2(\hat{\mathcal{Y}}), \ \chi \in \hat{\mathcal{Y}}.$$

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Consider now a CCR representation

$$\mathcal{Y} \ni y \mapsto W^\pi(y) \in U(\mathcal{H}).$$

On the Hilbert space $\mathcal{H} \otimes l^2(\mathcal{Y}) \simeq l^2(\mathcal{Y}, \mathcal{H})$ we introduce the unitary operator U defined by

$$U\Phi(x) := W^\pi(x)\Phi(x), \quad \Phi \in l^2(\mathcal{Y}, \mathcal{H}), \quad x \in \mathcal{Y}.$$

Note that

$$U\ W^\pi(y) \otimes R(y)\ U^* = \mathbb{1} \otimes W^{\mathrm{d}}(y). \tag{8.37}$$

Now

$$\begin{aligned} \left\| \sum \lambda_i W^{\mathrm{d}}(y_i) \right\| &= \left\| \sum \lambda_i W^\pi(y_i) \otimes R(y_i) \right\| \\ &= \left\| \sum \lambda_i W^\pi(y_i) \otimes \hat{R}(y_i) \right\| \\ &= \sup_{\chi \in \hat{\mathcal{Y}}} \left\| \sum \lambda_i W^\pi(y_i)\chi(y_i) \right\| \\ &= \sup_{x \in \mathcal{Y}} \left\| \sum \lambda_i W^\pi(y_i)\mathrm{e}^{-\mathrm{i}x\cdot\omega y_i} \right\| \\ &= \left\| \sum \lambda_i W^\pi(y_i) \right\|. \end{aligned}$$

First we applied (8.37). Next we used that $\mathcal{F}$ is unitary. Then we used that the set of characters $\chi_x(y) := \mathrm{e}^{-\mathrm{i}x\cdot\omega y}$ for $x \in \mathcal{Y}$ is dense in $\hat{\mathcal{Y}}$, since ω is non-degenerate. Finally we noted that

$$W^\pi(y_i)\mathrm{e}^{-\mathrm{i}x\cdot\omega y_i} = W^\pi(x)W^\pi(y_i)W^\pi(-x).$$

(2) immediately implies (1).

By Subsect. 6.2.3, (2) implies (3). □

8.4 Weyl–Wigner quantization

The Weyl–Wigner quantization has a long and complicated history. It also has many names.

It was first proposed by Weyl in 1927 in his book on group theory in quantum mechanics (Weyl (1931)). Hence it is commonly called the *Weyl quantization*.

Wigner was the first who considered its inverse, at least in the case of an operator of the form $|\Psi)(\Psi|$; see Wigner (1932b). Hence the name *Wigner function* is commonly used to denote the inverse of the Weyl quantization.

Apparently, for some time the link between the Weyl quantization and the Wigner function was not understood. This link seems to have been clarified only in the late 1940s by Moyal (1949). Moyal also found a version of the formula (8.41). The non-commutative operation $*$ defined by $b := b_1 * b_2$ as in (8.41) is often called the *Moyal star*. Moyal also found the identity (8.44).

Our terminology, "the Weyl–Wigner quantization" and "the Weyl–Wigner symbol", is thus a compromise between the names "Weyl quantization" and "Wigner function". In the literature, one can also find the name *Weyl–Wigner–Moyal quantization.*

One can argue that the Weyl–Wigner quantization is the most important kind of quantization. It is certainly the most canonical quantization – its definition depends only on the symplectic structure of the phase space. It is, however, not so useful if the phase space has infinite-dimension.

Historically, Weyl introduced this quantization in the context of the Schrödinger representation, which hides the symplectic invariance of this concept. Therefore, in our presentation we start from manifestly symplectically invariant definitions, which involve a regular CCR representation. The case of the Schrödinger representation is discussed later, in Subsect. 8.4.3.

8.4.1 Quantization of polynomial symbols

In this subsection we will consider the Weyl–Wigner quantization only for polynomial symbols. More general symbols will be considered in the following subsections. (In the subsequent subsections we will, however, restrict ourselves to finite-dimensional symplectic spaces $\mathcal{Y}$).

Suppose that $(\mathcal{Y}, \omega)$ is an arbitrary pre-symplectic space. Let

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\mathcal{H}) \tag{8.38}$$

be a regular CCR representation. Let $\mathcal{H}^\infty$ denote the abstract Schwartz space for this representation introduced in Def. 8.28. Recall that $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ denotes the polynomial CCR algebra over $\mathcal{Y}$, which can be treated as an algebra of operators on $\mathcal{H}^\infty$.

Definition 8.65 *Let $y_1, \dots, y_n \in \mathcal{Y}$. We can treat these as polynomials on $\mathcal{Y}^\#$ and take their product $y_1 \cdots y_n \in \mathrm{Pol}_{\mathrm{s}}^n(\mathcal{Y}^\#)$. We define*

$$\mathrm{Op}(y_1 \cdots y_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \phi(y_{\sigma 1}) \cdots \phi(y_{\sigma n}). \tag{8.39}$$

The map extends uniquely to a linear bijective map

$$\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^\#) \ni b \mapsto \mathrm{Op}(b) \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}). \tag{8.40}$$

Theorem 8.66 (1) $\mathrm{Op}(b)^* = \mathrm{Op}(\overline{b})$, *for* $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^\#)$.
(2) *If $y \in \mathbb{C}\mathcal{Y}$, then*

$$\mathrm{Op}(y) = \phi(y).$$

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More generally, let $\mathcal{X}$ be an isotropic subspace in $\mathcal{Y}^{\#}$, so that the operators $\phi(y)$, $y \in \mathcal{X}$ commute with one another. Then, for $f \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$, $\mathrm{Op}(f)$ coincides with $f(\phi)$ defined by the functional calculus.

(3) *If $b_1, b_2 \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$, then $\mathrm{Op}(b_1)\mathrm{Op}(b_2) = \mathrm{Op}(b)$ for*

$$b(v) = \exp\Big(-\frac{\mathrm{i}}{2} D_{v_1} \cdot \omega D_{v_2}\Big) b_1(v_1) b_2(v_2)\big|_{v=v_1=v_2}. \tag{8.41}$$

(4) *If $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$ and $y \in \mathbb{C}\mathcal{Y}$, then*

$$\frac{1}{2}\big(\phi(y)\mathrm{Op}(b) + \mathrm{Op}(b)\phi(y)\big) = \mathrm{Op}(yb).$$

Remark 8.67 *We refer to Remark 8.27 for the notation used in (2). The r.h.s. of (3) can be interpreted as a finite sum of differential operators.*

The following theorem is a version of the Wick theorem adapted to the Weyl–Wigner quantization.

