Canad. Math. Bull. Vol. **38** (3), 1995 pp. 334–346

ON THE DUALITY OF OPERATOR SPACES

CHRISTIAN LE MERDY

ABSTRACT. We prove that given an operator space structure on a dual Banach space Y^*, it is not necessarily the dual one of some operator space structure on Y. This allows us to show that Sakai's theorem providing the identification between C^*-algebras having a predual and von Neumann algebras does not extend to the category of operator spaces. We also include a related result about completely bounded operators from $B(\ell_2)^*$ into the operator Hilbert space OH.

1. **Introduction.** Given a Hilbert space H, we denote by $B(H)$ (resp. $K(H)$) the C^*-algebra of all bounded (resp. compact) operators on H. We let $S_1(H)$ be the Banach space of all trace class operators on H and recall that $K(H)^* = S_1(H) = B(H)_*$. Let Y be a Banach space and let $A = Y^*$ be its dual space. Assume that A is a C^*-algebra. A celebrated theorem of Sakai [S1, S2 (Theorem 1.16.7); see also T, Theorem III.3.5] leading to the identification between C^*-algebras having a predual and von Neumann algebras asserts that there exist a Hilbert space H and a one-one C^*-algebraic representation $\pi: A \longrightarrow B(H)$ which is continuous with respect to the σ-weak topologies $\sigma(A, Y)$ and $\sigma\big(B(H), S_1(H)\big)$.

The theory of operator spaces which was recently developed by Blecher-Paulsen and Effros-Ruan (see [B1, B2, BP, ER1, ER2, ER3]) naturally leads to the problem of whether we may replace the C^*-algebra by an operator space and the faithful C^*-algebraic representation by a complete isometry in Sakai's theorem. Note that this problem makes sense even though a dual Banach space can have several non isomorphic preduals. The main purpose of this paper is the study of a problem posed by D. Blecher about the duality of operator spaces, from which we will deduce that Sakai's theorem does not extend to the category of operator spaces.

Before stating this problem, let us recall a few definitions. By an operator space we mean a closed subspace of $B(H)$ for some Hilbert space H. Given an operator space $E \subset B(H)$, we denote by $M_{n,m}(E)$ the vector space of all $n \times m$ matrices with entries in E and equip it with the norm induced by the embedding $M_{n,m}(E) \subset B\big(\ell_2^m(H), \ell_2^n(H)\big)$. The notation $M_n(E)$ stands for $M_{n,n}(E)$. Let E and F be operator spaces and let $u: E \longrightarrow F$ be a bounded linear map. We set $\|u\|_{\mathrm{cb}} = \sup_{n\geq 1} \|u \otimes I_{M_n}\|$ where $u \otimes I_{M_n}: M_n(E) \longrightarrow M_n(F)$ is defined by $u([x_{ij}]) = [u(x_{ij})]$ for all $[x_{ij}] \in M_n(E)$. We say that u is *completely bounded* (c.b. in short) provided that $\|u\|_{\mathrm{cb}} < +\infty$. We denote by $\mathrm{CB}(E, F)$ the set of all c.b. maps from E into F. It is easily checked that $\| \ \|_{\mathrm{cb}}$ is a norm on $\mathrm{CB}(E, F)$ with which this space is complete. Furthermore, we say that $u: E \longrightarrow F$ is a *complete isometry* provided that $u \otimes I_{M_n}$ is an isometry for all $n \geq 1$ and that u is a c.b. isomorphism provided that u is an isomorphism such that u, u^{-1} are both c.b. maps. Given a Banach space E, an operator

Received by the editors March 22, 1994; revised December 6, 1994.
AMS subject classification: Primary: 47C15; secondary: 46A20, 46B28.

space structure (o.s.s. for short) on E will be a sequence of norms, the n-th norm defined on $M_n(E)$, such that there exists an embedding $J\colon E \longrightarrow B(H)$ for which all the $J \otimes I_{M_n}$ are isometries. We refer the reader to [R] for the fundamental characterization of operator space structures.

We will be mainly concerned with duality of operator spaces. Let E be an operator space. Let $\varphi = [\varphi_{ij}] \in M_n(E^*)$. We can regard φ as a map from E into M_n by letting $\varphi(x) = [\varphi_{ij}(x)]$ for all $x \in E$. The resulting identification:

$$M_n(E^*) = \mathrm{CB}(E, M_n) \tag{1.1}$$

defines a norm on $M_n(E^*)$. It turns out that these matrix norms define an o.s.s. on E^* (see [B2, BP, ER1]) with which E^* is called the *dual operator space* of E. This duality has many nice properties. There were studied and developed in the papers [B2, BP, BS, ER1, ER2, ER4]. We will recall some of them in Section 2 below.

When dealing with operator spaces, it often happens that an o.s.s. is given on a dual Banach space Y^* and one wishes to know whether or not it "comes from" an o.s.s. on Y. This leads to the following problem, which was first raised by Blecher.

PROBLEM 1.1. Let Y be a Banach space and let $X = Y^*$ be its dual space. Assume that we are given an o.s.s. on X. Does there exist an o.s.s. on Y for which the dual operator space Y^* coincides (completely isometrically or completely isomorphically) with X?

In Section 3, we shall give necessary and sufficient conditions under which the completely isometric form of Problem 1.1 has a positive answer. We shall then construct an operator space structure on $X = B(\ell_2)^*$ for which the completely isomorphic form of Problem 1.1 has a negative answer. We will then deduce that Sakai's theorem cannot be generalized to operator spaces by proving:

THEOREM 1.2. *There exists a Banach space Y and an operator space structure on its dual space $X = Y^*$ such that:*

Whenever $J\colon X \longrightarrow B(H)$ is a complete isometry, J is not continuous with respect to the topologies $\sigma(X, Y)$ and $\sigma\big(B(H), S_1(H)\big)$.

We mention that the problem of whether Sakai's theorem extends to the category of unital operator algebras remains open. Given a unital operator algebra $A = Y^*$, the question is now: does there exist a unital completely isometric homomorphism $J\colon A \longrightarrow B(H)$ continuous w.r.t. $\sigma(A, Y)$ and $\sigma\big(B(H), S_1(H)\big)$? The only fact we know on this subject is that for any unital operator algebra B, the bidual operator space $A = B^{**}$ (equipped with the Arens product) is a unital operator algebra which can be represented (in the above sense) by a σ-weak $J\colon A \longrightarrow B(H)$.

