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A GENERAL AND SHARPENED FORM OF OPIAL'S INEQUALITY

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1. **Introduction.** Z. Opial [11] proved in 1960 the following theorem:

THEOREM 1. *If u is a continuously differentiable function on* $[0, b]$, *and if* $u(0)=u(b)=0$ *and* $u(x)>0$ *for* $x \in (0, b)$, *then*

$$\int_0^b |u(x)u'(x)|\,dx \leq \frac{b}{4}\int_0^b [u'(x)]^2\,dx, \tag{1}$$

where the constant $b/4$ is the best possible.

Equality holds in (1) *if and only if*

$$u(x) = cx, \quad for \quad 0 \leq x \leq b/2 \quad and$$
$$u(x) = c(b-x), \quad for \quad b/2 \leq x \leq b,$$

where c is a constant.

In a note published at the same time, C. Olech [10] showed that (1) is valid for any function $u(x)$ which is absolutely continuous on $[0, b]$, and satisfies the boundary conditions $u(0)=u(b)=0$.

We also note that in order to prove (1) it suffices to prove the following (see also [10])

THEOREM 2. *If u is an absolutely continuous function on* $[0, b]$ *and if* $u(0)=0$, *then*

$$\int_0^b |u(x)u'(x)|\,dx \leq \frac{b}{2}\int_0^b [u'(x)]^2\,dx, \tag{2}$$

where $b/2$ is the best possible constant.

Equality holds in (2) *if and only if $u(x)=cx$, where c is a constant.*

In 1962 P. R. Beesack published a paper [2] which gives a different proof of Opial's inequality, and which shows that (2) is contained in the following

THEOREM 3. *If u is an absolutely continuous function on* $[0, b]$ *and if* $u(0)=0$, *then*

$$\int_0^b |u(x)u'(x)|\,dx + \frac{b}{2}\int_0^b \frac{1}{x^2}\left\{2\int_0^x |u(t)u'(t)|\,dt - [u(x)]^2\right\}dx \leq \frac{b}{2}\int_0^b [u'(x)]^2\,dx. \tag{3}$$

Equality in (3) *holds if and only if $u(x)=cx$, where c is a constant.*

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Since

$$g_1(x) = 2\int_0^x |u(t)u'(t)|\,dt - [u(x)]^2 \geq 0,$$

(3) is an improvement of (2), though not explicitly pointed out by the author.

The result of Z. Opial has led to numerous articles; we shall discuss some of these in section 3 (see also [3] or [9] for more complete background). The purpose of this note is to prove Theorem 4, which is a generalization of Theorem 3. The method of the proof of Theorem 4 was first used by Benson [4], and was modified in a recent paper [12].

2. Preliminary lemmas.

LEMMA 1. *If u is absolutely continuous on $[a, b]$ and if $u(a)=0$, then, for all $p>-1$,*

$$(4) \qquad g_2(x) = (p+1)\int_a^x |u|^p |u'|\,dt - |u(x)|^{p+1} \geq 0, \qquad (a \leq x \leq b).$$

Equality is attained in (4), if and only if u' does not change sign on $[a, b]$.

The proof of this lemma is left to the reader.

Before giving our second lemma we first state two elementary algebraic inequalities [1, or 7, TH 41] to which we shall apply Benson's method:

$$(5) \qquad s^{p+1} + pt^{p+1} - (p+1)st^p \geq 0 \qquad (p > 0, \text{ or } p < -1);$$

$$(6) \qquad s^{p+1} + pt^{p+1} - (p+1)st^p \leq 0 \qquad (-1 < p < 0).$$

Here, s and t are nonnegative (positive if $p<-1$), and in both cases strict inequality holds unless $s=t$. (We also note that when $p=0$ or $p=-1$, the left sides of both (5) and (6) become identically zero for all s and t.)

The associated integral inequalities are stated in the following lemma. The special case $p+1=2n$ of (7) below gives the basic integral inequality used by Benson in [4].