Theorem 8.68 *If $b, b_1, \dots, b_n \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$ and*

$$\mathrm{Op}(b) = \mathrm{Op}(b_1) \cdots \mathrm{Op}(b_n),$$

then

$$b(v) = \exp\Big(\tfrac{\mathrm{i}}{2} \sum_{i<j} \nabla_{v_i} \cdot \omega \nabla_{v_j}\Big) b_1(v_1) \cdots b_n(v_n)\big|_{v=v_1=\cdots=v_n}.$$

8.4.2 Quantization of distributional symbols

In this subsection we assume that the form ω is symplectic and $\mathcal{Y}$ is finite-dimensional. We set $2d = \dim \mathcal{Y}$. Denote by $\mathrm{d}y$ the Liouville measure on $\mathcal{Y}$ defined in Subsect. 3.6.3. The dual space $\mathcal{Y}^{\#}$ is equipped with the symplectic form ω^{-1} and the dual measure $\mathrm{d}v$.

Consider a regular irreducible CCR representation (8.38). In this subsection we extend the Weyl–Wigner quantization to $\mathcal{S}'(\mathcal{Y}^{\#})$.

Recall that for $b \in \mathcal{S}'(\mathcal{Y}^{\#})$ the Fourier transform of b, denoted $\hat{b} \in \mathcal{S}'(\mathcal{Y})$, satisfies

$$b(v) = (2\pi)^{-2d} \int_{\mathcal{Y}} \hat{b}(y) \mathrm{e}^{\mathrm{i}y\cdot v} \mathrm{d}y, \quad v \in \mathcal{Y}^{\#}.$$

Definition 8.69 *If $b \in \mathcal{S}'(\mathcal{Y}^{\#})$, then $\mathrm{Op}(b) \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$ is defined by the formula*

$$\begin{aligned}\big(\Psi_1|\mathrm{Op}(b)\Psi_2\big) &:= (2\pi)^{-2d} \int_{\mathcal{Y}} \hat{b}(y)\big(\Psi_1|W(y)\Psi_2\big)\mathrm{d}y \qquad (8.42)\\ &= (2\pi)^{-2d} \int_{\mathcal{Y}} \int_{\mathcal{Y}^{\#}} b(v)\big(\Psi_1|W(y)\Psi_2\big)\mathrm{e}^{-\mathrm{i}v\cdot y}\mathrm{d}y\mathrm{d}v, \qquad \Psi_1, \Psi_2 \in \mathcal{H}^{\infty}.\end{aligned}$$

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Recall that $\mathcal{H}^\infty$ is the space of $\mathcal{S}$-type vectors for the representation (8.38). We know from Thm. 8.29 that if $\Psi_1, \Psi_2 \in \mathcal{H}^\infty$, then $\mathcal{Y} \ni y \mapsto \big(\Psi_1 | W(y)\Psi_2\big)$ is a Schwartz function. Therefore, the integral (8.42) is well defined.

The following theorem extends some of statements of Thm. 8.66 to the case of distributional symbols:

Theorem 8.70 (1) *If $b \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^\#)$, then the definition (8.39) coincides with (8.42).*

(2) $W(y) = \mathrm{Op}(\mathrm{e}^{\mathrm{i}y(\cdot)})$. *More generally, if $\mathcal{X}$ is an isotropic subspace of $\mathcal{Y}$, and $f \in \mathcal{S}'(\mathcal{X}^\#) \subset \mathcal{S}'(\mathcal{Y}^\#)$ is a measurable function, then $\mathrm{Op}(f)$ coincides with $f(\phi)$ defined by the functional calculus.*

(3) $\mathrm{Op}(b)^* = \mathrm{Op}(\overline{b})$.

(4) *If $b_1 \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^\#)$, $b_2, b \in \mathcal{S}'(\mathcal{Y}^\#)$ and $\mathrm{Op}(b_1)\mathrm{Op}(b_2) = \mathrm{Op}(b)$, then*

$$\begin{aligned} b(v) &:= \exp\Big(-\frac{\mathrm{i}}{2} D_{v_1} \cdot \omega D_{v_2}\Big) b_1(v_1) b_2(v_2)\Big|_{v=v_1=v_2} \\ &= \pi^{-2d} \int\limits_{\mathcal{Y}^\#} \int\limits_{\mathcal{Y}^\#} \mathrm{e}^{2\mathrm{i}(v-v_1)\cdot\omega^{-1}(v-v_2)} b_1(v_1) b_2(v_2) \mathrm{d}v_1 \mathrm{d}v_2. \end{aligned}$$

(5) *For $v \in \mathcal{Y}^\#$, $W(-\omega^{-1}v)\mathrm{Op}(b)W(\omega^{-1}v) = \mathrm{Op}(b(\cdot - v))$.*

(6) *The map*

$$\mathcal{S}'(\mathcal{Y}^\#) \ni b \mapsto \mathrm{Op}(b) \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y}) \tag{8.43}$$

is bijective.

(7) $\mathrm{Op}(b) \in B^2(\mathcal{H})$ *iff* $b \in L^2(\mathcal{Y}^\#)$, *and*

$$\mathrm{Tr}\,\mathrm{Op}(b)^*\mathrm{Op}(a) = (2\pi)^{-d} \int \overline{b(v)} a(v) \mathrm{d}v, \quad a, b \in L^2(\mathcal{Y}^\#). \tag{8.44}$$

Proof To prove (1), it is enough to consider $y_0 \in \mathcal{Y}$ and $b(v) = (y_0 \cdot v)^n$, because such polynomials span $\mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^\#)$. The Fourier transform of b is $\hat{b} = (2\pi)^{2d} \mathrm{i}^n (y_0 \cdot \nabla_y)^n \delta_0$. Hence

$$\mathrm{Op}(b) = (-\mathrm{i})^n (y_0 \cdot \nabla_y)^n W(y)\big|_{y=0} = \phi(y_0)^n.$$

(2) follows from the spectral theorem and (3) is immediate.