In order to construct an o.s.s. on $B(\ell_2)^*$ for which Problem 1.1 has a negative answer (see Theorem 3.2), we will take advantage of ideas from Pisier's paper [P1]. We will appeal to some of the properties of the operator Hilbert space OH and will use the lack of local reflexivity of operator spaces in the following form: the identity map $\mathrm{Id}\colon \mathrm{OH} \otimes B(\ell_2)^{**} \longrightarrow \big(\mathrm{OH} \otimes B(\ell_2)\big)^{**}$ is not bounded, when the tensor products are both endowed with the spatial tensor norms. Section 2 below is devoted to the necessary preliminaries about OH. We will also give some background about the duality of operator spaces. In Section 4, we will apply some of the ideas of Section 3 to the study of c.b. maps from $B(\ell_2)^*$ into OH. We will especially prove that there exist c.b. maps $T: B(\ell_2)^* \longrightarrow \mathrm{OH}$ which are not (2, oh)-summing in the sense of [P1]. This is a first step in the research of possible analogues of Grothendieck's inequalities in the framework of operator spaces.

2. **Preliminaries.** We start this section by recalling a few facts about the duality of operator spaces.

Let E, F be two operator spaces. We denote by $E \check{\otimes} F$ the spatial (or minimal) tensor product of E and F. Unless otherwise specified, the norms of elements of a tensor product of operator spaces will always be defined with respect to this spatial tensor norm. Duality and $\check{\otimes}$ are related by the isometric embedding [BP]:

$$E^* \check{\otimes} F \subset \mathrm{CB}(E, F) \tag{2.1}$$

under the canonical identification of $E^* \otimes F$ as the space of finite rank operators from E into F.

The two following relations, which were first established in [B2], will play an important role below. Let E be an operator space. Then under natural identification, we have:

$$M_{n,m}(E)^{**} = M_{n,m}(E^{**}). \tag{2.2}$$

Furthermore, let F be another operator space and let $u \in \mathrm{CB}(E, F)$. Then the transposed map $u^*: F^* \longrightarrow E^*$ is c.b. and

$$\|u\|_{\mathrm{cb}} = \|u^*\|_{\mathrm{cb}}. \tag{2.3}$$

We now state a very useful formula about column matrices. We denote by C the operator space $B(\mathbb{C}, \ell_2)$ and by $C_n = B(\mathbb{C}, \ell_2^n)$ its n-dimensional version. For any operator space E, we let $C_n(E) = M_{n,1}(E) = C_n \check{\otimes} E$. Similarly we will use the notations $R = B(\overline{\ell_2}, \mathbb{C})$ and $R_n = B(\overline{\ell_2^n}, \mathbb{C})$. In the case of column matrices, formula (1.1) becomes:

$$C_n(E^*) = \mathrm{CB}(E, C_n). \tag{2.4}$$

Thus the norm on $C_n(E^*)$ can be computed by a very simple formula. For any $\varphi_1, \ldots, \varphi_n$ in E^*:

$$\left\| \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right\|_{C_n(E^*)} = \sup \left\{ \Big(\sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} |\langle \varphi_i, x_j \rangle|^2 \Big)^{1/2} / x_j \in E, \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{C_n(E)} \le 1 \right\}. \tag{2.5}$$

See [L, Proposition 2.5] for example.

We now come to the operator Hilbert space OH. It is a very important o.s.s. on ℓ_2 which was introduced and investigated in [P1, P2]. We wish to point out some of the main features of this operator space that will be used in the sequel. Let us denote by $(e_i)_{i \ge 1}$ the canonical basis of ℓ_2. The two fundamental properties of OH to be used later on are:

https://doi.org/10.4153/CMB-1995-049-9 Published online by Cambridge University Press

(i) The canonical linear identification between OH and $\overline{\mathrm{OH}^*}$ is a complete isometry. We write:

$$\overline{\mathrm{OH}^*} = \mathrm{OH} \tag{2.6}$$

(ii) For any operator space E and any finite sequence $(x_1,\ldots,x_N)$ in E:

$$\Big\|\sum_{i=1}^{N} e_i \otimes x_i\Big\| = \Big\|\sum_{i=1}^{N} x_i \otimes \overline{x_i}\Big\|^{1/2} \tag{2.7}$$

(with $\sum_{i=1}^N e_i \otimes x_i \in \mathrm{OH} \overset{\smile}{\otimes} E, \sum_{i=1}^N x_i \otimes \overline{x_i} \in E \overset{\smile}{\otimes} \bar{E}$).

Actually OH is the only o.s.s. on ℓ_2 for which (2.6) holds but we won't make use of this uniqueness property. Note that the identity (2.7) contains the definition of OH. Indeed if we apply (2.7) with $E = M_n$, we obtain an explicit description of $M_n(\mathrm{OH})$. For any $n \geq 1$, the notation OH_n will stand for the subspace $\mathrm{Span}\{e_1,\ldots,e_n\} \subset \mathrm{OH}$.

Let E be an operator space. As a straightforward consequence of (2.1), (2.6) and (2.7), we have:

$$\forall u \in CB(\mathrm{OH}_n, E), \quad \|u\|_{\mathrm{cb}} = \Big\|\sum_{i=1}^{n} u(e_i) \otimes \overline{u(e_i)}\Big\|^{1/2}. \tag{2.8}$$

All this can be found in [P1]. The reader is referred to this publication for further details and information.

We now introduce a new definition which is convenient for our purpose. In order to motivate it, note that when E, F are operator spaces with F finite dimensional, the identity map $CB(F,E)^{**} \longrightarrow CB(F,E^{**})$ is always a contraction. However, it was discovered in [EH] that the converse map $CB(F,E^{**}) \longrightarrow CB(F,E)^{**}$ can have a norm > 1. This striking phenomenon (in view of the Banach space case [D]) gave rise to the problem of local reflexivity of operator spaces [EH, ER3, ER4]. In the definition below, we focus on the case when F runs over the spaces OH_n.

DEFINITION 2.1. Let E be an operator space. We will say that E is OH-*locally reflexive* provided that there exists a constant $C \geq 1$ such that:

(2.9)

For all $n \geq 1$, the identity map $CB(\mathrm{OH}_n, E^{**}) \longrightarrow CB(\mathrm{OH}_n, E)^{**}$ has norm less than C.

The following result is implicit in [P1, Problem 10.4]. It was brought to our attention by G. Pisier.

