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Testing Bi-orderability of Knot Groups

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Abstract. We investigate the bi-orderability of two-bridge knot groups and the groups of knots with 12 or fewer crossings by applying recent theorems of Chiswell, Glass and Wilson. Amongst all knots with 12 or fewer crossings (of which there are 2977), previous theorems were only able to determine bi-orderability of 499 of the corresponding knot groups. With our methods we are able to deal with 191 more.

1 Introduction

A group G is called *bi-orderable* if there exists a strict total ordering of the elements of G such that $g < h$ implies $fg < fh$ and $gf < hf$ for all $f, g, h \in G$.

With the ongoing investigation into the connection between left-orderability and Heegaard–Floer homology [3], one is naturally led to ask about other orderability conditions on the fundamental group of a 3-manifold, for example, local indicability and bi-orderability. For compact, connected, P^2-irreducible 3-manifolds other than S^3, it turns out that the fundamental group is locally indicable if and only if the first homology is infinite [2], but the question of when the fundamental group can be bi-ordered is not as well understood. There are, however, a few available theorems [1, 5, 6, 10, 11, 15, 16], although with the exception of [5] all of these theorems apply only to 3-manifolds which fibre over S^1.

As a test of the effectiveness of the theorems available at the time, the authors of [6] computationally investigated the knot groups for knots with twelve or fewer crossings appearing in the table available from *KnotInfo* [4]. At that time, all of the available theorems applied only to manifolds fibring over S^1. Therefore, of the 2977 knots with twelve or fewer crossings, since 1246 of them fibre over S^1, the available theorems were applicable; from these only 12 were found to have bi-orderable knot groups while 487 were found to have non-bi-orderable knot groups. We could interpret this as a success rate of about $\frac{12+487}{2977} \sim 17\%$.

This paper extends the efforts of [6] to the nonfibered case by using the newly available theorems of [5]. The authors of [5] have already observed that their methods can be used to show that the knot group of 5_2 is not bi-orderable[1] (5_2 is a non-fibred knot), and they produce families of Alexander polynomials of degree four for which the corresponding knots will always have non-bi-orderable knot groups.

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[1]Naylor and Rolfsen showed independently via a computer program that this group is not bi-orderable. In fact, their program showed that the group admits generalized torsion [14].

As the theorems of [5] apply only to groups with two generators and one relator, we focus our attention on two sources of knot groups having these properties. Our first family of such knot groups comes from two-bridge knots, whose groups are known to have a presentation with two generators and one relator of the form $\langle a, b \mid aw = wb\rangle$ (see Section 3). The second family is all knots from the the *KnotInfo* table having twelve or fewer crossings, bridge number greater than two, and a group presentation (as computed by SnapPy [7]) with two generators and one relator.

With these techniques we were able to determine bi-orderability of many two-bridge knot groups, all twist knot groups, and find 6 new bi-orderable knot groups and 185 new non-bi-orderable knot groups arising from knots with fewer than 12 crossings. This extends the previous success rate of 17% by $\frac{6+185}{2977} \sim 6\%$. Despite these advances, at the time of this writing the first knot appearing in the tables for which bi-orderability of the knot group cannot be determined is the knot 8_2.

The difficulty in the case of 8_2 is that its Alexander polynomial has two real roots and four complex roots, and the available theorems are only applicable when all the roots of the Alexander polynomial are real and positive, or all the roots are complex. The same difficulty arises in the cases of 6_2 and 7_6. Their Alexander polynomials also have both real and complex roots, and as such non-bi-orderability of their groups had to be determined directly from their group presentations [1]. This difficulty highlights a problem that is made explicit in [14, Corollary 12], namely that the Alexander polynomial alone is not enough to detect non-bi-orderability of knot groups. As such, one may not be able to address knots whose Alexander polynomials have both real and complex roots by strengthening existing theorems, as they all depend solely on the Alexander polynomial.

Question 1.1 Can non-bi-orderability of the knot group be determined by examining knot invariants other than the Alexander polynomial?