To prove (4), set

$$b_0(v_1, v_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} D_{v_1} \cdot \omega D_{v_2}} b_1(v_1) b_2(v_2).$$

Clearly,

$$\hat{b}_0(y_1, y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \cdot \omega y_2} \hat{b}(y_1) \hat{b}(y_2).$$

Moreover,

$$\begin{aligned} b(v) = b_0(v, v) &= (2\pi)^{-4d} \int \hat{b}(y_1, y_2) \mathrm{e}^{\mathrm{i}(y_1+y_2)\cdot v} \mathrm{d}y_1 \mathrm{d}y_2 \\ &= (2\pi)^{-4d} \int \hat{b}_1(y_1) \hat{b}_2(y_2) \mathrm{e}^{-\frac{\mathrm{i}}{2} y_1 \cdot \omega y_2} \mathrm{e}^{\mathrm{i}(y_1+y_2)\cdot v} \mathrm{d}y_1 \mathrm{d}y_2. \end{aligned}$$

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Hence

$$\begin{aligned}\mathrm{Op}(b) &= (2\pi)^{-4d} \iint \hat{b}_1(y_1)\hat{b}_2(y_2)\mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\cdot\omega y_2} W(y_1+y_2)\mathrm{d}y_1\mathrm{d}y_2 \\ &= (2\pi)^{-4d} \iint \hat{b}_1(y_1)\hat{b}_2(y_2)W(y_1)W(y_2)\mathrm{d}y_1\mathrm{d}y_2 \\ &= \mathrm{Op}(b_1)\mathrm{Op}(b_2).\end{aligned}$$

To prove the last two items of the theorem it is convenient to use the Schrödinger representation, considered in the next subsection. □

Definition 8.71 *The inverse of (8.43) will be called the* Weyl–Wigner symbol. *If $B \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$, its Weyl–Wigner symbol will be denoted by $\mathrm{s}_B \in \mathcal{S}'(\mathcal{Y}^{\#})$.*

8.4.3 Weyl–Wigner quantization in the Schrödinger representation

Let $\mathcal{X}$ be a finite-dimensional real vector space. Consider the Schrödinger representation

$$\mathcal{X}^{\#} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot D)} \in U\big(L^2(\mathcal{X})\big).$$

Remark 8.72 *In the Schrödinger representation one often writes $b^{w}(x,D)$ instead of $\mathrm{Op}(b)$.*

Theorem 8.73 (1) *Let $b \in \mathcal{S}'(\mathcal{X}\oplus\mathcal{X}^{\#})$. The distributional kernel of $B = \mathrm{Op}(b)$ can be computed as follows:*

$$B(x,y) = (2\pi)^{-d}\int b\Big(\frac{x+y}{2},\xi\Big)\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\mathrm{d}\xi. \qquad (8.45)$$

(2) *Let $B \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$. The symbol of B can be obtained from its distributional kernel by the formula*

$$\mathrm{s}_B(x,\xi) = \int B\Big(x+\frac{y}{2}, x-\frac{y}{2}\Big)\mathrm{e}^{-\mathrm{i}\xi\cdot y}\mathrm{d}y.$$

(3) *The relationship between the x, D-symbol and the Weyl–Wigner symbol is as follows: If $\mathrm{Op}^{x,D}(b_+) = \mathrm{Op}(b)$, then*

$$\begin{aligned}b_+(x,\xi) &= \mathrm{e}^{\frac{\mathrm{i}}{2}D_x\cdot D_\xi} b(x,\xi) \\ &= \pi^{-d}\int \mathrm{e}^{-\mathrm{i}2(x-x_1)\cdot(\xi-\xi_1)} b(x_1,\xi_1)\mathrm{d}x_1\mathrm{d}\xi_1.\end{aligned}$$

(4) *If* $\mathrm{Op}(b) = \mathrm{Op}(b_1)\mathrm{Op}(b_2)$, *then*

$$\begin{aligned}
&b(x,\xi)\\
&= \exp \frac{\mathrm{i}}{2}\Big(D_{\xi_1}\cdot D_{x_2} - D_{\xi_2}\cdot D_{x_1}\Big) b_1(x_1,\xi_1) b_2(x_2,\xi_2)\Big|_{\substack{x = x_1 = x_2 \\ \xi = \xi_1 = \xi_2}}\\
&= \pi^{-2d}\int \mathrm{e}^{2\mathrm{i}(x-x_1)\cdot(\xi-\xi_2)-(x-x_2)\cdot(\xi-\xi_1)} b_1(x_1,\xi_1) b_2(x_2,\xi_2) \mathrm{d}x_1 \mathrm{d}\xi_1 \mathrm{d}x_2 \mathrm{d}\xi_2.
\end{aligned}$$

Proof Let us prove (1). It is enough to check (8.45) for $b(x,\xi) := \mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot\xi)}$. We know that

$$\mathrm{Op}(b) = W(\eta, q) = \mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot D)} = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot x},$$

which has the integral kernel

$$B(x,y) = (2\pi)^{-d}\int_{\mathcal{X}^\#} \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot(x+y)}\mathrm{e}^{\mathrm{i}q\cdot\xi + \mathrm{i}\xi\cdot(x-y)}\mathrm{d}\xi.$$

Properties (2), (3), (4) then follow from (1). □

Example 8.74 *Let P_0 be the operator considered in Example 4.42 (the orthogonal projection onto $\pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}$). Then*

$$\mathrm{s}_{P_0}(x,\xi) = 2^d \mathrm{e}^{-x^2-\xi^2}.$$

8.4.4 Parity operator

Let $(\mathcal{Y},\omega)$ be a symplectic space of dimension $2d$. Consider a regular irreducible CCR representation (8.38). Let δ_v denote the delta function at $v \in \mathcal{Y}^\#$.

Definition 8.75 *Define the* parity operator

$$I := \mathrm{Op}(\pi^d \delta_0). \tag{8.46}$$

Theorem 8.76 (1) *I is self-adjoint and $I^2 = \mathbb{1}$.*
(2) *$I\mathrm{Op}(b)I = \mathrm{Op}(b_0)$, where $b_0(v) = b(-v)$.*
(3) *In the Schrödinger representation,*

$$I\Psi(x) = \Psi(-x). \tag{8.47}$$

Proof Let us show (3) first. The distributional kernel of I in the Schrödinger representation is

$$\begin{aligned}
I(x,y) &= 2^{-d}\int \delta\Big(\frac{x+y}{2},\xi\Big)\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\mathrm{d}\xi\\
&= 2^{-d}\delta\Big(\frac{x+y}{2}\Big) = \delta(x+y).
\end{aligned}$$

This proves(3).

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In the case of the Schrödinger representation, (1) and (2) follow immediately from (3). But every regular irreducible CCR representation is equivalent to the Schrödinger representation. □

Definition 8.77 *Define the* parity operator centered at v *as*

$$I_v := \mathrm{Op}(\pi^d \delta_v) = W(-\omega^{-1}v)IW(\omega^{-1}v), \quad v \in \mathcal{Y}^{\#}.$$

Theorem 8.78 (1) *I_v is self-adjoint and $I_v^2 = 1\!\!1$.*
(2) *$I_v \mathrm{Op}(b) I_v = \mathrm{Op}(b_v)$, where $b_v(w) = b(2v - w)$.*
(3) *In the Schrödinger representation,*

$$I_{(q,\eta)}\Psi(x) = \mathrm{e}^{2\mathrm{i}\eta\cdot(x-q)}\Psi(2q - x). \tag{8.48}$$

The following theorem is an analog of Prop. 4.31.