LEMMA 2. *Let $v(x)$ be absolutely continuous on $[\alpha, \beta]$ with $v'(x)\geq 0$* a.e. *Also, suppose that $Q(x)$ is nonnegative* a.e. *and measurable on $[\alpha, \beta]$, and $G(v, x)$ is continuously differentiable for $x \in [\alpha, \beta]$ and v in the range of the function $v(x)$, with $G_v(v, x)\geq 0$* (or $G_v(v, x)>0$ in case $p<0$). Then, if the integrals exist,*

$$(7) \qquad \int_\alpha^\beta [Qv'^{p+1} + p(G_v)^{(p+1)/p}Q^{-1/p} + (p+1)G_x]\,dx \geq (p+1)\{G(v(\beta), \beta) - G(v(\alpha), \alpha)\} \qquad (p > 0 \text{ or } p < -1),$$

$$(8) \qquad \int_\alpha^\beta [Qv'^{p+1} + p(G_v)^{(p+1)/p}Q^{-1/p} + (p+1)G_x]\,dx \leq (p+1)\{G(v(\beta), \beta) - G(v(\alpha), \alpha)\} \qquad (-1 < p < 0),$$

* We note that in case $p+1=2n$ with n a positive integer, the restrictions $v'(x)\geq 0$ a.e., and $G_v(v, x)\geq 0$ may be removed.

where $G_v=(\partial/\partial v)G(v,x)$, $G_x=(\partial/\partial x)G(v,x)$. *Equality in both* (7) *and* (8) *holds if and only if the differential equation*

$$v' = (G_v/Q)^{1/p} \tag{9}$$

is satisfied almost everywhere.

Proof. (The following proof illustrates how Benson's method works.) By taking $s=v'Q^{1/(p+1)}$, $t=(G_v)^{1/p}Q^{-1/[p(p+1)]}$ in (5), we have, almost everywhere,

$$Qv'^{p+1}+p(G_v)^{(p+1)/p}Q^{-1/p}\geq (p+1)v'G_v.$$

That is,

$$Qv'^{p+1}+p(G_v)^{(p+1)/p}Q^{-1/p}+(p+1)G_x\geq (p+1)\frac{d}{dx}G(v,x),$$

proving (7) by integrating both sides of the above inequality from α to β.

The proof of (8) follows from the above argument, but using (6) instead of (5). The proof of (9) follows at once from the remarks after (5) and (6).

3. The main theorem.

THEOREM 4. *Let u be an absolutely continuous function on* $[a, b]$, *and* $u(a)=0$. *If* $p>0$, *and* $\int_a^b |u'|^{p+1}\,dx<\infty$, *then*

$$\int_a^b |u|^p\,|u'|\,dx+\frac{p(b-a)^p}{p+1}\int_a^b\frac{g_2(x)\,dx}{(x-a)^{p+1}}\leq\frac{(b-a)^p}{p+1}\int_a^b |u'|^{p+1}\,dx, \tag{10}$$

where $g_2(x)$ is defined in (4). *If either* $p<-1$ *and both* $\int_a^b |u|^p\,|u'|\,dx<\infty$, *and* $\int_a^b |u'|^{p+1}\,dx<\infty$, *or* $-1<p<0$ *and* $\int_a^b |u|^p\,|u'|\,dx<\infty$, *the reverse inequality holds.*

For $p>0$, *equality holds in* (10) *if and only if* $u(x)=c(x-a)$ *for some constant c; for* $p<-1$ *equality never holds; for* $-1<p<0$, *equality holds if and only if* $u(x)=c(x-a)$ *for some constant* $c\neq 0$.

Proof. First we note that with u defined above we have (see Remark 2 below), for $p>0$,

$$\int_a^b |u|^p\,|u'|\,dx\leq\frac{(b-a)^p}{p+1}\int_a^b |u'|^{p+1}\,dx<\infty. \tag{11}$$

Now, from (11), with b replaced by x, it follows that

$$\lim_{x\to a+}\frac{p+1}{(x-a)^p}\int_a^x |u|^p\,|u'|\,dt=0,\qquad (p>0). \tag{12}$$

