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THE CLASSIFICATION OF GROUPS WITH THE SMALL SQUARING PROPERTY ON 3-SETS

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Let G be a group and k an integer greater than 1. We say that G has the square property of k-sets and we write $G \in DS(k)$ if $\left|X^2\right| < k^2$ for any subset X of G of order k. The groups in $DS(2)$ are exactly the *Dedekind* groups, as Freiman showed. The class $DS(3)$ has been recently studied by Berkovich, Freiman and Praeger. In this paper we complete the classification of $DS(3)$-groups by characterising finite 2-groups of exponent 4 in $DS(3)$.

1. Introduction

Let G be a group and k an integer greater than 1.

Following [1] we say that G has the square property of k-sets and we write $G \in DS(k)$ if $\left|X^2\right| < k^2$ for any subset X of G of order k.

The groups in $DS(2)$ are exactly the *Dedekind* groups, as Freiman showed in [2]. The class $DS(3)$ has been recently studied by Berkovich, Freiman and Praeger. They came close to a classification of the finite $DS(3)$-groups. There was only one case left, the case where G is a finite 2-group of exponent 4.

In this paper we complete the classification of $DS(3)$-groups. We prove the following:

Theorem A. *Let G be a finite 2-group of exponent 4. Then $G \in DS(3)$ if and only if one of the following holds:*

(1) *G is abelian,*
(2) *$\mho_1(G) = \langle x^2 / x \in G\rangle$ has order 2,*
(3) *$G = A\langle x\rangle$, where A is abelian, $x^2 \in A$, $a^x = a^{-1}$ for every $a \in A$,*
(4) *$G = D \times \langle a, b, c\rangle$, where D is an elementary abelian 2-group, $|a| = |b| = |c| = 4$, $a^b = a^{-1}$, $[a, c] = [b, c] = 1$, $c^2 = a^2b^2$.*
(5) *$G = D \times \langle a, b, c, d\rangle$, where D is an elementary abelian 2-group, $a^b = a^{-1}$, $c^b = c^{-1}$, $c^d = c^{-1}$, $[a, c] = [a, d] = [b, d] = 1$, $|a| = |b| = |c| = |d| = 4$, $c^2 = d^2 = a^2b^2$.*

Theorem A together with the results of [1] gives the following:

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THEOREM B. *Let G be a finite group. Then $G \in DS(3)$ if and only if G is abelian, or $G = A\langle x\rangle$, where A is abelian, $x^2 \in A$, $a^x = a^{-1}$, for any $a \in A$, or G is a 2-group of exponent 4 satisfying (2) or (4) or (5) of Theorem A.*

It is not difficult to show that in fact Theorem B holds for any group (not necessarily finite) as we prove in Section 3.

Our notation is the usual one (see for instance [4]).

If G is a finite 2-group we put

$$\Omega_1(G) = \langle x \in G / |x| = 2\rangle, \quad \mho_1(G) = \langle x^2 / x \in G\rangle.$$

2. FINITE 2-GROUPS OF EXPONENT 4 IN $DS(3)$

LEMMA 2.1. *Let $G \in DS(3)$ be a finite 2-group of exponent 4. Then $G/Z(G)$ has exponent 2. In particular G is nilpotent of class at most 2.*

PROOF: We prove that $x^2 \in Z(G)$ for every $x \in G$. Write $\Phi(G)$ for the *Frattini* subgroup of G. Then it suffices to show that $x^2y = yx^2$ for every $y \in G - \Phi(G)$ such that $x\Phi(G) \neq y\Phi(G)$. Assume by contradiction that there exist $x, y \in G$ with $y\Phi(G) \neq \Phi(G)$, $x^2y \neq yx^2$ and $x\Phi(G) \neq y\Phi(G)$. If $|y| = 2$, then $|x^2y| = 4$; hence we may assume, replacing y with x^2y if necessary, that $|y| = 4$.

For any $g \in G - \langle x^2, y\rangle$ consider the set $X = \{x^2, y, g\}$. Then from $|X^2| < 9$ it follows that $g \in C_G(x^2) \cup C_G(y)$ or $g^2 = y^2$ or $g^2 = 1$ (from $g^2 = x^2y$ or $g^2 = yx^2$ we get $y \in \Phi(G)$, a contradiction).

In particular either $(xy)^2 = 1$ or $(xy)^2 = y^2$ since $[xy, x^2] \neq 1$ and $[xy, y] \neq 1$. But $(xy)^2 = y^2$ implies $xyxy = y^2$, $y^{-1}xy = x^{-1}$ and $x^2y = yx^2$, a contradiction. Therefore $(xy)^2 = 1$; hence $\langle x^2, xy\rangle \cong D_4$ and $x^2xy = x^{-1}y$ has order 4. Thus, as before, from $x^{-1}y \notin C_G(x^2) \cup C_G(y)$ it follows that $(x^{-1}y)^2 = y^2$, and hence $x^{-1}yx^{-1}y = y^2$, $y^{-1}x^{-1}y = x$ and $x^2y = yx^2$, again a contradiction. ∎

In the following G will always be a finite 2-group of exponent 4. We apply Lemma 2.1 to get the following useful result:

LEMMA 2.2. *Let $G \in DS(3)$ and x, y be elements of G such that $[x, y] \neq 1$ and $x^2 \neq y^2$. Then for every $g \in G$ we have $g \in C_G(x) \cup C_G(y)$ or $g^2 = x^2$ or $g^2 = y^2$.*

PROOF: The result follows easily from $|\{x, y, g\}^2| < 9$, using the fact that $G' \leqslant Z(G)$ and $g^2 \in Z(G)$. ∎

The following lemma gives some more precise information.