Theorem 8.79 (1) *If $b \in L^1(\mathcal{Y}^{\#})$, then $\mathrm{Op}(b)$ is a compact operator. In terms of an absolutely norm convergent integral, we can write*

$$\mathrm{Op}(b) = \pi^{-d} \int I_v b(v) \mathrm{d}v. \tag{8.49}$$

Hence,

$$\|\mathrm{Op}(b)\| \le \pi^{-d}\|b\|_1. \tag{8.50}$$

(2) *If $B \in B^1(\mathcal{H})$, then $s_B \in C_\infty(\mathcal{Y}^{\#})$ and*

$$s_B(v) = 2^d \mathrm{Tr} I_v B. \tag{8.51}$$

Hence

$$|s_B(v)| \le 2^d \mathrm{Tr}|B|.$$

Proof Clearly, $b = \int b(v)\delta_v \mathrm{d}v$. Therefore, (8.49) follows from $I_v = \mathrm{Op}(\pi^d \delta_v)$. (8.49) implies (8.50).

Let $b \in L^1(\mathcal{Y}^{\#})$. Let $b_n \in L^2(\mathcal{Y}^{\#}) \cap L^1(\mathcal{Y}^{\#})$ such that $b_n \to b$ in $L^1(\mathcal{Y}^{\#})$. By Thm. 8.70 (6), the operators $\mathrm{Op}(b_n)$ are Hilbert–Schmidt and hence compact. By (8.50), we have $\mathrm{Op}(b_n) \to \mathrm{Op}(b)$ in norm. Therefore, b is compact.

Let us prove (2). Let $a \in L^2 \cap L^1$ and let B be trace-class. Then B is also Hilbert–Schmidt. Using first Thm. 8.70 (7), then (8.49), and finally the trace-class property of B, we obtain

$$\begin{aligned}(2\pi)^{-d} \int \overline{a(v)} s_B(v) \mathrm{d}v = \mathrm{Tr}\, \mathrm{Op}(a)^* B = \pi^{-d} \mathrm{Tr}\Big(\int \overline{a(v)} I_v \mathrm{d}v B\Big) \\ = (2\pi)^{-d} \int \overline{a(v)} 2^d \mathrm{Tr} I_v B \mathrm{d}v.\end{aligned}$$

This proves the identity (8.51) for almost all v.

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Using the fact that $v \mapsto I_v$ is strongly continuous and B is trace-class we see that $v \mapsto 2^d \mathrm{Tr} I_v B$ is continuous. Using $\mathrm{w} - \lim\limits_{\|v\|\to\infty} I_v = 0$ and Prop. 2.40, we conclude that $\lim\limits_{\|v\|\to\infty} 2^d \mathrm{Tr} I_v B = 0$. □

Remark 8.80 *The Weyl–Wigner symbol of a quantum state can be measured. The first such experiment involved the motional degrees of freedom of an ion and was performed by Leibfried* et al. *(1996).*

In the case of a light mode this was first done in a simple and elegant experiment by Wódkiewicz, Radzewicz, Banaszek and Krasiński (described in Banaszek et al. *(1999)). A mode of a laser light was trapped between two mirrors. By applying an external source of light its state was "translated" in the phase space. The parity was measured by counting the number of scattered photons. Then the formula (8.51) was used to compute the Weyl–Wigner symbol of a given quantum state.*

8.5 General coherent vectors

By translating a fixed normalized vector with Weyl operators we obtain a family of vectors parametrized by the phase space. These vectors will be called *coherent vectors*. The family of coherent vectors has properties similar in some respects to those of an o.n. basis.

Coherent vectors can be used to define two kinds of quantizations. These two quantizations go under various names. We use the names proposed by Berezin (1966): the *covariant* and *contravariant quantizations*. These two quantizations are often used in applications.

In the literature the name "coherent vector" (or "coherent state") usually has a narrower meaning, of a Gaussian vector translated in phase space. Up to a phase factor, Gaussian coherent vectors can be also defined as eigenvectors of the annihilation operator. The covariant, resp. contravariant quantization w.r.t. Gaussian coherent vectors are also known as the *Wick*, resp. *anti-Wick quantization*. (Other names are used as well.)

In this section we describe the properties of general coherent vectors. We also discuss the covariant and contravariant quantization related to a given family of coherent vectors.

Gaussian coherent vectors, as well as Wick and anti-Wick quantizations, will be discussed in Chap. 9 about the Fock representation.

Throughout this section $\mathcal{Y}$ is a finite-dimensional symplectic space of dimension $2d$. $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ is an irreducible regular CCR representation. $\Psi_0 \in \mathcal{H}$ is a fixed normalized vector and $P_0 := |\Psi_0)(\Psi_0|$ is the corresponding orthogonal projection.

8.5.1 Coherent states transformation

Definition 8.81 *The family of* coherent vectors associated with the vector Ψ_0 *is defined by*

$$\Psi_v := W(-\omega^{-1}v)\Psi_0, \quad v \in \mathcal{Y}^{\#}.$$

The orthogonal projection onto Ψ_v*, called the* coherent state*, will be denoted*

$$P_v := W(-\omega^{-1}v)P_0 W(\omega^{-1}v), \quad v \in \mathcal{Y}^{\#}.$$

Remark 8.82 *One often assumes that, for any* $y \in \mathcal{Y}$, $\Psi_0 \in \mathrm{Dom}\,\phi(y)$ *and*

$$\big(\Psi_0|\phi(y)\Psi_0\big) = 0.$$

This assumption implies that $\Psi_v \in \mathrm{Dom}\,\phi(y)$ *and*

$$\big(\Psi_v|\phi(y)\Psi_v\big) = v{\cdot}y, \quad v \in \mathcal{Y}^{\#}.$$

Thus Ψ_v *is localized in the phase space around* $v \in \mathcal{Y}^{\#}$*. Note, however, that we will not use the above assumption in this section.*

Definition 8.83 *The* coherent states transform of $\Phi \in \mathcal{H}$ *is defined as*

$$\mathcal{Y}^{\#} \ni v \mapsto T^{\mathrm{FBI}}\Phi(v) := (2\pi)^{-\frac{d}{2}}(\Psi_v|\Phi).$$

The coherent state transform is sometimes also called the FBI transform*, for Fourier, Bros and Iagolnitzer.*

