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ON GROUP RINGS OF FINITE METABELIAN GROUPS

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Abstract

It is proved that a finite metabelian group G is determined by its group ring KG where the ring K satisfies the following conditions.
(*) K is an integral domain of characteristic 0 in which no prime dividing the order of G is invertible.
(**) Z/mZ is a homomorphic image of K where $m =$ exponent of G'.
It is also shown that all groups of order 2^n, $n \leqslant 7$ are determined by their integral group rings.

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The aim of this paper is to prove that a finite metabelian group G is determined by its group ring KG where the ring K satisfies the following conditions

(*) K is an integral domain of characteristic 0 in which no prime dividing the order of G is invertible

(**) Z/mZ is a homomorphic image of K where $m =$ exponent of G'

This theorem generalizes Whitcomb's result (Whitcomb (1968)) which states that a finite metabelian group is determined by its integral group ring and also a result of Sehgal (1970) who proved that a finite metabelian p-group is determined by its p-adic group ring. We present our theorem in a generalized form which covers (for the case $K = Z$) Obayashi's result (Obayashi (1970)) established by cohomological methods. We also prove that a 2-group which is abelian-by-dihedral of order 8 is determined by its integral group ring. The corresponding result for the case of a 2-group which is elementary abelian-by-dihedral of order 8 was proved by Obayashi (1970). Finally, we show that all groups of order 2^n, $n \leqslant 7$, are determined by their integral group rings.

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In what follows KG denotes a group ring of a group G over an associative ring K with 1, $I(K,G)$ denotes the augmentation ideal of KG, $KG=KH$ means that H is a normalized group basis of KG. We shall often write $I(G)$ instead of $I(K,G)$ when the precise situation will be clear from the content. If S is a subset of KG and Λ is an ideal of KG then $S+\Lambda=\{s+\Lambda \mid s\in S\}$. Finally, O_p (respectively $Z_{(p)}$) stands for the ring of p-adic integers (respectively p-integral rationals).

LEMMA 1. *Let G be an arbitrary group, K an arbitrary ring with* 1, *M and N arbitrary subgroups of G. Then the following equalities hold*

(1) $$I(M)\cdot I(N)\cap I(N)=I(M\cap N)\cdot I(N).$$

(2) $$I(G)\cdot I(N)\cap I(N)=I(N)^2.$$

(3) $$G\cap 1+I(G)\cdot I(N)=N\cap 1+I(N)^2.$$

PROOF. By taking $M=G$ we see that (1) $\Rightarrow$ (2). Since $G\cap 1+KG\cdot I(N)=N$ it follows that $G\cap 1+I(G)\cdot I(N)=N\cap 1+I(G)\cdot I(N)=N\cap 1+(I(G)\cdot I(N)\cap I(N))$ and therefore (2) $\Rightarrow$ (3). Let T be a set of all coset representatives of M with respect to $M\cap N$. If $m=tn$, $n\in M\cap N$, then for $n'\in N$ we have

$$(m-1)(n'-1)=(t-1)(n-1)(n'-1)+(t-1)(n'-1)+(n-1)(n'-1).$$

Since the first and the second summand belong to $(t-1)I(N)$ and since

$$(n-1)(n'-1)\in I(M\cap N)\cdot I(N),\quad (m-1)(n'-1)\in I(M\cap N)\cdot I(N)+(t-1)\cdot I(N)$$

then

$$I(M)\cdot I(N)=I(M\cap N)\cdot I(N)+\sum_{1\neq t\in T}(t-1)I(N).$$

Let

$$x=y+(t_1-1)[\alpha_{11}(n_1-1)+\dots+\alpha_{1s}(n_s-1)]+\dots+(t_k-1)[\alpha_{k1}(n_1-1) +\dots+\alpha_{ks}(n_s-1)],$$

where

$$y\in I(M\cap N)\cdot I(N),\quad t_j\in T,\ n_i\in N,\quad 1\leqslant i\leqslant s,\quad 1\leqslant j\leqslant k.$$

If $x\in I(N)$ then

$$z=\alpha_{11}t_1(n_1-1)+\dots+\alpha_{1s}t_s\quad(n_s-1)+\dots+\alpha_{ks}t_k(n_s-1)\in I(N)$$

and since all elements of N have coefficient 0 in z, $z=0$.