LEMMA 2.2. *$B(\ell_2)$ is not* OH-*locally reflexive.*

PROOF. Assume that $B(\ell_2)$ satisfies (2.9). We wish to show that $B(\ell_2)^{**}$ satisfies the condition (v) of [P1, Theorem 2.9]. Since $B(\ell_2)^{**}$ is not injective, this will yield the desired contradiction.

Let $x_1,\ldots,x_n$ in $B(\ell_2)^{**}$ with $\|\sum_{i=1}^n x_i \otimes \overline{x_i}\| \leq 1$. Let $u{:}\, \mathrm{OH}_n \longrightarrow B(\ell_2)^{**}$ be defined by $u(e_i) = x_i$. From (2.8) we obtain that $\|u\|_{\mathrm{cb}} \leq 1$ hence by (2.9) the norm of u in $CB\big(\mathrm{OH}_n, B(\ell_2)\big)^{**}$ is less than C. Therefore there exists a net u^α

in $\mathrm{CB}\big(\mathrm{OH}_n, B(\ell_2)\big)$ with $\|u^\alpha\|_{\mathrm{cb}} \leq C$ such that u^α converges to u in the topology $\sigma\Big(\mathrm{CB}\big(\mathrm{OH}_n, B(\ell_2)\big)^{**}, \mathrm{CB}\big(\mathrm{OH}_n, B(\ell_2)\big)^{*}\Big)$. Letting $x_i^\alpha = u^\alpha(e_i)$ and applying (2.8) once more, we thus obtain a net of n-tuples $(x_1^\alpha, \ldots, x_n^\alpha)$ valued in $B(\ell_2)$ such that $\| \sum_{i=1}^n x_i^\alpha \otimes \overline{x_i^\alpha}\|^{1/2} \leq C$ and x_i^α converges to x_i in the ω^*-topology of $B(\ell_2)^{**}$. By [P1, (2.11)] we then obtain a_i^α, b_i^α in $B(\ell_2)$ such that

$$\Big\|\Big(\sum_{i=1}^n a_i^\alpha a_i^{\alpha *}\Big)^{1/2}\Big\| \leq C, \quad \Big\|\Big(\sum_{i=1}^n b_i^{\alpha *} b_i^\alpha\Big)^{1/2}\Big\| \leq C \quad \text{and} \quad x_i^\alpha = a_i^\alpha + b_i^\alpha.$$

Refining the net we can assume that each a_i^α has a limit $a_i \in B(\ell_2)^{**}$ in the ω^*-topology of $B(\ell_2)^{**}$ and, similarly, $b_i^\alpha \longrightarrow b_i$. Hence we have

$$\Big\|\Big(\sum_{i=1}^n a_i a_i^*\Big)^{1/2}\Big\| \leq C, \quad \Big\|\Big(\sum_{i=1}^n b_i^* b_i\Big)^{1/2}\Big\| \leq C.$$

Since $x_i = a_i + b_i$, this completes the proof. ∎

3. **The predual operator space structure.** This section is devoted to Problem 1.1 and Theorem 1.2. Let Y be a Banach space and let $X = Y^*$ be its dual space. We give ourselves an o.s.s. on X and endow X^* with the associated dual o.s.s. The canonical inclusion $Y \subset X^*$ induces an o.s.s. on Y and it is clear from (2.2) that this o.s.s. is the only one on Y for which we could have $X = Y^*$ completely isometrically. This o.s.s. will be called the *predual operator space structure* of Y.

In the following, the notation $\mathcal{B}_Z$ stands for the closed unit ball of some Banach space Z.

PROPOSITION 3.1. *Let Y be a Banach space. Assume that its dual space $X = Y^*$ is equipped with an* o.s.s. *We endow Y with its predual* o.s.s. *Then the following are equivalent.*

(i) $X = Y^*$ completely isometrically.
(ii) For all $n \geq 1$, $\mathcal{B}_{M_n(X)}$ is $\sigma(X, Y)$-closed.
(iii) For all $n \geq 1$, $\mathcal{B}_{M_n(Y)}$ is $\sigma(X^*, X)$-dense in $\mathcal{B}_{M_n(X^*)}$.
(iv) For all $n \geq 1$, $M_n(Y)^{**} = M_n(X^*)$.

PROOF. We will use the Haagerup tensor product $\overset{h}{\otimes}$ for which the reader is referred to [BP, B1, BS, ER1, ER2] and will appeal to the formulae (3.2) and (3.3). Before stating them, note that we define a duality pairing between $M_n \otimes X$ and $M_n \otimes X^*$ by letting

$$\langle \varphi, x\rangle = \sum_{1 \leq i,j \leq n} \langle \varphi_{ij}, x_{ij}\rangle \tag{3.1}$$

for any $x = [x_{ij}] \in M_n \otimes X$ and $\varphi = [\varphi_{ij}] \in M_n \otimes X^*$. By restriction to the subspace $Y \subset X^*$, (3.1) also defines a pairing between $M_n \otimes Y$ and $M_n \otimes X$. Of course, in the assertion (ii) (resp. (iii)), the notation $\sigma(X, Y)$ (resp. $\sigma(X^*, X)$) stands for the topology $\sigma\big(M_n(X), M_n \otimes Y\big)$ (resp. $\sigma\big(M_n(X^*), M_n \otimes X\big)$) induced by (3.1).

https://doi.org/10.4153/CMB-1995-049-9 Published online by Cambridge University Press

In the following we linearly identify $M_n \otimes Y$ with $R_n \otimes Y \otimes C_n$ by letting $[y_{ij}] = \sum_{1\leq i,j\leq n} e_i \otimes y_{ij} \otimes e_j$. Then we have:

$$\left(R_n \overset{h}{\otimes} Y \overset{h}{\otimes} C_n\right)^* = M_n(Y^*) \tag{3.2}$$

under the duality (3.1). Similarly we have:

$$M_n(X)^* = R_n \overset{h}{\otimes} X^* \overset{h}{\otimes} C_n. \tag{3.3}$$

We first prove (i) $\Leftrightarrow$ (ii). Note that the injectivity of the Haagerup tensor product yields an isometric embedding $R_n \overset{h}{\otimes} Y \overset{h}{\otimes} C_n \subset R_n \overset{h}{\otimes} X^* \overset{h}{\otimes} C_n$. Therefore it follows from (3.3) that the polar in $M_n \otimes Y$ of the convex set $\mathcal{B}_{M_n(X)}$ is the closed unit ball of $R_n \overset{h}{\otimes} Y \overset{h}{\otimes} C_n$. Hence the bipolar theorem and (3.2) imply that:

$$\overline{\mathcal{B}_{M_n(X)}}^{\sigma(X,Y)} = \mathcal{B}_{M_n(Y^*)}$$

whence the result.