Example 8.84 *Assume for the moment that* $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$ *and* $\mathcal{H} = L^2(\mathcal{X})$*. Consider the Schrödinger representation* $\mathcal{X}^{\#} \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot D)} \in U\big(L^2(\mathcal{X})\big)$*. Fix a normalized vector* $\Psi \in L^2(\mathcal{X})$*. Let* $(q,\eta) \in \mathcal{Y}^{\#} = \mathcal{X} \oplus \mathcal{X}^{\#}$*. The coherent vectors and states are then given by*

$$\Psi_{(q,\eta)}(x) = \mathrm{e}^{\mathrm{i}(-q\cdot D + \eta\cdot x)}\Psi(x) = \mathrm{e}^{\mathrm{i}\eta\cdot x - \frac{\mathrm{i}}{2}q\cdot\eta}\Psi(x-q),$$
$$P_{(q,\eta)}(x_1,x_2) = \Psi(x_1-q)\overline{\Psi}(x_2-q)\mathrm{e}^{\mathrm{i}(x_1-x_2)\cdot\eta}.$$

Theorem 8.85 (1)

$$(2\pi)^{-d}\int P_v \mathrm{d}v = 1\!\!1, \quad \textit{as a weak integral.} \tag{8.52}$$

(2) *If* $\Phi \in \mathcal{H}$*, then* $T^{\mathrm{FBI}}\Phi \in L^2(\mathcal{Y}^{\#}) \cap C_\infty(\mathcal{Y}^{\#})$ *and*

$$\|T^{\mathrm{FBI}}\Phi\|_2 = \|\Phi\|_{\mathcal{H}}, \quad \|T^{\mathrm{FBI}}\Phi\|_\infty \le (2\pi)^{-\frac{d}{2}}\|\Phi\|_{\mathcal{H}}. \tag{8.53}$$

In particular, T^{FBI} *is an isometry from* $\mathcal{H}$ *into* $L^2(\mathcal{Y}^{\#})$*.*

(3) *The FBI transformation intertwines the representation* W *with a certain representation of CCR on* $L^2(\mathcal{Y}^{\#})$*:*

$$\mathrm{e}^{\mathrm{i}y\cdot(\frac{1}{2}v - \omega D_v)}T^{\mathrm{FBI}} = T^{\mathrm{FBI}}W(y), \quad y \in \mathcal{Y}.$$

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Proof To prove (1) we use the Schrödinger representation. Let $\Phi \in L^2(\mathcal{X})$. Then

$$\begin{aligned}&\int\limits_{\mathcal{X}\oplus\mathcal{X}^\#} (\Phi|P_{(q,\eta)}\Phi)\mathrm{d}q\mathrm{d}\eta \\ &= \int\limits_{\mathcal{X}\oplus\mathcal{X}^\#}\int\limits_{\mathcal{X}}\int\limits_{\mathcal{X}} \overline{\Phi(x_1)}\Psi_0(x_1-q)\overline{\Psi_0(x_2-q)}\mathrm{e}^{\mathrm{i}(x_1-x_2)\cdot\eta}\Phi(x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}q\mathrm{d}\eta \\ &= (2\pi)^d\int\limits_{\mathcal{X}}\int\limits_{\mathcal{X}} \overline{\Phi(x)}\Psi_0(x-q)\overline{\Psi_0(x-q)}\Phi(x)\mathrm{d}x\mathrm{d}q = (2\pi)^d\|\Phi\|^2\|\Psi_0\|^2.\end{aligned}$$

The first statement from (2) follows immediately from (1), the second from the definition of T^{FBI} and the fact that $W(y)$ tends weakly to 0 when $y \to \infty$.

To prove (3) we compute

$$\begin{aligned}\big(T^{\mathrm{FBI}}W(y)\Phi\big)(v) &= \big(\Psi_0|W(\omega^{-1}v)W(y)\Phi\big) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}y\cdot v}\big(\Psi_0|W(\omega^{-1}(v+\omega y))\Phi\big) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}y\cdot v}\mathrm{e}^{\mathrm{i}(\omega y)\cdot D_v}\big(\Psi_0|W(\omega^{-1}v)\Phi\big) = \Big(\mathrm{e}^{\mathrm{i}y\cdot(\frac{1}{2}v-\omega D_v)}T^{\mathrm{FBI}}\Phi\Big)(v).\end{aligned}$$

□

8.5.2 Contravariant quantization

Recall that $\mathrm{Meas}(\mathcal{Y}^\#)$ denotes the space of complex Borel pre-measures on $\mathcal{Y}^\#$. The subspace of $\mathrm{Meas}(\mathcal{Y}^\#)$ consisting of finite Borel measures is denoted $\mathrm{Meas}^1(\mathcal{Y}^\#)$. If $b \in L^1_{\mathrm{loc}}(\mathcal{Y}^\#)$, then $\mathrm{d}\mu = b\mathrm{d}v$ belongs to $\mathrm{Meas}(\mathcal{Y}^\#)$ and $\|\mu\|_1 = \|b\|_1$. In such a case, μ is absolutely continuous w.r.t. the Lebesgue measure $\mathrm{d}v$ and b is its Radon–Nikodym derivative w.r.t. $\mathrm{d}v$. Thus $L^1_{\mathrm{loc}}(\mathcal{Y}^\#)$, resp. $L^1(\mathcal{Y}^\#)$ can be viewed as subspaces of $\mathrm{Meas}(\mathcal{Y}^\#)$, resp. $\mathrm{Meas}^1(\mathcal{Y}^\#)$. In such a case, we will abuse notation and write simply $b \in \mathrm{Meas}(\mathcal{Y}^\#)$.

Actually, we will abuse the notation even further. We will write $b\mathrm{d}v$ instead of $\mathrm{d}\mu$ even if $\mu \in \mathrm{Meas}(\mathcal{Y}^\#)$ is not absolutely continuous w.r.t. the Lebesgue measure $\mathrm{d}v$. Thus b will denote the "Radon–Nikodym derivative of μ w.r.t. $\mathrm{d}v$", even if strictly speaking such a derivative does not exist.

By smearing out coherent states with a classical symbol we obtain the so-called contravariant quantization. In the following proposition we describe properties of the contravariant quantization. Note in particular that positive symbols correspond to positive operators.