But $\{t_1(n_1-1),\dots,t_s(n_s-1)\}$ is a linearly independent set and therefore

$$\alpha_{11}=\dots=\alpha_{1s}=\dots=\alpha_{ss}=0.$$

Hence $x=y\in I(M\cap N)\cdot I(N)$, proving the lemma.

Let A be an abelian group of exponent n and let $K=Z/mZ$ where $m\equiv 0\pmod n$. As in the case $K=Z$ the formula

$$f\Big(\sum_{a\in A}(\alpha_a\cdot 1)(a-1)\Big)=\prod_{a\in A}a^{\alpha_a}(\alpha_a\in Z)$$

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defines a homomorphism of $I(A)$ onto A with kernel $I(A)^2$. From this follows: $A \cap 1+I(A)^2 = 1$ and $A \cong I(A)/I(A)^2$. Moreover, since

$$f\Big(\sum_{a\in A}(\alpha_a\cdot 1)(a-1)\Big) = f\Big(\prod_{a\in A} a^{\alpha_a}-1\Big)$$

the following congruence holds

$$\sum_{a\in A}(\alpha_a\cdot 1)(a-1) \equiv \prod_{a\in A} a^{\alpha_a}-1 \quad (\operatorname{mod} I(A)^2\ (\alpha_a\in Z)). \tag{4}$$

In general if G is a group and n is the exponent of G/G' then

$$G/G' \cong I(K,G)/I^2(K,G) \tag{5}$$

and

$$G \cap 1+I^2(K,G) = G', \tag{6}$$

where $K = Z/mZ$ and $m \equiv 0\ (\operatorname{mod} n)$.

LEMMA 2. *Let A be a subgroup G and let $K = Z/mZ$ where*

$$m \equiv 0\ (\operatorname{mod} n), \quad n = \text{exponent of } A/A'.$$

Then the following properties hold

$$KG\cdot I(A)/I(G)\cdot I(A) \cong A/A' \tag{7}$$

and

(8) *If A is abelian and $x \equiv g\ (\operatorname{mod} KG\cdot I(A))$ for some $g \in G$, then there exists a unique element $g_x = ga\ (a\in A)$ such that $x \equiv g_x\ (\operatorname{mod} I(G)\cdot I(A))$.*

PROOF. It follows from $KG = K+I(G)$ that $KG\cdot I(A) = I(A)+I(G)\cdot I(A)$ and the application of (2) and (5) yields

$$KG\cdot I(A)/I(G)\cdot I(A) \cong I(A)/I(A)\cap I(G)\cdot I(A) = I(A)/I(A)^2 \cong A/A',$$

proving (7). Finally, $x \equiv g\ (\operatorname{mod} KG\cdot I(A))$ implies $x \equiv g+t\ (\operatorname{mod} I(G)\cdot I(A)$ for some $t = \sum_{s\in A}(\alpha_s\cdot 1)(s-1) \in I(A)$.

Therefore $x \equiv g+(a-1) = (1-g)(a-1)+ga = g_x\ (\operatorname{mod} I(G)\cdot I(A))$ where

$$a = \prod_{s\in A} s^{\alpha_s}.$$

Since $G \cap 1+I(G)\cdot I(A) = A \cap 1+I(A)^2 = 1$ the element g_x is unique, proving the lemma.

Let $\alpha \to \bar{\alpha}$ be a ring homomorphism from K onto $\bar{K}$ with kernel Λ and let G be an arbitrary group. Then the mapping ϕ: $KG \to \bar{K}G$ defined by

$$\phi\Big(\sum_g \alpha_g g\Big) = \sum_g \bar{\alpha}_g g$$

determines the epimorphism of rings KG and $\bar{K}G$ with $\operatorname{Ker}\phi = \Lambda G$.