Let us now check that (iii) $\Leftrightarrow$ (iv). For all $n \geq 1$, the Banach space equality $X^* = Y^{**}$ induces an isomorphic identification between $M_n(X^*)$ and $M_n(Y^{**})$. Moreover $M_n(Y) \subset M_n(X^*)$ isometrically. Therefore Goldstine's theorem implies that:

$$\mathcal{B}_{M_n(Y)^{**}} = \overline{\mathcal{B}_{M_n(Y)}}^{\sigma(X^*,X)}$$

whence the result.

The implication (i) $\Rightarrow$ (iv) is an obvious consequence of (2.2). Thus it remains to show that (iv) $\Rightarrow$ (i). Assume (iv). It follows from (2.2) that $Y^{**} = X^*$ completely isometrically hence $Y^{***} = X^{**}$ completely isometrically. Since the canonical embeddings $X \subset X^{**}$ and $Y^* \subset Y^{***}$ are complete isometries (see [BP, ER1]), we obtain (i). ■

We now turn to a counterexample.

THEOREM 3.2. *Let $Y = B(\ell_2)$ and let $X = Y^* = B(\ell_2)^*$. There exists an* o.s.s. *on X such that for any* o.s.s. *on Y, the canonical identity map $j\colon Y^* \longrightarrow X$ is not a* c.b. *isomorphism.*

It will be clear from its proof that Theorem 3.2 remains valid when Y is any injective von Neumann algebra whose bidual Y^{**} is not injective. However we do not know whether Theorem 3.2 is true for any non-reflexive Banach space Y.

We should notice here that the natural o.s.s. on $B(\ell_2)$ induces via (1.1) a natural o.s.s. on $B(\ell_2)^*$. Of course this structure does not satisfy our Theorem 3.2 so we warn the reader that we will have to use different operator space structures on the Banach space $B(\ell_2)^*$ or on $B(\ell_2)$. In order to avoid confusion we shall denote by Y, X the new structures (to be defined later) on $B(\ell_2), B(\ell_2)^*$ and we shall use the notation

$$Y_o = B(\ell_2), \quad X_o = B(\ell_2)^* \tag{3.4}$$

for the natural operator space structures. The key idea in proving Theorem 3.2 is to construct an o.s.s. X on $B(\ell_2)^*$ for which $C_n(X^*) = \mathrm{CB}(\mathrm{OH}_n, X_o^*)$ under natural identification.

https://doi.org/10.4153/CMB-1995-049-9 Published online by Cambridge University Press

Then if Y is the predual o.s.s. on $B(\ell_2)$ associated to X, Lemma 2.2 shows that $C_n(X^*)$ and $C_n(Y^{**})$ are different for large n. Hence X and Y^* are not completely isometric. Although we cannot restrict ourselves to the study of the predual o.s.s. of Y in studying the completely isomorphic form of Problem 1.1, we will see that the idea above still works.

Before going through the proof of Theorem 3.2, we need to recall the definition of $(2,\mathrm{oh})$-summing maps which was introduced in [P1, Chapter 5]. Let E be an operator space. Let F be a Banach space and let $T\colon E \longrightarrow F$ be a linear map. We say that T is $(2,\mathrm{oh})$-*summing* provided that there exists a constant $C \geq 0$ such that for any $x_1,\ldots,x_n$ in E, we have:

$$(3.5) \qquad \Big(\sum_{j=1}^{n} \|T(x_j)\|^2\Big)^{1/2} \leq C\Big\|\sum_{j=1}^{n} x_j \otimes \overline{x_j}\Big\|^{1/2}.$$

We denote by $\pi_{2,\mathrm{oh}}(T)$ the smallest $C \geq 0$ for which (3.5) holds and by $\Pi_{2,\mathrm{oh}}(E,F)$ the space of all $(2,\mathrm{oh})$-summing maps $T\colon E \longrightarrow F$. It is easy to see that $\pi_{2,\mathrm{oh}}$ is a norm on $\Pi_{2,\mathrm{oh}}(E,F)$ with which this space is a Banach space.

We will make use of the following ideal property of $(2,\mathrm{oh})$-summing maps. Given two operator spaces E, E_1 and two Banach spaces F, F_1, if $T\colon E \longrightarrow F$ is $(2,\mathrm{oh})$-summing, if $R\colon F \longrightarrow F_1$ is bounded and if $S\colon E_1 \longrightarrow E$ is completely bounded, then RTS is $(2,\mathrm{oh})$-summing and:

$$(3.6) \qquad \pi_{2,\mathrm{oh}}(RTS) \leq \|R\|\pi_{2,\mathrm{oh}}(T)\|S\|_{\mathrm{cb}}.$$

Indeed, for all $x_j \in E_1$, the inequality $\|\sum S(x_j) \otimes \overline{S(x_j)}\| \leq \|S\|_{\mathrm{cb}}^2 \|\sum x_j \otimes \overline{x_j}\|$ holds (see *e.g.* [BP, Proposition 5.11]).

In the proof of the lemma below, we will use the following notation. Given any Hilbert space H, and $p \in [1,+\infty[$, we denote by $S_p(H)$ the Schatten space of all compact operators $T\colon H \longrightarrow H$ such that $tr|T|^p < +\infty$ equipped with the norm $T \longmapsto (tr|T|^p)^{1/p}$.

LEMMA 3.3. *Let E be an operator space. Let $n \geq 1$ and let $(\varphi_1,\ldots,\varphi_n)$ be a finite sequence in the dual operator space E^*. Then we have:*

$$(3.7) \qquad \Big\|\sum_{i=1}^{n} \varphi_i \otimes \overline{\varphi_i}\Big\|^{1/2} = \sup\Big\{\Big(\sum_{i=1}^{n} \|u(\varphi_i)\|^2\Big)^{1/2}\Big\}$$

where the supremum runs over all ω^*-continuous $u\colon E^* \longrightarrow \ell_2^n$ for which $\pi_{2,\mathrm{oh}}(u) \leq 1$.