Proposition 8.86 *Let $b \in L^\infty(\mathcal{Y}^\#) + \mathrm{Meas}^1(\mathcal{Y}^\#)$. Then the formula*

$$\big(\Phi|\mathrm{Op}^{\mathrm{ct}}(b)\Phi\big) := (2\pi)^{-d}\int(\Phi|P_v\Phi)b(v)\mathrm{d}v \tag{8.54}$$

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defines $\mathrm{Op}^{\mathrm{ct}}(b) \in B(\mathcal{H})$. *We have*

$$\|\mathrm{Op}^{\mathrm{ct}}(b)\| \leq (2\pi)^{-d}\|b\|_1, \quad b \in \mathrm{Meas}^1(\mathcal{Y}^{\#}), \tag{8.55}$$

$$\|\mathrm{Op}^{\mathrm{ct}}(b)\| \leq \|b\|_\infty, \qquad b \in L^\infty(\mathcal{Y}^{\#}). \tag{8.56}$$

Definition 8.87 $\mathrm{Op}^{\mathrm{ct}}(b) \in B(\mathcal{H})$ *defined in (8.54) is called the* contravariant quantization of b.

Proof of Prop. 8.86. If $b \in \mathrm{Meas}^1(\mathcal{Y}^{\#})$, then the integral on the r.h.s. of (8.54) is finite and we obtain (8.55).

If $b \in L^\infty(\mathcal{Y}^{\#})$, we can write

$$\mathrm{Op}^{\mathrm{ct}}(b) = T^{\mathrm{FBI}*}b(v)T^{\mathrm{FBI}}, \tag{8.57}$$

where on the r.h.s. $b(v)$ has the meaning of a multiplication operator on $L^2(\mathcal{Y}^{\#})$, and we obtain (8.56).

In the general case, we can write

$$b = b_0 + b_1, \quad b_0 \in L^\infty(\mathcal{Y}^{\#}), \quad b_1 \in \mathrm{Meas}^1(\mathcal{Y}^{\#}), \tag{8.58}$$

and set

$$\mathrm{Op}^{\mathrm{ct}}(b) := \mathrm{Op}^{\mathrm{ct}}(b_0) + \mathrm{Op}^{\mathrm{ct}}(b_1). \tag{8.59}$$

It is easy to see that (8.59) does not depend on the decomposition (8.58). □

Proposition 8.88 (1) $\mathrm{Op}^{\mathrm{ct}}(1) = 1\!\!1_{\mathcal{H}}$.

(2) $\mathrm{Op}^{\mathrm{ct}}(b)^* = \mathrm{Op}^{\mathrm{ct}}(\bar{b})$.

(3) *If* $v \in \mathcal{Y}^{\#}$, *then*

$$W(-\omega^{-1}v)\mathrm{Op}^{\mathrm{ct}}(b)W(\omega^{-1}v) = \mathrm{Op}^{\mathrm{ct}}(b(\cdot - v)).$$

(4) *If* $b \in L^\infty(\mathcal{Y}^{\#})$ *is real-valued, then*

$$\operatorname{ess\,inf} b \leq \mathrm{Op}^{\mathrm{ct}}(b) \leq \operatorname{ess\,sup} b. \tag{8.60}$$

(5) *Let* $b \geq 0$. *Then* $\mathrm{Op}^{\mathrm{ct}}(b) \geq 0$. *Moreover,* $b \in \mathrm{Meas}^1(\mathcal{Y}^{\#})$ *iff* $\mathrm{Op}^{\mathrm{ct}}(b) \in B^1(\mathcal{H})$, *and*

$$\mathrm{Tr}\,\mathrm{Op}^{\mathrm{ct}}(b) = (2\pi)^{-d}\int b(v)\mathrm{d}v. \tag{8.61}$$

(6) *If* $b \in \mathrm{Meas}^1(\mathcal{Y}^{\#})$, *then* $\mathrm{Op}^{\mathrm{ct}}(b) \in B^1(\mathcal{H})$ *and (8.61) is true.*

(7) *Suppose that* $b \in L^\infty_\infty(\mathcal{Y}^{\#}) + \mathrm{Meas}^1(\mathcal{Y}^{\#})$, *where* $L^\infty_\infty(\mathcal{Y}^{\#})$ *denotes the set of* $b \in L^\infty(\mathcal{Y}^{\#})$ *such that* $\lim\limits_{|v|\to\infty} b(v) = 0$. *Then* $\mathrm{Op}^{\mathrm{ct}}(b)$ *is compact.*

Proof (3) follows from $W(-\omega^{-1}v)P_w W(\omega^{-1}v) = P_{w+v}$. (8.60) follows immediately from (8.57).

We will now prove (5). Let b be positive. Let $\{e_i\}_{i\in I}$ be an o.n. basis of $\mathcal{H}$. By Fubini's theorem, we get

$$\begin{aligned}\operatorname{Tr}\operatorname{Op}^{\mathrm{ct}}(b) &= \sum_{i\in I}\big(e_i|\operatorname{Op}^{\mathrm{ct}}(b)e_i\big) = (2\pi)^{-d}\int\sum_{i\in I} b(v)(e_i|P_v e_i)\mathrm{d}v\\ &= (2\pi)^{-d}\int b(v)\operatorname{Tr}P_v\mathrm{d}v = (2\pi)^{-d}\int b(v)\mathrm{d}v,\end{aligned}$$

which proves (5).

To show (6) we use (5) and the decomposition $b = b_1 + \mathrm{i}b_2 - b_3 - \mathrm{i}b_4$, where $b_i \in \mathrm{Meas}^1$ and $b_i \geq 0$. Finally, if $b = b_0 + \mu$ for $b_0 \in L^\infty_\infty(\mathcal{Y}^\#)$, $\mu \in \mathrm{Meas}^1(\mathcal{Y}^\#)$, we write $b_n = \mathbb{1}_{[0,n]}\big(|v|\big)b_0 + \mu$, so that $b_n \in \mathrm{Meas}^1(\mathcal{Y}^\#)$, $\operatorname{Op}^{\mathrm{ct}}(b_n) \in B^1(\mathcal{H})$, and $\|\operatorname{Op}^{\mathrm{ct}}(b_n - b)\| \leq \|b_n - b\|_\infty \to 0$ when $n \to \infty$. This proves (7). $\square$

Definition 8.89 *If the map*

$$L^\infty(\mathcal{Y}^\#) + \mathrm{Meas}^1(\mathcal{Y}^\#) \ni b \mapsto \operatorname{Op}^{\mathrm{ct}}(b) \in B(\mathcal{H})$$

is injective, then its inverse is called the contravariant symbol. *For $B \in B(\mathcal{H})$, its contravariant symbol will be denoted $\mathrm{s}^{\mathrm{ct}}_B$.*

8.5.3 Covariant quantization

In this subsection we describe the covariant quantization, which in a sense is the operation dual to the contravariant quantization. Strictly speaking, the operation that has a natural definition and good properties is not the covariant quantization but the covariant symbol of an operator.