Now, let $v(x)=\int_a^x |u|^p\,|u'|\,dt$, which is well defined by (11), $Q^{-1}=|u|^{p(p+1)}$ and $G=v(x-a)^{-p}$. Then from (7) with $[\alpha,\beta]$ replaced by $[\alpha, b]$ we obtain, for $p>0$ and $a<\alpha<b$,

$$\frac{p+1}{(b-a)^p}\int_a^b |u|^p\,|u'|\,dx-\frac{p+1}{(\alpha-a)^p}\int_a^\alpha |u|^p\,|u'|\,dx$$
$$\leq\int_\alpha^b |u'|^{p+1}\,dx+p\int_\alpha^b\left\{\frac{1}{(x-a)^{p+1}}\left(|u|^{p+1}-(p+1)\int_a^x |u|^p\,|u'|\,dt\right)\right\}dx,$$

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or

$$(13)\quad \frac{p+1}{(b-a)^p}\int_a^b |u|^p\,|u'|\,dx+p\int_\alpha^b \frac{g_2(x)\,dx}{(x-a)^{p+1}} \le \frac{p+1}{(\alpha-a)^p}\int_a^\alpha |u|^p\,|u'|\,dx+\int_\alpha^b |u'|^{p+1}\,dx.$$

Now, on taking limits as $\alpha\to a+$ on both sides of the above inequality one obtains (10) by using (12), on noting that $g_2(x)$ is nonnegative by Lemma 1, so that both integrals on the left side of (10) exist (finite).

For $p<-1$, the inequality (13) is still valid, and hence we have

$$(14)\quad \frac{-(p+1)}{(\alpha-a)^p}\int_a^\alpha |u|^p\,|u'|\,dx+p\int_\alpha^b \frac{g_2(x)\,dx}{(x-a)^{p+1}} \le \frac{-(p+1)}{(b-a)^p}\int_a^b |u|^p\,|u'|\,dx+\int_\alpha^b |u'|^{p+1}\,dx, \qquad (p<-1).$$

Now, since $p<-1$ it follows from the definition (4) of g_2 that $pg_2(x)\ge 0$, hence both limits on the left side of (14) exist (finite) as $\alpha\to a+$, and the first limit is zero since $p<0$, proving the case $p<-1$ without the equality condition.

The proof of the case $-1<p<0$ is essentially the same as above except that instead of (7) we now use (8).

The proof of the equality condition begins by employing (9) in (13). The details are left to the reader, and are similar to those used in [12].

REMARK 1. From (10) we may deduce (3) on setting $a=0$, $p=1$ in (10). Thus, Theorem 3 is a corollary of Theorem 4.

REMARK 2. We note that (11) with $a=0$ is a special case of an inequality first proved by G. S. Yang [14] who proved that if $p\geqq 0$, $q\geqq 1$, then

$$\int_0^b |u|^p\,|u'|^q\,dx \le \frac{qb^p}{p+q}\int_0^b |u'|^{p+q}\,dx$$

for any u which is absolutely continuous on $[0, b]$ with $u(0)=0$. Yang stated his result only for $p\ge 1$, $q\ge 1$, but the proof also holds for $p\ge 0$, $q\ge 1$, as noticed by P. R. Beesack. The result is sharp only for $q=1$. An earlier paper by Hua [8] proved (11) with p a positive integer. The inequality (11) is included in a (later) generalization of Calvert [6], and about the same time a short direct proof of the latter was given by Wong [13]. In addition, the inequality (11) is also contained in the paper [5] of Boyd and Wong.

Finally, we note that the inequality (10), with $p>0$, can further be generalized to

$$(15)\quad \int_a^b s\,|u'|^{p+1}\,dx \ge (p+1)(b-a)^{-p}\int_a^b s\,|u|^p\,|u'|\,dx + p\int_a^b \frac{s(x)}{(x-a)^{p+1}}\left\{\frac{p+1}{s(x)}\int_a^x s\,|u|^p\,|u'|\,dt-|u(x)|^{p+1}\right\}dx,$$

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valid for any u such that $u(x)=\int_a^b u'\,dt$ and $\int_a^x s\,|u'|^{p+1}\,dx<\infty$. Here s is positive and nonincreasing on (a, b) with $-\infty<a<b<\infty$.

The proof of (15) may be completed by setting $v=\int_a^x s\,|u|^p\,|u'|\,dt$, $Q^{-1}=|u|^{p(p+1)}s^p$, and $G=v(x-a)^{-p}$ in (7).

REMARK 3. By setting $s\equiv 1$ in (15) we get (10). The case of equality in (15) may also be discussed by using (9).

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