LEMMA 3. *Let G be a group and let K be a ring. Then $KG = KH$ implies $\bar{K}G = \bar{K}\phi(H)$ and $\phi(H) \cong H$. Therefore if A and N (respectively B and T) are subgroups of G (respectively of H) such that*

$$KG \cdot I(A) = KG \cdot I(B) \quad \textit{and} \quad N + KG \cdot I(A) = T + KG \cdot I(A)$$

then

$$\bar{K}G \cdot I(\bar{K}, A) = \bar{K}G \cdot I(\bar{K}, \phi(B)) \quad \textit{and} \quad N + \bar{K}G \cdot I(\bar{K}, A) = \phi(T) + \bar{K}G \cdot I(\bar{K}, A).$$

PROOF. All we have to do is to prove that there exists an isomorphism λ of group rings $\bar{K}H$ and $\bar{K}G$ such that $\lambda(H) = \phi(H)$. Clearly, $KG = KH$ implies $\Lambda G = \Lambda H$. Consider the mappings

$$\bar{K}H \xrightarrow{\lambda_1} KH/\Lambda H = KG/\Lambda G \xrightarrow{\lambda_2} \bar{K}G$$

where

$$\lambda_1\left(\sum_{h \in H} \bar{\alpha}_h \cdot h\right) = \sum_{h \in H} \alpha_h \cdot h + \Lambda H \quad \text{and} \quad \lambda_2\left(\sum \alpha_g g + \Lambda G\right) = \sum_{g \in G} \bar{\alpha}_g g.$$

Then λ_1 and λ_2 are ring isomorphisms and therefore $\lambda_2 \lambda_1$ is also a ring isomorphism. It is easy to see that $\lambda_2 \lambda_1$ is also a $\bar{K}$-module isomorphism and that $(\lambda_2 \lambda_1)(h) = \phi(h)$ for any $h \in H$, proving the lemma.

Let R be the ring of algebraic integers of a number field and let $u = \sum_g \alpha_g g$ be a unit of finite order in RG. It is well known (Berman (1955)) that if $\alpha_1 \neq 0$ then $u = \alpha_1 \cdot 1$. From this follows that every central unit of finite order in RG is trivial. The following extension of this result belongs to Saksonov (1971).

LEMMA 4. *Let K be an integral domain of characteristic 0 in which no prime dividing the order of G is invertible. If $u^m = 1$ where $u = \sum_g \alpha_g g \in KG$ and if $\alpha_1 \neq 0$ then $u = \alpha_1 \cdot 1$. In particular, all central units of finite order in KG are trivial.*

PROOF. Consider the regular representation of the group ring $\Phi(\varepsilon) G$, where Φ is the quotient field of K and ε a primitive mth root of unity. Then

$$\mathrm{tr}(u) = \alpha_1 \cdot |G| = \varepsilon_1 + \dots \varepsilon_{|G|},$$

where $\varepsilon_i^m = 1$ and $\varepsilon_i \in \Phi(\varepsilon)$, $i = 1, 2, \dots, |G|$. Therefore $\alpha_1 = |G|^{-1}(\varepsilon_1 + \dots \varepsilon_{|G|})$. All we have to do is to prove that α_1 is an algebraic integer. By looking at the $\mathrm{tr}(u^r)$ where $(r, m) = 1$ we see that the set $\{\beta_1 = \alpha_1, \beta_2, \dots, \beta_t\}$ of all Q-conjugates to α_1, belongs to K and therefore $Z[\beta_1, \dots, \beta_t] \leqslant K$.