2 Background

For the reader's convenience, we state the results of [5] that we will use here. We write b^a in place of $a^{-1}ba$, and for a word $w \in F(a, b)$ in the free group on generators a and b, write w_b and w_a for the total exponent sum (which we will call the weight) of b and a in the word w. Given such a word w, if $w_a = 0$, then we can rewrite w as

$$w = b^{m_1 a^{d_1}} \cdots b^{m_r a^{d_r}}$$

for some integers m_i, d_i, and $r \geq 1$. For all $j \in \mathbb{Z}$, set $\tau_j(w) = \{i : d_i = j\}$ and let $S_w = \{j : \sum_{i\in\tau_j(w)} m_i \neq 0\}$.

We say that a word w of the form above is *tidy* if $\tau_j(w) = \varnothing$ for all j satisfying either $j > \max\{S_w\}$ or $j < \min\{S_w\}$. Set $\ell = \max\{S_w\}$; we say that w is *principal* if it is tidy and $|\tau_\ell(w)| = 1$. In the case that w is principal and $\tau_\ell(w) = \{k\}$, we call w *monic* if in addition $m_k = 1$. Set $s = \min\{S_w\}$. When $\pi_1(S^3 \setminus K) = \langle a, b \mid w\rangle$ with w as above, the Alexander polynomial has formula $\Delta_K(t) = \sum_{i=1}^r m_i t^{d_i - s}$. We can group like powers and rewrite this as

$$(*) \qquad \Delta_K(t) = \sum_{j\in\mathbb{Z}} \Big(\sum_{i\in\tau_j(w)} m_i \Big) t^{j-s},$$

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where we understand that the coefficient of t^{j-s} is zero when $\tau_j(w) = \varnothing$.

Theorem 2.1 ([5, Corollary 2.5]) *Let K be a knot in S^3, and suppose that $\pi_1(S^3 \smallsetminus K)$ has a presentation of the form $\langle a, b \mid w\rangle$ where w is tidy. Let $\Delta_K(t)$ denote the Alexander polynomial of K.*

(i) *If $\pi_1(S^3 \smallsetminus K)$ is bi-orderable, then $\Delta_K(t)$ has a positive real root.*
(ii) *If w is monic and all the roots of $\Delta_K(t)$ are real and positive, then $\pi_1(S^3 \smallsetminus K)$ is bi-orderable.*
(iii) *If w is principal, $\Delta_K(t) = a_0 + \cdots + a_{d-1}t^{d-1} - mt^d$ where $\gcd\{a_0, \ldots, a_{d-1}\} = 1$ and a_{d-1} is not divisible by m, and all the roots of $\Delta_K(t)$ are real and positive, then $\pi_1(S^3 \smallsetminus K)$ is bi-orderable.*

3 Two-Bridge Knots

Recall that according to Schubert, 2-bridge knots are classified by coprime pairs of odd integers p and q, with $0 < p < q$. Thus every two-bridge knot may be written as $K_{\frac{p}{q}}$ where $\frac{p}{q}$ is a reduced fraction. The fundamental group has presentation

$$\pi_1(S^3 \smallsetminus K_{\frac{p}{q}}) = \langle a, b \mid aw = wb\rangle,$$

where $w = b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{q-2}}a^{\epsilon_{q-1}}$ and $\epsilon_i = (-1)^{\lfloor \frac{ip}{q} \rfloor}$. This formula follows from Schubert's normal form [18] (see the discussion in [13]).

Lemma 3.1 *If $\Delta_{K_{\frac{p}{q}}}(t) = a_0 + a_1 t + \cdots + a_n t^n$, then $\Delta_{K_{\frac{p}{q}}}(-1) = \sum_{i=1}^{n} |a_i| = q$.*

Proof From the group presentation above, one can use Fox calculus to compute (see [9]) $\Delta_{K_{p/q}}(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \cdots + t^{\epsilon_1+\cdots+\epsilon_{q-1}}$. Since $\epsilon_i = \pm 1$, $\sum_{i=1}^{\ell} \epsilon_i$ is odd if and only if ℓ is odd. From this we draw two conclusions. First, if we regroup terms and write $\Delta_{K_{p/q}}(t) = a_0 + a_1 t + \cdots + a_n t^n$, then a_i is negative if and only if i is odd. Therefore $\Delta_{K_{p/q}}(-1) = \sum_{i=1}^{n} |a_i|$. Second, we may compute from the above formula that

$$\Delta_{K_{\frac{p}{q}}}(-1) = \underbrace{1 + 1 + \cdots + 1}_{q \text{ times}} = q. \qquad \blacksquare$$