Definition 8.90 *Let $B \in B(\mathcal{H})$. Then we define its* covariant symbol *by*

$$\begin{aligned}\mathrm{s}^{\mathrm{cv}}_B(v) &:= \operatorname{Tr}P_v B\\ &= \big(W(-\omega^{-1}v)\Psi_0|BW(-\omega^{-1}v)\Psi_0\big), \qquad v \in \mathcal{Y}^\#.\end{aligned}$$

Theorem 8.91 (1) $\mathrm{s}^{\mathrm{cv}}_{\mathbb{1}} = 1$, $\mathrm{s}^{\mathrm{cv}}_{B^*} = \overline{\mathrm{s}^{\mathrm{cv}}_B}$.
(2) *If $B \in B(\mathcal{H})$, $B_v := W(-\omega^{-1}v)BW(\omega^{-1}v)$, then*

$$\mathrm{s}^{\mathrm{cv}}_{B_v} = \mathrm{s}^{\mathrm{cv}}_B(\cdot - v).$$

(3) *If $B \in B(\mathcal{H})$, then $\mathrm{s}^{\mathrm{cv}}_B \in C(\mathcal{Y}^\#) \cap L^\infty(\mathcal{Y}^\#)$ and*

$$\|\mathrm{s}^{\mathrm{cv}}_B\|_\infty \leq \|B\|. \tag{8.62}$$

(4) *Let $B \geq 0$. Then $\mathrm{s}^{\mathrm{cv}}_B \geq 0$. Moreover, $B \in B^1(\mathcal{H})$ iff $\mathrm{s}^{\mathrm{cv}}_B \in L^1(\mathcal{Y}^\#)$, and*

$$\operatorname{Tr} B = (2\pi)^{-d}\int \mathrm{s}^{\mathrm{cv}}_B(v)\mathrm{d}v. \tag{8.63}$$

(5) *If $B \in B^1(\mathcal{H})$, then $\mathrm{s}^{\mathrm{cv}}_B \in L^1(\mathcal{Y}^\#)$ and (8.63) is true.*
(6) *If B is compact, then $\mathrm{s}^{\mathrm{cv}}_B \in C_\infty(\mathcal{Y}^\#)$.*

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Proof (1) and (2) are immediate. Let us show (3). It is easy to see that the inequality (8.62) is true. Moreover

$$v \mapsto W(\omega^{-1}v)BW(-\omega^{-1}v) \in B(\mathcal{H})$$

is strongly continuous. Hence $v \mapsto s_B(v)$ is continuous. To prove (6), we note that Ψ_v goes weakly to zero as $|v| \to \infty$. Hence, for compact B, $\mathrm{s}_B(v) \to 0$ as $|v| \to \infty$.

To show (4), we use (8.52) and apply the trace to the identity

$$B = (2\pi)^{-d} \int_{\mathcal{Y}^{\#}} B^{\frac{1}{2}} P_v B^{\frac{1}{2}} \mathrm{d}v.$$

The interchange of trace and integral is justified by Fubini's theorem. To prove (5), we note that, if $B \in B^1(\mathcal{H})$, we can decompose it as $B = B_1 + \mathrm{i}B_2 - B_3 - \mathrm{i}B_4$, with $B_i \geq 0$, $B_i \in B^1(\mathcal{H})$. □

Definition 8.92 *If the map*

$$B(\mathcal{H}) \ni B \mapsto \mathrm{s}_B^{\mathrm{cv}} \in C(\mathcal{Y}^{\#}) \cap L^{\infty}(\mathcal{Y}^{\#})$$

is injective, then its inverse will be called the covariant quantization. *If b is a function on $\mathcal{Y}^{\#}$, its covariant quantization will be denoted* $\mathrm{Op}^{\mathrm{cv}}(b)$.

8.5.4 Connections between various quantizations

In this subsection we show how to pass between the covariant, Weyl–Wigner and contravariant quantizations. Note that there is a preferred direction: from contravariant to Weyl–Wigner, and then from Weyl–Wigner to covariant. Going back is less natural.

Let $w \in \mathcal{Y}^{\#}$. Let us compute various symbols of P_w:

$$\begin{aligned} s_{P_w}^{\mathrm{cv}}(v) &= |(\Psi_{w-v}|\Psi_0)|^2, \\ s_{P_w}(v) &= 2^d(\Psi_{w-v}|I\Psi_{w-v}), \\ s_{P_w}^{\mathrm{ct}}(v) &= (2\pi)^d\delta(v-w). \end{aligned}$$

The functions described in the following proposition will be used in formulas connecting various quantizations:

Proposition 8.93 *Set*

$$k_1(v) := (2\pi)^{-d} s_{P_0}(v) = \pi^{-d}(\Psi_0|I_v\Psi_0), \tag{8.64}$$

$$k_2(v) := (2\pi)^{-d} s_{P_0}^{\mathrm{cv}}(v) = (2\pi)^{-d}|(\Psi_v|\Psi_0)|^2. \tag{8.65}$$

Then k_2 is an even function in $C_\infty(\mathcal{Y}^{\#})$, $k_1 \in L^1(\mathcal{Y}^{\#}) \cap C_\infty(\mathcal{Y}^{\#})$ and

$$k_2(v) = \int k_1(w-v)k_1(w)\mathrm{d}w. \tag{8.66}$$

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Proof Assume first that $\Psi_0 \in \mathcal{H}^\infty$. (Recall that $\mathcal{H}^\infty$ is defined in Def. 8.28.) Then $s_{P_0} \in \mathcal{S}(\mathcal{Y}^\#)$ and using (8.49) we have

$$P_0 = \pi^{-d} \int s_{P_0}(w) I_w \mathrm{d}w$$

as a norm convergent integral. Next, by (8.51),

$$\begin{aligned} s_{P_0}(w - v) &= 2^d \mathrm{Tr}(I_{w-v} P_0) \\ &= 2^d \mathrm{Tr}(I_w P_v). \end{aligned}$$

Hence,

$$\begin{aligned} s^{\mathrm{cv}}_{P_0}(v) \; &= \; \mathrm{Tr}(P_0 P_v) = \pi^{-d} \mathrm{Tr} \int s_{P_0}(w) I_w P_v \mathrm{d}w \\ &= (2\pi)^{-d} \int s_{P_0}(w) s_{P_0}(w - v) \mathrm{d}w. \end{aligned}$$

If $\Psi_0 \in \mathcal{H}$, we choose a sequence (Ψ_n) of normalized vectors in $\mathcal{H}^\infty$, such that $\Psi_n \to \Psi_0$ when $n \to \infty$. Then $s_{P_n} \to s_{P_0}$ in $L^2(\mathcal{Y}^\#)$, and $s^{\mathrm{cv}}_{P_n} \to s^{\mathrm{cv}}_{P_0}$ in $C_\infty(\mathcal{Y}^\#)$. (8.66) holds for Ψ_n. By letting $n \to \infty$, it also holds for Ψ_0. □

Define the integral operator

$$K\Psi(v) := \int k_1(v - w) \Psi(w) \mathrm{d}w. \tag{8.67}$$

Then the identity (8.66) means that

$$K^* K \Psi(v) = \int k_2(v - w) \Psi(w) \mathrm{d}w,$$

where K^* is the adjoint w.r.t. the scalar product of $L^2(\mathcal{Y}^\#)$.