PROOF. Let us denote by C the right hand side of (3.7). Applying [B2, Corollary 3.2], we obtain that there exist a Hilbert space H and a complete quotient map $Q\colon S_1(H) \longrightarrow E$. This means that all the $Q \otimes I_{M_n}$ are quotient maps. The dual map $Q^*\colon E^* \longrightarrow B(H)$ is then a complete isometry hence we have:

$$(3.8) \qquad \Big\|\sum_{i=1}^{n} \varphi_i \otimes \overline{\varphi_i}\Big\|^{1/2} = \sup\Big\{\Big(\sum_{i=1}^{n} \|aQ^*(\varphi_i)b\|_{S_2(H)}^2\Big)^{1/2}\Big\}$$

where the supremum runs over all a, b in $S_4(H)$ such that $\|a\|_{S_4(H)}, \|b\|_{S_4(H)} \leq 1$ (see [P1, (5.9)]). Now fix a, b in the unit ball of $S_4(H)$. The linear map $\theta\colon B(H) \longrightarrow S_2(H)$ defined by $\theta(c) = acb$ is $(2,\mathrm{oh})$-summing with $\pi_{2,\mathrm{oh}}(\theta) \leq 1$ (see [P1, Corollary 5.8]). Moreover, θ is

ω^*-continuous. Let $p: S_2(H) \longrightarrow S_2(H)$ be an orthogonal projection onto an n-dimensional subspace $K \subset S_2(H)$ containing all the $\theta\big(Q^*(\varphi_i)\big)$. Then the map $u = p\theta Q^*$ is a ω^*-continuous map from E^* into an n-dimensional Hilbert space and it satisfies

$$\sum_{i=1}^{n} \|u(\varphi_i)\|^2 = \sum_{i=1}^{n} \|aQ^*(\varphi_i)b\|^2_{S_2(H)}. \tag{3.9}$$

Furthermore, the ideal property (3.6) ensures that $\pi_{2,\mathrm{oh}}(u) \leq 1$. Therefore, (3.9) and (3.8) imply that $\| \sum_{i=1}^n \varphi_i \otimes \overline{\varphi_i}\|^{1/2} \leq C$. The converse inequality is obvious. ■

PROOF OF THEOREM 3.2. We use the notation (3.4). First we claim that there exists an o.s.s. X on $B(\ell_2)^*$ such that for any finite sequence $(x_1, \dots, x_n)$ in X, we have:

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{C_n(X)} = \sup\Big\{ \Big(\sum_{j=1}^{n} \|S(x_j)\|^2\Big)^{1/2} / m \geq 1, S \in \mathrm{CB}(X_o, \mathrm{OH}_m), \|S\|_{\mathrm{cb}} \leq 1 \Big\}. \tag{3.10}$$

Indeed, if we denote by $\mathcal{B}_m$ the closed unit ball of $\mathrm{CB}(X_o, \mathrm{OH}_m)$, the map

$$J: B(\ell_2)^* \longrightarrow \bigoplus_{m\geq 1}^{\infty} \bigoplus_{\mathcal{B}_m}^{\infty} C_m, \quad J(x) = \big(S(x)\big)_{m \geq 1, S \in \mathcal{B}_m}$$

is an isometry. Therefore the natural o.s.s. on $\oplus_{m\geq 1}^{\infty} \oplus_{\mathcal{B}_m}^{\infty} C_m$ induces via J an o.s.s. X on $B(\ell_2)^*$ and this structure clearly satisfies (3.10). We refer the reader to [P1, Proposition 4.8] or [BP, pp. 279–281] for more about such constructions.

We wish to show that the o.s.s. defined above (in fact, any o.s.s. on $B(\ell_2)^*$ for which (3.10) holds) satisfies Theorem 3.2. For we assume that there exists an o.s.s. Y on $B(\ell_2)$ for which the canonical map $j: Y^* \longrightarrow X$ is a c.b. isomorphism. Our aim is to prove that $\mathrm{CB}(\mathrm{OH}_n, Y_o)$ and $C_n(Y)$ are uniformly isomorphic, see (3.16). We let $K_1 = \|j\|_{\mathrm{cb}}$ and $K_2 = \|j^{-1}\|_{\mathrm{cb}}$.

From (2.3), we have $K_1 = \|j^*\|_{\mathrm{cb}}$ hence for all $n \geq 1, j^* \otimes I_{C_n}: C_n(X^*) \longrightarrow C_n(Y^{**})$ has norm less than K_1. Since $C_n(Y)^{**} = C_n(Y^{**})$ isometrically (see (2.2)) we obtain that:

(3.11) The inclusion maps $C_n(X^*) \longrightarrow C_n(Y)^{**}$ have norm $\leq K_1$.

We now need a few computations on the norms. First let us consider $x_1, \dots, x_n$ in X. We can form the map $u = \sum_{j=1}^n x_j \otimes e_j: X_o^* \longrightarrow \ell_2^n$ by letting $u(\varphi) = \big(\varphi(x_j)\big)_{1 \leq j \leq n}$ for all $\varphi \in X_o^*$. Then we have:

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{C_n(X)} = \pi_{2,\mathrm{oh}}(u). \tag{3.12}$$

Indeed, given $m \geq 1$, let $S: X_o \longrightarrow \mathrm{OH}_m$ be a linear map. It can be written as $S = \sum_{i=1}^m \varphi_i \otimes e_i$ with $\varphi_i = S^*(e_i) \in X_o^*$ and it then follows from (2.3), (2.6) and (2.8) that $\|S\|_{\mathrm{cb}} = \| \sum_{i=1}^m \varphi_i \otimes \overline{\varphi_i}\|^{1/2}$. Therefore the equality (3.10) becomes:

$$\begin{aligned} \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{C_n(X)} &= \sup\Big\{ \Big(\sum_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} |\langle \varphi_i, x_j \rangle|^2 \Big)^{1/2} / \varphi_1, \dots, \varphi_m \in X_o^*, \Big\| \sum_{i=1}^m \varphi_i \otimes \overline{\varphi_i} \Big\| \leq 1 \Big\} \\ &= \sup\Big\{ \Big(\sum_{i=1}^m \|u(\varphi_i)\|^2 \Big)^{1/2} / \varphi_1, \dots, \varphi_m \in X_o^*, \Big\| \sum_{i=1}^m \varphi_i \otimes \overline{\varphi_i} \Big\| \leq 1 \Big\} \end{aligned}$$

whence (3.12). Note that the latter equality establishes an isometry from $C_n(X)$ onto the closed subspace of $\Pi_{2,\mathrm{oh}}(X_o^*, \ell_2^n)$ of all ω^*-continuous maps.