Lemma 3.2 *Every two-bridge knot group admits a presentation of the form $\langle x, y \mid r\rangle$ where the relator r is a tidy word in the generators x, y.*

Proof Consider an arbitrary two-bridge knot $K_{p/q}$, and the presentation of its knot group with notation as defined above $\pi_1(S^3 \smallsetminus K_{p/q}) = \langle a, b \mid aw = wb\rangle$. Define a homomorphism $\phi\colon F(a,b) \to F(x,y)$ by $\phi(a) = xy$, $\phi(b) = y$. This descends to an isomorphism of the group $\langle a, b \mid aw = wb\rangle$ with the group presented by $\langle x, y \mid R\rangle$ where $R = xy\phi(w)y^{-1}\phi(w)^{-1}$. Note that y has weight zero in R. We claim that R is a tidy word in $\{x, y\}$.

Observe that since each occurrence of either a or a^{-1} in the word $awb^{-1}w^{-1}$ results in exactly one occurrence of x or x^{-1} in R, the letters x and x^{-1} occur a total of q times in R. Thus upon rewriting R in the form $x^{m_1 y^{d_1}} \cdots x^{m_r y^{d_r}}$, we have $\sum_{i=1}^{r} |m_i| = q$, since the rewriting is accomplished without cancelling any powers of x.

Now suppose that R is not tidy, so there exists j_0 such that $\tau_{j_0}(R) \neq \varnothing$ and $j_0 \notin S_R$, and therefore $\sum_{i \in \tau_{j_0}(R)} m_i = 0$. We compute

$$\begin{aligned} q &= \sum_{j \in \mathbb{Z}} \Big| \sum_{i \in \tau_j(R)} m_i \Big| && \text{by Lemma 3.1 and } (*) \\ &= \sum_{j \neq j_0} \Big| \sum_{i \in \tau_j(R)} m_i \Big| && \text{since } \sum_{i \in \tau_{j_0}(R)} m_i = 0 \\ &\leq \sum_{\substack{i \in \mathbb{Z} \\ i \notin \tau_{j_0}(R)}} |m_i| \\ &< \sum_{i=1}^{r} |m_i| = q \end{aligned}$$

and this contradiction completes the proof. ∎

As an immediate consequence, we can apply Theorem 2.1 (i) to all two-bridge knot groups.

Theorem 3.3 *Suppose that K is a two-bridge knot with Alexander polynomial $\Delta_K(t)$. If $\pi_1(S^3 \setminus K)$ is bi-orderable, then $\Delta_K(t)$ has a positive real root.*

4 Twist Knots

Twist knots are a subfamily of two-bridge knots whose diagrams appears as in Figure 1. Considered as a two-bridge knot $K_{p/q}$, the twist knot with r positive half-twists corresponds to the case $p = 2r - 1$ and $q = 2r + 1$. We simplify notation by writing K_r instead of $K_{\frac{2r-1}{2r+1}}$. By *positive half-twists* we mean twisting in the direction illustrated in Figure 2.

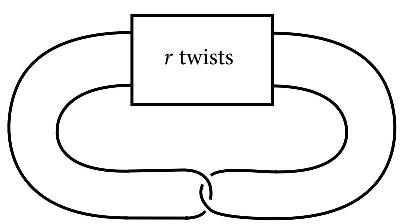

Figure 1: The twist knot K_r.

Theorem 4.1 *Let $r > 0$ be an integer. If r is even, then $\pi_1(S^3 \setminus K_r)$ is bi-orderable, otherwise $\pi_1(S^3 \setminus K_r)$ is not bi-orderable.*

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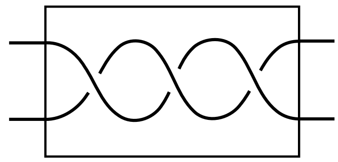

Figure 2: Our twisting convention illustrated in the case $r = 3$.