Suppose that α_1 is not an algebraic integer. Then there exists an elementary symmetric function f of t variables such that $f(\beta_1, \dots, \beta_t)$ is not a rational integer. On the other hand, for some $l \in Z$ and some $k, 1 \leqslant k \leqslant t$, $f(\beta_1, \dots, \beta_t) = |G|^{-k} l$ and therefore $f(\beta_1, \dots, \beta_t) = (a/b) \in Z[\beta_1, \dots, \beta_t]$ where $a, b \in Z$, $(a, b) = 1$, $b > 1$

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and every prime divisor p of b also divides $|G|$. This shows that $(a/p) \in Z[\beta_1, \dots, \beta_t]$. Since $(a,p) = 1$ there exist $c, d \in Z$ such that $ac+dp = 1$ whence

$$\frac{1}{p} = \frac{ac+dp}{p} = \frac{a}{p} \cdot c + d \in Z[\beta_1, \dots, \beta_t] \leqslant K,$$

which is a contradiction. This proves the lemma.

COROLLARY. *Let $N \Delta G$ and let $\pi: KG \to K(G/N)$ be the canonical homomorphism where K is as in Lemma 4. If $KG = KH$ then*

$$K(G/N) = K\pi(H) \quad \textit{and} \quad KG \cdot I(N) = KH \cdot I(N^*),$$

where $N^* = H \cap 1 + KG \cdot I(N)$.

PROOF. It suffices to show that $\pi(H)$ is a group basis for $K(G/N)$. Indeed, in this case π can be regarded as the extension of the epimorphism $H \to \pi(H)$ by K-linearity, hence $\operatorname{Ker} \pi = KG \cdot I(N) = KH \cdot I(N^*)$.

Let $\sum_{x \in H} \alpha_x \cdot x = 0 (\alpha_x \in K)$ and let $\alpha_h \neq 0$ for some $h \in \pi(H)$. Then

$$\alpha_h \cdot h = - \sum_{x \neq h} \alpha_x \cdot x$$

and therefore there exists $y \neq h$ such that the coefficient of 1 in yh^{-1} is nonzero. But in this case $yh^{-1} = \alpha \cdot 1$ $(\alpha \in K)$ and since H is normalized, $y = h$—contradiction. This proves the corollary.

THEOREM. *Let A be an abelian normal subgroup of a finite group G and let $KG = KH$ where the ring K satisfies (*) and (**) for $n =$ exponent of A. If N/A (respectively M/A^* where $A^* = H \cap 1 + KG \cdot I(A)$) is the centre of G/A (respectively H/A^*) then there exists an isomorphism of M onto N carrying A^* onto A.*

PROOF. We shall prove even a more general result, namely if N and M are subgroups of G and H respectively such that

$$N \geqslant A, \quad M \geqslant A^* \quad \text{and} \quad N + KG \cdot I(A) = M + KG \cdot I(A)$$

then there exists an isomorphism of M onto N carrying A^* onto A. It follows from the corollary of Lemma 4 that $KG \cdot I(A) = KG \cdot I(A^*)$ and hence $|A| = |A^*|$.

By applying Lemma 3 we may assume that $K = Z/mZ$ where m is a multiple of both the exponent of A and the exponent of A^*. Multiplying both sides of the equality $KG \cdot I(A) = KG \cdot I(A^*)$ by $I(G) = I(H)$ we obtain $I(G) \cdot I(A) = I(H) \cdot I(A^*)$. It follows from (7) that $A \cong A^*/(A^*)'$. Since $|A| = |A^*|$ then $A \cong A^*$ and therefore the application of (3) and (6) yields $H \cap 1 + I(H) \cdot I(A^*) = 1$. Let

$$\pi: KG \to K(G/A)$$

be the canonical homomorphism. Since $N+KG\cdot I(A)=M+KG\cdot I(A)$ then

$$\pi(N)=\pi(M)$$

and therefore $|N/A|=|M/A^*|$. Hence $|M|=|N|$. The same equality also implies that for every $h\in M$ there exists $g\in N$ such that $h\equiv g \pmod{KG\cdot I(A)}$. By (8) there exists a unique $g_h\in N$ such that $h\equiv g_h \pmod{I(G)\cdot I(A)}$.