Now let $\varphi_1, \ldots, \varphi_n$ in X^* and let $v\colon \mathrm{OH}_n \longrightarrow X_o^* = Y_o^{**}$ be defined by $v(e_i) = \varphi_i$ for all $1 \leq i \leq n$. We claim that:

$$\left\| \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right\|_{C_n(X^*)} = \|v\|_{\mathrm{cb}}. \tag{3.13}$$

It should be noticed that this equality gives rise to the isometric identification:

$$C_n(X^*) = \mathrm{CB}(\mathrm{OH}_n, Y_o^{**}). \tag{3.14}$$

To check (3.13), we apply the duality formula (2.5) with $E = X$. Then by (3.12),

$$\left\| \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right\|_{C_n(X^*)} = \sup\Big\{ \Big(\sum_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} |\langle \varphi_i, x_j\rangle|^2 \Big)^{1/2} / x_j \in X_o, \pi_{2,\mathrm{oh}}\Big(\sum x_j \otimes e_j \colon X_o^* \longrightarrow \ell_2^n\Big) \leq 1 \Big\}.$$

However the w^*-continuous maps $u\colon X_o^* \longrightarrow \ell_2^n$ are exactly those which can be written as $u = \sum x_j \otimes e_j$ hence:

$$\left\| \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right\|_{C_n(X^*)} = \sup\Big\{ \Big(\sum_{i=1}^n \|u(\varphi_i)\|^2 \Big)^{1/2} \Big\}$$

where the supremum runs over all ω^*-continuous $u\colon X_o^* \longrightarrow \ell_2^n$ for which $\pi_{2,\mathrm{oh}}(u) \leq 1$. Applying Lemma 3.3 with $E = X_o$ together with (2.8), we obtain (3.13).

We are now ready to conclude the proof. Since the map $j^{*-1}\colon Y^{**} \longrightarrow X^*$ satisfies $\|j^{*-1}\|_{\mathrm{cb}} = K_2$, we deduce from (3.13) that:

(3.15) The inclusion maps $C_n(Y^{**}) \longrightarrow \mathrm{CB}(\mathrm{OH}_n, Y_o^{**})$ have norm $\leq K_2$.

Note that we have commutative diagrams

$$\begin{array}{ccc} C_n(Y^{**}) & \longrightarrow & \mathrm{CB}(\mathrm{OH}_n, Y_o^{**}) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ C_n(Y) & \longrightarrow & \mathrm{CB}(\mathrm{OH}_n, Y_o) \end{array}$$

with isometric maps α, β. Therefore (3.15) implies that:

(3.16) The inclusion maps $C_n(Y) \longrightarrow \mathrm{CB}(\mathrm{OH}_n, Y_o)$ have norm $\leq K_2$.

Combining (3.11), (3.14) and (3.16), we obtain that the inclusion maps $\mathrm{CB}(\mathrm{OH}_n, Y_o^{**}) \longrightarrow \mathrm{CB}(\mathrm{OH}_n, Y_o)^{**}$ have norm less than K_1K_2 and this contradicts Lemma 2.2. ∎

PROOF OF THEOREM 1.2. Let $Y = B(\ell_2)$. We equip $X = Y^*$ with an o.s.s. which satisfies Theorem 3.2. We give ourselves a complete isometry $J\colon X \longrightarrow B(H)$ and assume that J is $(\sigma(X,Y),\sigma\big(B(H),S_1(H)\big))$-continuous. To reach a contradiction, it then suffices to define an o.s.s. on Y for which $X = Y^*$ completely isometrically.

Let $E = J(X)$, $N = E_\perp \subset S_1(H)$ and let $\bar{J}: X \longrightarrow E$ be the onto map induced by J. Then the σ-weak continuity of J implies that E is $\sigma\big(B(H),S_1(H)\big)$-closed. Hence

$$(S_1(H)/N)^* = E \tag{3.17}$$

and moreover $\bar{J}$ is the adjoint map of an isometry

$$\theta: S_1(H)/N \longrightarrow Y \tag{3.18}$$

Let us equip $S_1(H)/N$ with the quotient o.s.s. of $S_1(H)$. This means that once $S_1(H)$ is equipped with its natural o.s.s., we let $M_n\big(S_1(H)/N\big)$ be the quotient space $M_n\big(S_1(H)\big)/M_n(N)$. As is well-known (see *e.g.* [B2, Corollary 2.4]), (3.17) then holds completely isometrically. Via (3.18), the quotient o.s.s. on $S_1(H)/N$ defines an o.s.s. on Y. By the complete isometry (3.17), we thus have $X = Y^*$ completely isometrically, whence the result. ∎

4. **Completely bounded maps from $B(\ell_2)^*$ into OH.** The aim of this section is to give one more application of the lack of OH-local reflexivity of $B(\ell_2)$. Namely we will prove that c.b. maps from $B(\ell_2)^*$ into OH are not necessarily $(2,\mathrm{oh})$-summing. In order to explain the relevance of this result, we recall that the notion of $(2,\mathrm{oh})$-summing operators is a generalization to the framework of operator spaces of the classical notion of 2-summing operators between Banach spaces. We refer the reader to [P3] for the definition and the most classical results about 2-summing operators. We merely recall that whenever E is a minimal operator space (see [BP] for example), then for any operator $u: E \longrightarrow F$ from E into a Banach space F, u is 2-summing iff u is $(2,\mathrm{oh})$-summing and (in this case), $\pi_{2,\mathrm{oh}}(u)$ is the 2-summing norm of u [P1, (9.3)]. The little Grothendieck Theorem (see *e.g.* [P3, Theorem 5.4]) states that for all compact sets K, any bounded map from $C(K)$ into a Hilbert space in 2-summing. Equivalently (see [P3, th. 5.10]) any bounded map from a L^1-space into a Hilbert space is 2-summing. We are thus led to study what are the operator spaces X for which we have:

$$\mathrm{CB}(X,\mathrm{OH}) = \Pi_{2,\mathrm{oh}}(X,\ell_2). \tag{4.1}$$

Note that by [P1, Corollary 7.5], the inclusion "$\supset$" is always achieved. Because of the little Grothendieck Theorem, the question of whether (4.1) holds is especially interesting when X is a C^*-algebra or a non-commutative L^1-space, *i.e.* the predual of a von Neumann algebra. This problem is raised in [P1, Problem 10.2]. Let us mention a simple counterexample.