Proof The Alexander polynomial of the twist knot with r twists is [17, p. 80]:

$$\Delta(t) = \begin{cases} -\left(\frac{r}{2}\right) + (1+r)t - \left(\frac{r}{2}\right)t^2 & \text{if } r \text{ is even,} \\ \left(\frac{1+r}{2}\right) - rt + \left(\frac{1+r}{2}\right)t^2 & \text{if } r \text{ is odd.} \end{cases}$$

One can check that when r is odd, the Alexander polynomial has no real roots and thus $\pi_1(S^3 \setminus K_r)$ is not bi-orderable by Theorem 3.3.

Henceforth we assume r is even. Then both roots of the Alexander polynomial are real and positive, so we can prove bi-orderability by applying Theorem 2.1 (ii) and (iii). Note that $1 + r = 2(\frac{r}{2}) + 1$, so $\frac{-r}{2}$ and $r + 1$ are relatively prime. Thus when $r > 2$, the coefficients of the Alexander polynomial satisfy the necessary divisibility conditions to apply Theorem 3.3 (iii); when $r = 2$, we can apply Theorem 3.3 (ii). To do this, we show that $\pi_1(S^3 \setminus K_r)$ admits a presentation with two generators and a single principal relator, and when $r = 2$, the relator is also monic. We begin with the standard two-bridge presentation of K_r.

We recall that a twist knot with r twists has a two-bridge representation as $K_{\frac{2r-1}{2r+1}}$. We begin with the standard two-bridge presentation $\pi_1(S^3 \setminus K_{p/q}) = \langle a, b \mid aw = wb\rangle$ and first determine the values of ϵ_i appearing in the formula for w.

For i satisfying $1 \le i \le r$ we have $i > \frac{i(2r-1)}{2r+1} > i-1$, and so $\lfloor \frac{i(2r-1)}{2r+1} \rfloor = i-1$. Whereas for i satisfying $r < i \le 2r$, we have $i - 1 > \frac{i(2r-1)}{2r+1} > i - 2$,and thus $\lfloor \frac{i(2r-1)}{2r+1} \rfloor = i - 2$. Recalling that r is even, this allows us to compute

$$w = (\underbrace{ba^{-1}ba^{-1}\cdots ba^{-1}ba^{-1}}_{r \text{ letters}})(\underbrace{b^{-1}ab^{-1}a\cdots b^{-1}ab^{-1}a}_{r \text{ letters}})$$

As in the proof of Lemma 3.2, we will apply the homomorphism

$$\phi\colon F(a,b) \to F(x,y)$$

with $\phi(a) = xy$, $\phi(b) = y$ to create a new presentation $\langle x, y \mid R\rangle$ of $\pi_1(S^3 \setminus K_r)$ with relator $R = xy\phi(w)y^{-1}\phi(w)^{-1}$. From the calculations above we find

$$\phi(w) = x^{-\frac{r}{2}}y^{-1}x^{\frac{r}{2}}y$$

and thus $R = xyx^{-\frac{r}{2}}y^{-1}x^{\frac{r}{2}}y^{-1}x^{-\frac{r}{2}}yx^{\frac{r}{2}}$. We replace R with the relator $y^{-1}Ry$ to get

$$y^{-1}Ry = (y^{-1}xy)(x^{-\frac{r}{2}})(y^{-1}x^{\frac{r}{2}}y)(y^{-2}x^{-\frac{r}{2}}y^2)(y^{-1}x^{\frac{r}{2}}y)$$

from which we read off the sets τ_j for $j = 0, 1, 2$, finding

$$\tau_0(y^{-1}Ry) = \{2\}, \quad \tau_1(y^{-1}Ry) = \{1,3,5\}, \quad \tau_2(y^{-1}Ry) = \{4\},$$

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and $S_{y^{-1}Ry} = \{0,1,2\}$. Since $|\tau_2(y^{-1}Ry)| = 1$, the relator $y^{-1}Ry$ is principal. Note that when $r = 2$, we find $\tau_2(y^{-1}Ry) = \{4\}$ and $m_4 = -\frac{r}{2} = -1$ so that the word is not monic. We can fix this by considering, in the case $r = 2$, the inverse relator $y^{-1}R^{-1}y$. This relator is monic, and thus $\pi_1(S^3 \setminus K_r)$ is bi-orderable when $r = 2$ as well. ∎

