# 24
# Chern–Simons theories

In $2+1$ dimensions it is possible to construct actions which include a 'topological' interaction called the Chern–Simons term. The Chern–Simons term takes the form

$$\begin{aligned} S_{\text{CS}} &= \int \frac{(\mathrm{d}x)}{\sqrt{g}} \; \frac{1}{2}\mu\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \\ &= \int \mathrm{d}^{n+1}x \; \frac{1}{2}\mu\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \end{aligned} \tag{24.1}$$

for Abelian theories, and the extended form,

$$S_{\text{CS-NA}} = \int \frac{\mathrm{d}V_t}{\sqrt{g}} \frac{kg^2}{2\hbar^2 C_2(G_{\text{adj}})} \epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda - \mathrm{i}\frac{2}{3}A_\mu A_\nu A_\lambda\right), \tag{24.2}$$

in the non-Abelian case, where Hermitian generators are used. This action is real, as may be seen by applying the Lie algebra relation in eqn. (23.3). The effect of the Chern–Simons term on the dynamics of a field theory depends on whether the Maxwell or Yang–Mills term is also present. Since the Chern–Simons term is purely linear in all derivatives, and there are no additional constraints, as in the Dirac equation, it does not carry any independent dynamics of its own.

In the absence of dynamics from a Maxwell or Yang–Mills-like contribution to the action, the effect of this term is to induce a duality of variables, i.e. an equivalence relation between $F_\mu$ and $J^\mu$. Coupled together with a Maxwell or Yang–Mills term, the Chern–Simons term endows the vector potential with a gauge-invariant mass [35, 36, 64, 110].

An unusual but important feature of the Chern–Simons action is that it is independent of the spacetime metric. Since the Levi-Cevita tensor transforms like a tensor density, a factor of $\sqrt{g}$ is therefore required to cancel the one



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already in the volume element. This has obvious implications for the usefulness of the variational definition of energy and momentum, by $T_{\mu\nu}$.

## 24.1 Parity- and time reversal invariance

The Chern–Simons terms violates parity- and time reversal invariance, both of which are defined for $2+1$ dimensions as the reflection in the axis of a single coordinate [64]. This is clearly not a fundamental property of nature. It is therefore only expected to play a role in physical systems where such a breakdown of parity invariance is present by virtue of special physical conditions. There are several such situations. Ferromagnetic states of spin fields, in the Hall effect, strong magnetic fields and vortices are examples [54].

The presence of a Chern–Simons term in the action of a field theory would lead to a rotation of the plane of polarization of radiation passing through a two-dimensional system, as in the Faraday effect (see sections A.6.1 and 7.3.3 and refs. [14, 16]). In ref. [24], the authors use the formalism of parity-violating terms to set limits on parity-violation from astronomical observations of distant galaxies. Spin polarized systems can be made into junctions, where Chern–Simons coefficients can appear with variable strength and sign [11, 14].

## 24.2 Gauge invariance

The transformation of the Chern–Simons action under gauge transformations, with its independence of the metric tensor, is what leads to its being referred to as a topological term. Consider the transformation of the non-Abelian action under a gauge transformation

$$A_\mu \to U A_\mu U^{-1} - \mathrm{i}\frac{\hbar}{g}(\partial_\mu U)U^{-1}; \tag{24.3}$$

it transforms to

$$\begin{aligned} S \to\ & S + \int (\mathrm{d}x)\,(\partial_\mu V^\mu) \\ &+ \frac{k}{6C_2(G_{\mathrm{adj}})}\varepsilon^{\mu\nu\lambda}\,\mathrm{Tr}\int (\mathrm{d}x)\,\left[U(\partial_\mu U^{-1})U(\partial_\nu U^{-1})U(\partial_\lambda U^{-1})\right]. \end{aligned} \tag{24.4}$$

The second term in the transformed expression is a total derivative and therefore vanishes, provided $U(\infty) = U(0)$: for instance, if $U \to 1$ in both cases (this effectively compactifies the spacetime to a sphere). The remaining term is:

$$\begin{aligned} \delta S &= \frac{\hbar k}{6C_2(G_{\mathrm{adj}})}\varepsilon^{\mu\nu\lambda}\,\mathrm{Tr}\int (\mathrm{d}x)\,\left[U(\partial_\mu U^{-1})U(\partial_\nu U^{-1})U(\partial_\lambda U^{-1})\right] \\ &= 8\hbar\pi^2 k\, W(U), \end{aligned} \tag{24.5}$$

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where $W(U)$ is the winding number of the mapping of the spacetime into the group, which is determined by the cohomology of the spacetime manifold. It takes integer values $n$. Clearly the action is not invariant under large gauge transformations. However, if $k$ is quantized, such that $4\pi k$ is an integer, the action only changes by an integral multiple of $2\pi$, leaving the phase $\exp(\mathrm{i}S/\hbar + 2\pi \mathrm{i}\, n)$ invariant. This quantization condition has been discussed in detail by a number of authors [35, 36, 39, 40, 43].

## 24.3 Abelian pure Chern–Simons theory

The Abelian Chern–Simons theory is relatively simple and has been used mainly in connection with studies of fractional statistics and the quantum Hall effect, where it gives rise to 'anyons' [4].