Therefore the mapping $f\colon h\to g_h$ defines a homomorphism of M into N. Since $|M|=|N|$ and since $\operatorname{Ker} f\leqslant H\cap 1+I(H)\cdot I(A^*)=1$, f is actually an isomorphism of M onto N. Let $b\in A^*$. Then $b-1\in\operatorname{Ker}\pi=KG\cdot I(A)=I(A)+I(G)\cdot I(A)$ and it follows from (4) that $b-1\equiv a-1 \pmod{I(G)\cdot I(A)}$ for some $a\in A$. Hence $b\equiv a \pmod{I(G)\cdot I(A)}$ and $f(b)=a$. Thus it remains only to prove that N and M satisfy $N+KG\cdot I(A)=M+KG\cdot I(A)$. It follows from the corollary of Lemma 4 that $K(G/A)=K\pi(H)$. Since $\pi(N)$ (respectively $\pi(M)$) is the centre of G/A (respectively H/A^*) the application of Lemma 4 yields $\pi(N)=\pi(M)$. Hence $N+KG\cdot I(A)=M+KG\cdot I(A)$, proving the theorem.

Corollary 1. *Let G be a finite metabelian group and let the ring K satisfy* (*) *and* (**). *Then G is determined by its group ring KG.*

Proof. Take $A=G'$. Then $N=G$ and $M=H$. Now apply the theorem.

The following proposition was suggested to me by Dr K. R. Pearson.

Corollary 2. *Let G be a finite metabelian group and let $K=S^{-1}Z$ be the ring of fractions of Z with respect to S, where $S=\{a\in Z\mid (a,|G|)=1\}$. Then the group ring KG determines G.*

Proof. All we have to do is to check that K satisfies (*) and (**). It is obvious that K satisfies (*) and that the mapping $S^{-1}Z\to Z/mZ$ defined by $(a/b)\to\bar{a}(\bar{b})^{-1}$ where $m=$ exponent of G' and $\bar{a}=a+mZ$ is a ring epimorphism, proving the corollary.

Berman and Rossa (1966) and Whitcomb (1968) gave an example of two group bases in ZD_4 which are not conjugate in $U(ZD_4)$ but are conjugate in $U(0_2\,D_4)$. (In fact they are conjugate in $U(Z_{(2)}\,D_4)$. On the other hand, Weller (1972) proved that there are only two nonconjugate classes of group bases in $U(ZD_4)$.

Corollary 3. *Let G be a 2-group with abelian normal subgroup A such that $G/A\cong D_4$. Then G is determined by its integral group ring.*

Proof. Let $ZG=ZH$ and let $\pi\colon ZG\to Z(G/A)$ be the canonical homomorphism. Then $Z\pi(G)=Z\pi(H)$, $0_2\,\pi(G)=0_2\,\pi(H)$ and there exists a unit $u\in 0_2\,G$ such that

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$u^{-1}\pi(H)u = \pi(G)$. Since $0_2 G$ is a local ring there exists a unit $t \in 0_2 G$ such that $\pi(t) = u$ and therefore $\pi(t^{-1}Ht) = \pi(G)$. Thus if $\tilde{H} = t^{-1}Ht$ then $0_2 G = 0_2 \tilde{H}$ and $G + 0_2 G \cdot I(A) = \tilde{H} + 0_2 G \cdot I(A)$. It follows from the proof of the theorem that in this case $G \cong \tilde{H}$, proving the corollary.

COROLLARY 4. *Let* $|G| = 2^n$, $n \leqslant 7$. *Then the group G is determined by its integral group ring.*

PROOF. Every group of order 2^n $n \leqslant 6$, is metabelian and group of order 2^7 has a normal abelian subgroup A of index 8 (Miller, Blichfeldt and Dickson (1961)). Suppose that G is not metabelian. It follows from Berman (1955) that we can restrict ourself to the case when $G/A \cong D_4$. Now apply Corollary 3.

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