5 Computational Results and Knots With Bridge Number Greater Than Two

Our computational methods for knots with fewer than 12 crossings involved two steps. First, we attempted to find a presentation of the knot group having a tidy, principal or monic relator. In the event that our methods succeeded, we then determined the number of positive real roots of the Alexander polynomial using a combination of Rouché's Theorem and Sturm's Theorem [8, §24]. This approach to root counting allows for all the computations to be done symbolically, and avoids the problem of rounding error (which produced two erroneous claims in [6], see §B.1). These two steps are detailed below, combining them yields the results of Appendix B.

5.1 Finding a Suitable Presentation of the Knot Group

In order to apply Theorem 2.1 to a group $G = \langle a, b \mid w\rangle$ one must find a presentation of G where the relator w has the form $w = b^{m_1 a^{d_1}} \cdots b^{m_r a^{d_r}}$. Finding a presentation with a relator of this form is always possible when the group admits two generators and one relator, though there are possibly many different ways of doing so. The key to our method is contained in the following lemma.

Lemma 5.1 ([12, Chapter V, Lemma 11.8]) *Suppose that $G = \langle a, b \mid w\rangle$ and denote the weight of a and b in w by w_a, w_b, respectively. Assume without loss of generality that $0 < w_a \le w_b$. Set $k = -\lfloor \frac{w_b}{w_a} \rfloor$, $\phi(a) = xy^k$, and $\phi(b) = y$. This defines an isomorphism $\phi\colon \langle a, b \mid w\rangle \to \langle x, y \mid \phi(w)\rangle$. Moreover the weights $\phi(w)_x$ and $\phi(w)_y$ of x and y in $\phi(w)$ satisfy $|\phi(w)_x| + |\phi(w)_y| < |w_a| + |w_b|$.*

Proof The map ϕ is an isomorphism since it has inverse $\phi^{-1}(x) = ab^{-k}$, $\phi^{-1}(y) = b$. Note that the weights $\phi(w)_x$ and $\phi(w)_y$ satisfy $\phi(w)_x = w_a$ and $\phi(w)_y = w_b + kw_a$. In particular, there are bounds $0 \le w_b + kw_a < w_b$ so that

$$|\phi(w)_x| + |\phi(w)_y| = |\phi(w)_x| + |w_b + kw_a| < |w_a| + |w_b|.$$

as claimed. ∎

The conclusion $|\phi(w)_x| + |\phi(w)_y| < |w_a| + |w_b|$ means that upon iteratively applying this lemma to a group presentation (with appropriate variable changes made at each step to guarantee that the hypothesis $0 < w_a \le w_b$ is satisfied by the input), one will eventually produce a two generator, one relator presentation of the group G such that one of the generators has weight zero in the relator. Supposing that the generator a has weight zero, it is then possible to rewrite the relator in the form:

$$w = b^{m_1 a^{d_1}} \cdots b^{m_r a^{d_r}}.$$

Pseudocode for the procedure is found in Appendix A. We applied this algorithm to all knots with 12 or fewer crossings in order to produce group presentations having a single relator of the required form, and then checked if the relator was also tidy, principal, or monic.

Remark 5.2 Note that in Lemma 5.1 we could also have used the substitution $\phi(a) = xy^k$ and $\phi(b) = xyx^{-1}$ with k as above. Therefore one can also iterate the substitution $\phi(a) = xy^k$ and $\phi(b) = xyx^{-1}$ in order to find a relator having weight zero in one of the generators, although in practice we found that this yielded few new results. Indeed, SnapPy gives a presentation for the knot 9_{16} which has one non-tidy relator which becomes tidy upon iterating the substitution $\phi(a) = xy^k$ and $\phi(b) = xyx^{-1}$, but it does not become tidy by iterating $\phi(a) = xy^k$ and $\phi(b) = y$. Of all knots with twelve or fewer crossings, this was the only instance where one algorithm yielded a tidy relator while the other did not.