Theorem 8.94 *We have the following identities between various symbols of an operator B, valid for example if $s^{\mathrm{ct}}_B \in L^2(\mathcal{Y}^\#)$:*

$$\begin{aligned} s_B(v) &= \textstyle\int s^{\mathrm{ct}}_B(w) k_1(v - w) \mathrm{d}w, \quad &&\textit{or } s_B = K s^{\mathrm{ct}}_B, \\ s^{\mathrm{cv}}_B(v) &= \textstyle\int s_B(w) k_1(w - v) \mathrm{d}w, \quad &&\textit{or } s^{\mathrm{cv}}_B = K^* s_B, \\ s^{\mathrm{cv}}_B(v) &= \textstyle\int s^{\mathrm{ct}}_B(w) k_2(w - v) \mathrm{d}w, \quad &&\textit{or } s^{\mathrm{cv}}_B = K^* K s^{\mathrm{ct}}_B. \end{aligned}$$

8.5.5 Gaussian coherent vectors

Let us consider the Schrödinger representation on $L^2(\mathcal{X})$ and fix a Euclidean metric on $\mathcal{X}$. Consider the normalized Gaussian vector

$$\Psi_{(0,0)}(x) = \pi^{\frac{d}{4}} \mathrm{e}^{-\frac{1}{2}x^2}. \tag{8.68}$$

The corresponding coherent vectors are

$$\Psi_{(q,\eta)}(x) = \pi^{-\frac{d}{4}} \mathrm{e}^{\mathrm{i}\eta\cdot x - \frac{\mathrm{i}}{2} q\cdot\eta - \frac{1}{2}(x-q)^2}, \quad (q, \eta) \in \mathcal{X} \oplus \mathcal{X}^\#. \tag{8.69}$$

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In the literature, when one speaks about coherent states, one usually has in mind (8.69). They are also called *Gaussian or Glauber's coherent states*. We will say more about them in the next chapter, because they appear naturally in the context of the Fock representation; see Chap. 9.

The covariant, resp. contravariant quantization for Gaussian coherent states coincides with the so-called Wick, resp. anti-Wick quantization, which will be discussed in Sect. 9.4. The corresponding integral kernels k_1, k_2 introduced in (8.64) and (8.65), and the corresponding operators K and K^*K are

$$\begin{aligned} k_1(x,\xi) &= \pi^{-d}\mathrm{e}^{-x^2-\xi^2}, & K = K^* &= \mathrm{e}^{-\frac{1}{4}(D_x^2+D_\xi^2)}, \\ k_2(x,\xi) &= (2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}x^2-\frac{1}{2}\xi^2}, & K^*K &= \mathrm{e}^{-\frac{1}{2}(D_x^2+D_\xi^2)}. \end{aligned}$$

Thus in the Schrödinger representation one can distinguish five most natural quantizations. Their respective relations are nicely described by the following diagram, sometimes called the *Berezin diagram*:

$$\begin{tikzcd}
 & \begin{matrix}\text{anti-Wick}\\ \text{quantization}\end{matrix} \arrow[d, "\mathrm{e}^{-\frac{1}{4}(D_x^2+D_\xi^2)}"] & \\
\begin{matrix}D,x\text{-}\\ \text{quantization}\end{matrix} \arrow[r, "\mathrm{e}^{\frac{\mathrm{i}}{2}D_x\cdot D_\xi}"] & \begin{matrix}\text{Weyl–Wigner}\\ \text{quantization}\end{matrix} \arrow[r, "\mathrm{e}^{\frac{\mathrm{i}}{2}D_x\cdot D_\xi}"] \arrow[d, "\mathrm{e}^{-\frac{1}{4}(D_x^2+D_\xi^2)}"] & \begin{matrix}x,D\text{-}\\ \text{quantization}\end{matrix} \\
 & \begin{matrix}\text{Wick}\\ \text{quantization}\end{matrix} &
\end{tikzcd}$$

8.6 Notes

The relations

$$\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D} = \mathrm{e}^{-\mathrm{i}q\cdot\eta}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{e}^{\mathrm{i}\eta\cdot x},\ \eta, q \in \mathbb{R} \tag{8.70}$$

were first stated by Weyl (1931). The proof of the Stone–von Neumann theorem can be found in von Neumann (1931); see also Emch (1972) and Bratteli–Robinson (1996). The canonical commutation relations for systems with many degrees of freedom were used by Dirac (1927) to describe quantized electromagnetic field.

We sketched the early history of the Weyl–Wigner(–Moyal) quantization in the introduction, with basic references Weyl (1931), Wigner (1932b) and Moyal (1949). In pure mathematics it became well known quite late. It was recognized in the so-called microlocal analysis – a powerful approach to the study of partial

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differential equations; see especially Hörmander (1985). It is also very useful in closely related semi-classical analysis; see e.g. Robert (1987).

The fact that the Weyl–Wigner quantization of the delta function is proportional to the parity operator was discovered only in the 1970s by Grossman (1976).

The Weyl CCR algebra was studied by, among others, Manuceau (1968) and Slawny (1971). Thm. 8.64 comes from Slawny (1971); see also Bratteli–Robinson (1996).

The original and still the most common meaning of the term "coherent state" is what we call a "Gaussian coherent state". These were first studied by Schrödinger (1926). They were extensively applied in quantum optics by Glauber (1963), for which he was awarded the Nobel Prize. Glauber introduced the name "coherent state" and, together with Cahill, studied quantizations based on coherent states in Cahill–Glauber (1969).

Various forms of quantization involving a family of general coherent states, in particular the covariant and contravariant quantizations, were studied by Berezin (1966). For a discussion of quantization see also Berezin–Shubin (1991) and Folland (1989).

The concept of coherent states has been generalized even further to the context of a rather general Lie group with a distinguished subgroup by Perelomov (1972).

The name "FBI transformation" comes from Fourier–Bros–Iagolnitzer. The FBI transformation was used by Iagolnitzer (1975) to study microlocal properties of distributions.