### 24.3.1 Field equations and continuity

Pure Chern–Simons theory is described by the Chern–Simons action together with a gauged matter action. In the literature, Chern–Simons theory is usually analysed by coupling it only to some unspecified gauge-invariant source:

$$S = \int \left( -\frac{1}{2}\mu \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + J^\mu A_\mu \right). \tag{24.6}$$

The variation of the action is given by

$$\delta S = \int (\mathrm{d}x)\{-\mu \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu A_\lambda + J^\mu \delta A_\mu\} - \int \mathrm{d}\sigma_\nu \frac{1}{2}\mu A_\mu \delta A_\lambda, \tag{24.7}$$

implying that the field equations are

$$\frac{1}{2}\mu \epsilon^{\mu\nu\lambda} F_{\nu\lambda} = J^\mu, \tag{24.8}$$

with associated boundary (continuity) conditions

$$\Delta\left(\frac{1}{2}\mu \epsilon_{\mu\sigma\lambda} A_\mu\right) = 0, \tag{24.9}$$

where the boundary of interest points in the direction of $x^\sigma$. Notice that, whereas the field equations are gauge-invariant, the boundary conditions are not. The physical interpretation of this result requires a specific context.

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## 24.4 Maxwell–Chern–Simons theory

### 24.4.1 Field equations and continuity

In the literature, Chern–Simons theory is usually analysed by coupling it only to some unspecified gauge-invariant current:

$$S = \int \left( \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \mu \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + J^\mu A_\mu \right). \tag{24.10}$$

The same warnings about the generality of this notation apply as for pure Chern–Simons theory. The field equations are now given by

$$\frac{1}{\mu_0} \partial^\mu F_{\mu\nu} + \frac{1}{2} \mu \epsilon^{\mu\nu\lambda} F_{\nu\lambda} = J^\mu, \tag{24.11}$$

with associated boundary conditions

$$\Delta \left( -\mu_0^{-1} F_{\sigma\lambda} + \frac{1}{2} \mu \epsilon_{\mu\sigma\lambda} A^\mu \right) = 0. \tag{24.12}$$

### 24.4.2 Topological mass

To see that the derivative terms of the Chern–Simons action lead to a gauge-invariant massive mode, one may perform a diagonalization to the eigenbasis of the action operator:

$$\begin{aligned} S &= \frac{1}{2} \int (\mathrm{d}x)\ A^\mu (-\Box\, g^\mu_{\ \nu} + \mu \epsilon^{\mu\ \lambda}_{\ \nu} \partial_\lambda) \\ &= \int (\mathrm{d}x) A_\mu\, \mathcal{O}^\mu_{\ \nu}\, A^\nu. \end{aligned} \tag{24.13}$$

In a flat space, Cartesian coordinate basis, where all derivatives commute, this is seen most easily by writing the components in matrix form:

$$\mathcal{O}^\mu_{\ \nu} = \begin{pmatrix} -\Box & \mu\partial_2 & -\mu\partial_1 \\ -\mu\partial^2 & -\Box & \mu\partial_0 \\ \mu\partial^1 & \mu\partial^0 & -\Box \end{pmatrix}. \tag{24.14}$$

The determinant of this basis-independent operator is the product of its eigenvalues, which is the product of dispersion constraints. Noting that $-\Box = -\partial_0^2 + \partial_1^2 + \partial_2^2$, it is straightforward to show that

$$\det \mathcal{O} = (-\Box\,)^2 (-\Box\, + \mu^2), \tag{24.15}$$

showing that the dispersion of the Maxwell–Chern–Simons field contains two massless modes and one mode of mass $\mu^2$. This massive theory has been studied in refs. [35, 36, 64, 110].

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### 24.4.3 Energy–momentum tensors

In Chern–Simons theory, the energy–momentum tensors $\theta_{\mu\nu}$ and $T_{\mu\nu}$ do not agree. The reason for this is that the Chern–Simons action is independent of the metric tensor (it involves only anti-symmetric symbols), thus the variational definition of $T_{\mu\nu}$ inevitably leads to a zero value. If we use eqn. (11.68) and assume the gauge-invariant variation in eqn. (4.81), we obtain the following contributions to $\theta_{\mu\nu}$ from the action in eqn. (24.1),

$$\theta_{\mu\nu} = \frac{1}{4}\mu g_{\mu\nu}\epsilon^{\rho\sigma\lambda}A_\rho F_{\sigma\lambda} - \frac{1}{2}\mu\epsilon^{\rho\ \sigma}_{\ \mu}A_\rho F^{\nu}_{\ \sigma}. \tag{24.16}$$

The fact that these two tensors do not agree can be attributed to the failure of the variational definition of $T_{\mu\nu}$ in eqn. (11.79). Since the Chern–Simons term is independent of the spacetime metric, it cannot be used as a generator for the conformal symmetry.

The contribution in eqn. (24.16) is not symmetrical but, in using the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}$, it is seen to be gauge-invariant, provided the Chern–Simons coefficient is a constant [11, 12, 14].

## 24.5 Euclidean formulation

In its Wick-rotated, Hermitian form, the Chern–Simons action acquires a factor of $\mathrm{i} = \sqrt{-1}$, unlike most other action terms, since the Levi-Cevita tensor does not transform under Wick rotation. It has the Abelian form

$$S_{\mathrm{CS-E}} = \mathrm{i}\int \frac{(\mathrm{d}x)}{\sqrt{g}}\frac{1}{2}\mu\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \tag{24.17}$$

and the non-Abelian form, for Hermitian generators

$$S_{\mathrm{CS-NA-E}} = \mathrm{i}\int \frac{(\mathrm{d}x)}{\sqrt{g}}\frac{kg^2}{2C_2(G_{\mathrm{adj}})}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda - \mathrm{i}\frac{2}{3}A_\mu A_\nu A_\lambda\right). \tag{24.18}$$