5.2 Computing the Number of Real Roots of the Alexander Polynomial

By a straightforward application of Rouché's Theorem, all the roots of the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ are bounded above by

$$R = 1 + \frac{1}{|a_n|}\max\{|a_0|, |a_1|, \dots, |a_{n-1}|\},$$

and thus the positive real roots of $p(x)$ will be contained in the interval $[0, R]$. Sturm's Theorem can then be used to compute the number of roots, counted without multiplicity, of $p(x)$ in the interval $[0, R]$ [8, §24]. Thus, for each knot K having fewer than 12 crossings and a tidy, principal, or monic relator, we applied this approach to the Alexander polynomial $\Delta_K(t)$ in order to determine if all the roots of $\Delta_K(t)$ are real, or if none of them are.

Whenever Sturm's Theorem yields a count of 0 or $\deg \Delta_K(t)$ roots, we know that this quantity is exactly the total number of positive real roots of $\Delta_K(t)$. On the other hand, if Sturm's Theorem yields a count of m roots in $[0, R]$ with $0 < m < \deg \Delta_K(t)$, then we cannot say that $\Delta_K(t)$ has exactly m positive real roots, as some repeated real roots may have been counted without multiplicity. This occurred for the knots 10_{137}, $11a_5$, $11a_{103}$, $11a_{201}$, $12a_{348}$, $12a_{1202}$, $12n_{49}$, $12n_{145}$, $12n_{279}$, $12n_{394}$, $12n_{462}$, $12n_{553}$, and $12n_{838}$. In these cases we found it was possible to factor $\Delta_K(t)$ into a product of linear and quadratic factors and confirm the number of positive real roots directly.

Example 5.3 The knot 10_{52} is non-fibered, so none of the theorems available from [6, 11, 15, 16] apply. Moreover, it has bridge index 3, so Theorem 3.3 does not apply either; it is the first knot in the tables for which the algorithm in Appendix A succeeds where these theorems fail. SnapPy gives the group presentation $\pi_1(S^3 \setminus 10_{52}) = \langle a, b | w\rangle$, where

$$\begin{aligned} w = & ab^2a^2b^2aBab^2a^2bAB^2A^2B^2AbAB^2A^2Bab^2a^2b^2aBab^2a^2bAB^2A^2B^2Aba^2b^2 \\ & aBab^2a^2b^2aBA^2B^2AbAB^2A^2B^2Aba^2b^2aBab^2a^2b^2aBA^2B^2AbAB^2A^2B. \end{aligned}$$

Here we write capital letters in place of inverses in order to simplify notation. Note that in the above word, neither generator has weight zero: we find $w_a = 3$ and $w_b = 3$.

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Applying the algorithm described in Appendix A, yields the substitution $a \mapsto bA^3$ and $b \mapsto a^4B$, giving a new presentation of the form $\langle a, b \mid w' \rangle$, where

$$\begin{aligned} w' =& baBabaBabA^4baBabaBAbABAbABa^4BAbABAbaBabaBabA^4 \\ & baBabaBAbABAbABabaBabA^4baBabaBabABAbABa^4 \\ & BAbABAbABabaBabA^4baBabaBabABAbABa^4BAbABA. \end{aligned}$$

The generator a now has weight zero in w'. After writing w' in the form

$$w' = b^{m_1 a^{d_1}} \cdots b^{m_r a^{d_r}},$$

and conjugating so that $\min\{d_1, \dots, d_r\} = 0$, we find

$$\begin{aligned} w' =& b^{a^4} b^{-a^3} b^{a^2} b^{-a} b b^{a^4} b^{-a^3} b^{a^2} b^{-a} b^{a^2} b^{-a^3} b^{a^4} b^{-a^5} b^{-a} b^{a^2} b^{-a^3} b^{a^4} b^{-a^3} b^{a^2} b^{-a} b b^{a^4} \\ & b^{-a^3} b^{a^2} b^{-a} b^{a^2} b^{-a^3} b^{a^4} b^{-a^5} b^{a^4} b^{-a^3} b^{a^2} b^{a^6} b^{-a^5} b^{a^4} b^{-a^3} b^{a^2} b^{-a^3} b^{a^4} b^{-a^5} b^{-a} \\ & b^{a^2} b^{-a^3} b^{a^4} b^{-a^5} b^{-a^3} b^{a^2} b^{a^4} b^{a^6} b^{-a^5} b^{a^4} b^{-a^3} b^{a^2} b^{-a^3} b^{a^4} b^{-a^5} b^{-a} b^{a^2} b^{-a^3}. \end{aligned}$$