EXAMPLE 4.1. Let $\mathbb{F}_\infty$ be the free group with infinitely many generators $(f_n)_{n\geq 1}$ and let $C^*_\lambda(\mathbb{F}_\infty)$ be the reduced C^*-algebra of $\mathbb{F}_\infty$, *i.e.* the C^*-algebra generated by the left regular representation $\lambda: \mathbb{F}_\infty \longrightarrow B(\ell_{2,\mathbb{F}_\infty})$. Then $X = C^*_\lambda(\mathbb{F}_\infty)$ does not satisfy (4.1). To show this, let us denote by F the closed subspace of X spanned by the sequence $\big(\lambda(f_n)\big)_{n\geq 1}$ and let $R\cap C$ be the o.s.s. defined on ℓ_2 by letting

$$\|[x_{ij}]\|_{M_n(R\cap C)} = \max\{\|[x_{ij}]\|_{M_n(R)}, \|[x_{ij}]\|_{M_n(C)}\}$$

https://doi.org/10.4153/CMB-1995-049-9 Published online by Cambridge University Press

for all $[x_{ij}] \in M_n \otimes \ell_2$. Then Haagerup and Pisier [HP, Section 1] proved that the linear transformation which maps e_i to $\lambda(f_i)$ for all $i \geq 1$ is a c.b. isomorphism between $R \cap C$ and F and that the natural projection $P: X \longrightarrow F \approx R \cap C$ is completely bounded. Since $\| \mathrm{Id}: R \cap C \longrightarrow \mathrm{OH} \|_{\mathrm{cb}} \leq 1$ (see [P1, (2.11)]), P becomes a c.b. map from X into OH. However for any $n \geq 2$, $\sum_{i=1}^n \left\| P\big(\lambda(f_i)\big) \right\|^2 = \sum_{i=1}^n \|\lambda(f_i)\|^2 = n$ and by [AO] we have $\| \sum_{i=1}^n \lambda(f_i) \otimes \overline{\lambda(f_i)} \| = 2\sqrt{n-1}$. Therefore P is not (2, oh)-summing.

The result to be proved now is the following

THEOREM 4.2. *$B(\ell_2)^*$ does not satisfy (4.1), i.e.*

$$\mathrm{CB}\big(B(\ell_2)^*, \mathrm{OH}\big) \neq \Pi_{2,\mathrm{oh}}\big(B(\ell_2)^*, \ell_2\big).$$

The proof of the latter theorem is based on two lemmas.

LEMMA 4.3. *Let X be an operator space. If the dual operator space X^* satisfies (4.1), then X satisfies (4.1).*

PROOF. We assume that $\mathrm{CB}(X^*, \mathrm{OH}) = \Pi_{2,\mathrm{oh}}(X^*, \ell_2)$. Clearly this implies that there exists a constant $C > 0$ such that:

$$\forall n \geq 1,\ \forall S: X^* \longrightarrow \mathrm{OH}_n, \quad \pi_{2,\mathrm{oh}}(S) \leq C\|S\|_{\mathrm{cb}}. \tag{4.2}$$

Let $u: X \longrightarrow \mathrm{OH}$ be a c.b. map. We give ourselves a finite family $(x_1, \ldots, x_n)$ in X and denote by $T: \mathrm{OH}_n \longrightarrow X$ the operator defined by $T(e_i) = x_i$. Then we have:

$$\begin{aligned}
\Big(\sum_{i=1}^n \|u(x_i)\|^2\Big)^{1/2} &= \|uT\|_{HS} = \|T^*u^*\|_{HS} = \Big(\sum_{j=1}^\infty \|T^*(u^*e_j)\|^2\Big)^{1/2} \\
&\leq \pi_{2,\mathrm{oh}}(T^*)\|u^*\|_{\mathrm{cb}} \quad \text{by (2.8)} \\
&\leq C\|T\|_{\mathrm{cb}}\|u\|_{\mathrm{cb}} \quad \text{by (4.2) and (2.3)} \\
&\leq C\|u\|_{\mathrm{cb}} \Big\| \sum_{i=1}^n x_i \otimes \overline{x_i} \Big\|^{1/2} \quad \text{by (2.8).}
\end{aligned}$$

Consequently, the map u is (2, oh)-summing, which concludes the proof. ∎

LEMMA 4.4. *Let X be an operator space. For all $n \geq 1$, the identity map $\Pi_{2,\mathrm{oh}}(X^{**}, \ell_2^n) \longrightarrow \Pi_{2,\mathrm{oh}}(X, \ell_2^n)^{**}$ is a contraction.*

PROOF. We claim that there is an o.s.s. on X (which we denote by X_1) such that for all $n \geq 1$ and for all $x_1, \ldots, x_n$ in X we have:

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{C_n(X_1)} = \Big\| \sum_{i=1}^n x_i \otimes \overline{x_i} \Big\|^{1/2} \tag{4.3}$$

Indeed using (3.7) for example, we can write that for all $x_1, \ldots, x_n$,

$$\Big\| \sum_{i=1}^n x_i \otimes \overline{x_i} \Big\|^{1/2} = \sup\Big\{ \Big(\sum_{j=1}^n \|u(x_i)\|^2\Big)^{1/2} / m \geq 1, u \in \Pi_{2,\mathrm{oh}}(X, \ell_2^m), \pi_{2,\mathrm{oh}}(u) \leq 1 \Big\}.$$

https://doi.org/10.4153/CMB-1995-049-9 Published online by Cambridge University Press

Hence the claim (4.3) has the same proof as the claim (3.10). Together with (2.8), the above formula (4.3) yield $\mathrm{CB}(\mathrm{OH}_n, X) = C_n(X_1)$. Passing to the bidual via (2.2), we then obtain $C_n(X_1^{**}) = \mathrm{CB}(\mathrm{OH}_n, X)^{**}$. Since the identity map $\mathrm{CB}(\mathrm{OH}_n, X)^{**} \longrightarrow \mathrm{CB}(\mathrm{OH}_n, X^{**})$ is a contraction, we thus deduce from (2.8) that for any $\eta_1, \ldots, \eta_n$ in X^{**}:

$$(4.4) \qquad \Big\| \sum_{i=1}^{n} \eta_i \otimes \overline{\eta_i} \Big\|^{1/2} \leq \left\| \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \right\|_{C_n(X_1^{**})}.$$