One can easily check by hand from this expression that $\min S_{w'} = 0$ and $\max S_{w'} = 6$, and clearly τ_j is empty for $j > 6$ and $j < 0$. Therefore the word w' is tidy. The Alexander polynomial of 10_{52} is $\Delta(t) = 2-7t+13t^2-15t^3+13t^4-7t^5+2t^6$. An upper bound on the real roots of this polynomial is found, as in Section 5.2, to be $\frac{15}{2}$. Sturm's Theorem then shows that $\Delta(t)$ has no real roots in the interval $[0, \frac{15}{2}]$. Thus, $\Delta(t)$ has no positive real roots and by Theorem 2.1 (i) the knot 10_{52} does not have a bi-orderable knot group.

A Finding a Presentation Having a Relator With Zero Weight

function ZERO_WEIGHT(String w)
    int w_a;
    int w_b;
    while weight of a in w is not 0 **do**
        w_a = weight of a in w;
        w_b = weight of b in w;
        if $w_a < 0$ **then**
            swap a's and a^{-1}'s;
            $w_a = -w_a$;
        end if
        if $w_b < 0$ **then**
            swap b's and b^{-1}'s;
            $w_b = -w_b$;
        end if
        if $w_a > w_b$ **then**
            swap a's and b's in w;
            swap a^{-1}'s and b^{-1}'s in w;
            swap w_a and w_b;
        end if
        if weight of a in w is not 0 **then**
            replace a's with $ab^{-\lfloor w_b/w_a \rfloor}$;

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replace a^{-1} with $b^{\lfloor w_b/w_a \rfloor} a^{-1}$;
end if
end while
end function

Now that the weight of a in w is zero, w can be written as $b^{m_1 a^{d_1}} \cdots b^{m_r a^{d_r}}$. Additionally, we normalize by conjugating w by a until $\min\{d_1, \dots, d_r\} = 0$. Then it is possible to check if the relator is tidy, principal, or monic by directly applying the definitions from the paragraph preceding Theorem 2.1.

B Bi-orderable and Non-bi-orderable Knot Groups

All group presentations were initially calculated using SnapPy by Nathan Dunfield. The presentations were then changed using the algorithm of Appendix A in order to apply Theorem 2.1 (i) and (ii).

B.1 Fibered Knots

For fibered knots, our program found the same results as [6], with two exceptions which we believe came about due to rounding error in the numerical method used to solve for the roots of the Alexander polynomial. There are two exceptions.

- The knot $12n_{0019}$ is listed having a non-bi-orderable group, when in fact bi-orderability of its group cannot be determined by known theorems since its Alexander polynomial has both positive and negative real roots (and it is not a special polynomial, as defined in [11]).
- The knot $12a_{0477}$ is fibered, and its Alexander polynomial is

$$\Delta(t) = 1 - 11t + 41t^2 - 63t^3 + 41t^4 - 11t^5 + t^6,$$

which has all real positive roots. Therefore it has a bi-orderable knot group by the main theorem of [6], though it is listed as non-bi-orderable there.

B.2 Non-fibered Knots

The following 5 nonfibered knots have bi-orderable groups, by applying either Theorem 2.1 (ii) or (iii): 6_1, 8_1, 10_1, 10_{13}, $12a_{803}$.

The following 79 nonfibered knots are two-bridge and their Alexander polynomials have no positive real roots, and so their groups are not bi-orderable: 5_2, 7_2, 7_3, 7_4, 7_5, 8_8, 8_{13}, 9_2, 9_3, 9_4, 9_5, 9_6, 9_7, 9_9, 9_{10}, 9_{13}, 9_{14}, 9_{18}, 9_{19}, 9_{23}, 10_{10}, 10_{12}, 10_{15}, 10_{19}, 10_{23}, 10_{27}, 10_{28}, 10_{31}, 10_{33}, 10_{34}, 10_{37}, 10_{40}, $11a_{13}$, $11a_{75}$, $11a_{77}$, $11a_{85}$, $11a_{89}$, $11a_{90}$, $11a_{95}$, $11a_{98}$, $11a_{111}$, $11a_{119}$, $11a_{178}$, $11a_{183}$, $11a_{186}$, $11a_{188}$, $11a_{191}$, $11a_{192}$, $11a_{193}$, $11a_{195}$, $11a_{205}$, $11a_{210}$, $11a_{234}$, $11a_{235}$, $11a_{236}$, $11a_{238}$, $11a_{242}$, $11a_{243}$, $11a_{246}$, $11a_{247}$, $11a_{307}$, $11a_{333}$, $11a_{334}$, $11a_{335}$, $11a_{336}$, $11a_{337}$, $11a_{339}$, $11a_{341}$, $11a_{342}$, $11a_{343}$, $11a_{355}$, $11a_{356}$, $11a_{357}$, $11a_{358}$, $11a_{359}$, $11a_{360}$, $11a_{363}$, $11a_{364}$, $11a_{365}$.

The following 15 knots have bridge index greater than two, admit a two-generator presentation with a single tidy relator, and their Alexander polynomials have no positive real roots. Therefore their groups are not bi-orderable: 9_{16}, 10_{52}, 10_{57}, 10_{128}, 10_{129}, 10_{130}, 10_{134}, 10_{135}, $11a_{12}$, $11a_{32}$, $11a_{46}$, $11a_{241}$, $11a_{258}$, $11n_{18}$, $11n_{62}$.

The following 92 knots admit a two-generator presentation with a single tidy relator and their Alexander polynomials have no positive real roots, so their groups are not bi-orderable (we do not know their bridge index, as that information is not listed on *Knotinfo* for knots with twelve or more crossings): $12a_9$, $12a_{31}$, $12a_{32}$, $12a_{42}$, $12a_{81}$, $12a_{96}$, $12a_{143}$, $12a_{147}$, $12a_{148}$, $12a_{151}$, $12a_{169}$, $12a_{212}$, $12a_{235}$, $12a_{241}$, $12a_{247}$, $12a_{251}$, $12a_{302}$, $12a_{378}$, $12a_{379}$, $12a_{424}$, $12a_{511}$, $12a_{514}$, $12a_{534}$, $12a_{537}$, $12a_{580}$, $12a_{581}$, $12a_{582}$, $12a_{595}$, $12a_{596}$, $12a_{643}$, $12a_{669}$, $12a_{718}$, $12a_{720}$, $12a_{728}$, $12a_{732}$, $12a_{744}$, $12a_{759}$, $12a_{760}$, $12a_{761}$, $12a_{774}$, $12a_{791}$, $12a_{792}$, $12a_{826}$, $12a_{827}$, $12a_{836}$, $12a_{876}$, $12a_{879}$, $12a_{880}$, $12a_{882}$, $12a_{1029}$, $12a_{1030}$, $12a_{1033}$, $12a_{1034}$, $12a_{1129}$, $12a_{1130}$, $12a_{1132}$, $12a_{1133}$, $12a_{1138}$, $12a_{1139}$, $12n_{46}$, $12n_{78}$, $12n_{153}$, $12n_{154}$, $12n_{166}$, $12n_{167}$, $12n_{169}$, $12n_{170}$, $12n_{200}$, $12n_{236}$, $12n_{239}$, $12n_{241}$, $12n_{243}$, $12n_{244}$, $12n_{248}$, $12n_{250}$, $12n_{251}$, $12n_{288}$, $12n_{289}$, $12n_{305}$, $12n_{307}$, $12n_{308}$, $12n_{310}$, $12n_{374}$, $12n_{404}$, $12n_{501}$, $12n_{502}$, $12n_{503}$, $12n_{575}$, $12n_{594}$, $12n_{650}$, $12n_{723}$, $12n_{851}$.

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