Now let us apply the duality formula (2.5) with $E = X_1$. This yields $\Pi_{2,\mathrm{oh}}(X, \ell_2^n) = C_n(X_1^*)$ and consequently we have by (2.2):

$$(4.5) \qquad \Pi_{2,\mathrm{oh}}(X, \ell_2^n)^{**} = C_n(X_1^{***}).$$

Now let $T: X^{**} \longrightarrow \ell_2^n$ be a linear map. From (2.5) and (4.4), we have $\|T\|_{C_n(X_1^{***})} \leq \pi_{2,\mathrm{oh}}(T)$. Taking into account the identification (4.5), this completes the proof. ■

PROOF OF THEOREM 4.2. Assume that $B(\ell_2)^*$ satisfies (4.1). Then Lemma 4.3 implies that $S_1(\ell_2)$ satisfies (4.1) as well. Therefore applying Lemma 4.4 with $X = S_1(\ell_2)$, we obtain a constant $C > 0$ for which:

$$\text{The identity maps } \mathrm{CB}\big(B(\ell_2)^*, \mathrm{OH}_n\big) \longrightarrow \mathrm{CB}\big(S_1(\ell_2), \mathrm{OH}_n\big)^{**} \text{ have norm } \leq C.$$

But applying (2.3) and the self-duality of OH (2.6), this means that the maps $\mathrm{CB}\big(\mathrm{OH}_n, B(\ell_2)^{**}\big) \longrightarrow \mathrm{CB}\big(\mathrm{OH}_n, B(\ell_2)\big)^{**}$ have norm less than C and this contradicts Lemma 2.2. ■

REMARK 4.5. Clearly the above proof works as well if we replace $B(\ell_2)$ by any injective von Neumann algebra M whose bidual is non-injective. We thus obtain:

$$\mathrm{CB}(M^*, \mathrm{OH}) \neq \Pi_{2,\mathrm{oh}}(M^*, \ell_2).$$

Moreover, applying Lemma 4.3, we obtain the same negative result for duals of any order of such M. On the other hand, the problem of whether (4.1) is true for the spaces $B(H)$, $S_1(H)$ and $K(H)$ remains open.

It should be noticed here that the converse of Lemma 4.3 is not available. Actually, it seems likely that this is false. The obstacle to proving the converse of Lemma 4.3 is the fact that in general, the equality $\pi_{2,\mathrm{oh}}(u) = \pi_{2,\mathrm{oh}}(u^{**})$ does not hold. This feature is related to the lack of OH-local reflexivity. In order to emphasize this, we state a last result whose proof is left to the reader.

Let E be an operator space and let $C > 0$ be a constant. The following are equivalent.

(i) X is OH-locally reflexive and (2.9) holds.

(ii) For any finite dimensional Banach space F and any $u \in \Pi_{2,\mathrm{oh}}(X, F)$ we have $u^{**} \in \Pi_{2,\mathrm{oh}}(X^{**}, F)$ and $\pi_{2,\mathrm{oh}}(u^{**}) \leq C\pi_{2,\mathrm{oh}}(u)$.

(iii) For all $n \geq 1$, the identity map

$$\Pi_{2,\mathrm{oh}}(X, \ell_2^n)^{**} \longrightarrow \Pi_{2,\mathrm{oh}}(X^{**}, \ell_2^n) \quad \text{has norm } \leq C.$$

(iv) The identification $\Pi_{2,\mathrm{oh}}(X, \ell_2)^{**} = \Pi_{2,\mathrm{oh}}(X^{**}, \ell_2)$ is a C-isomorphism.

ACKNOWLEDGMENTS. I am grateful to Gilles Pisier for many fruitful and stimulating discussions. I also thank the referee for several valuable suggestions which improved the paper.

REFERENCES

[AO] C. Akeman and P. Ostrand, *Computing norms in group C^*-algebras*, Amer. J. Math. **98**(1976), 1015–1047.

[B1] D. Blecher, *Tensor products of operator spaces II*, Canad. J. Math. **44**(1992), 75–90.

[B2] ———, *The standard dual of an operator space*, Pacific J. Math. **153**(1992), 15–30.

[BP] D. Blecher and V. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. **99**(1991), 262–292.

[BS] D. Blecher and R. Smith, *The dual of the Haagerup tensor product*, J. London Math. Soc. **45**(1992), 126–144.

[D] D. Dean. *The equation $L(E, X^{**}) = L(E, X)^{**}$ and the principle of local reflexivity*, Proc. Amer. Math. Soc. **40**(1973), 146–148.

[EH] E. Effros and U. Haagerup, *Lifting problems and local reflexivity for C^*-algebras*, Duke Math. J. **52** (1985), 103–128.

[ER1] E. Effros and Z.-J. Ruan, *A new approach to operator spaces*, Canad. Math. Bull. **34**(1991), 329–337.

[ER2] ———, *Self duality for the Haagerup tensor product and Hilbert space factorization*, J. Func. Anal. **100**(1991), 257–284.

[ER3] ———, *Mapping spaces and liftings for operator spaces*, Proc. London Math. Soc. **69**(1994), 171–197.

[ER4] ———, *On approximation properties for operator spaces*, Internat. J. Math. **1**(1990), 163–187.

[HP] U. Haagerup and G. Pisier, *Bounded linear operators between C^*-algebras*, Duke Math. J. **71**(1993), 889–925.

[L] C. Le Merdy, *Analytic factorizations and completely bounded maps*, Israel J. Math. **88**(1994), 381–409.

[P1] G. Pisier, *The operator Hilbert space* OH, *complex interpolation and tensor norms*, to appear.

[P2] ———, *Espace de Hilbert d'opérateurs et interpolation complexe*, C. R. Acad. Sci. Paris Série I Math. **316**(1993), 47–52.

[P3] ———, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conf. Amer. Math. Soc. **60**(1986).

[R] Z.-J. Ruan. *Subspaces of C^*-algebras*, J. Funct. Anal. **76**(1988), 217–230.

[S1] S. Sakai. *A characterization of W^*-algebras*, Pacific J. Math. **6**(1956), 763–773.

[S2] ———, *C^*-algebras and W^*-algebras*, Springer Verlag, 1971.

[T] M. Takesaki, *Theory of operator algebras I*, Springer Verlag, 1979.

Equipe de Mathématiques
URA CNRS 741
Université de Franche-Comté
F-25030 Besançon Cedex
France