LEMMA 2.3. *Let $G \in DS(3)$ and x, y be elements of G such that $[x, y] \neq 1$ and $x^2 \neq y^2$. Assume $(xy)^2 = y^2$. Then for every $g \in G$ we have $g \in C_G(x)$ or $g^2 = x^2$ or $g^2 = y^2$.*

PROOF: Let $g \in G$. Then, by Lemma 2.2, $g \in C_G(x) \cup C_G(y)$ or $g^2 = x^2$ or $g^2 = y^2$. Similarly from $(xy)^2 = y^2$ it follows, by Lemma 2.2, that $g \in C_G(x) \cup C_G(xy)$ or $g^2 = x^2$ or $g^2 = y^2$. If $g \in C_G(y) \cap C_G(xy)$, then $g \in C_G(x)$. Hence $g \in C_G(x)$ or $g^2 = x^2$ or $g^2 = y^2$, as required. ∎

Let G be a finite non-abelian group in $DS(3)$ of exponent 4. If $\langle x \in G / |x| = 4 \rangle$ is a proper subgroup of G, then G is a D-group with the notation of [1]. Then, by the results of [1], (3) of Theorem A holds.

Thus in the following *we assume* $G = \langle x \in G / |x| = 4 \rangle$.

First we remark:

LEMMA 2.4. *Let $a, b \in G$ be such that $a^2 \neq b^2$, $a^b = a^{-1}$, $|a| = 4$. Then $\langle a \rangle$ is normal in G.*

PROOF: First we prove that $[a, g] \in \{a^2, b^2, 1\}$ for every $g \in G$. Let $g \in G - C_G(a)$. Then by Lemma 2.3 either $g^2 = a^2$ or $g^2 = b^2$, and similarly either $(ag)^2 = a^2$ or $(ag)^2 = b^2$. Furthermore we have $(ag)^2 = a^2 g^2 [a, g]$. If $g^2 = a^2$, then $[a, g] = (ag)^2 \in \{a^2, b^2\}$. If $g^2 = b^2$, then $[a, g] = (ag)^2 a^2 b^2$ and $[a, g] = b^2$ if $(ag)^2 = a^2$, while $[a, g] = a^2$ if $(ag)^2 = b^2$. Now, if $|b| = 2$ then $[a, g]$ is always in $\langle a \rangle$, and $\langle a \rangle$ is normal in G, as required. If $|b| = 4$ and $[a, g] = b^2$ for some $g \in G$, then $[a, bg] = a^2 b^2 \notin \{1, a^2, b^2\}$, a contradiction. ∎

Now we study groups satisfying our conditions and with $\Omega_1(G) \not\subseteq Z(G)$.

2.5. *Assume there exists $g \in G$ with $|g| = 2$, $g \notin Z(G)$. Then $|\mho_1(G)| = 2$ and (2) of Theorem A holds.*

PROOF: Let $g \notin Z(G)$, with $|g| = 2$. Then there exists $a \in G$, with $|a| = 4$, $ag \neq ga$, since $G = \langle x \in G / |x| = 4 \rangle$. By Lemma 2.2 either $(ag)^2 = a^2$ or $(ag)^2 = 1$. From $(ag)^2 = a^2$ it follows that $agag = a^2$, whence $ag = ga$, a contradiction. Hence $(ag)^2 = 1$, and $g^{-1} a g = a^{-1}$. Thus by Lemma 2.4 we have $\langle a \rangle$ normal in G. Moreover for every $x \in G$, $|x| = 4$, either $a^2 = x^2$ or $x \in C_G(a)$ by Lemma 2.3. There exists $b \in G$, with $|b| = 4$ and $ab \neq ba$ since $a \notin Z(G)$. Then $b^2 = a^2$ and $|ab| = 4$. If $[b, g] = 1$, then $[ab, g] \neq 1$. Without loss of generality we may assume $[b, g] \neq 1$. Then, arguing on b as previously on a, we get $\langle b \rangle$ normal in G and either $x^2 = b^2 = a^2$ or $x \in C_G(b)$ for every $x \in G$, $|x| = 4$.

Now assume by contradiction that there is an element $x \in G$, with $x^2 \neq a^2$ and $|x| = 4$. Thus $x \in C_G(a)$ and $x \in C_G(b)$. But then $ax \notin C_G(b)$ and $|ax| = 4$, hence $(ax)^2 = b^2 = a^2$ and $x^2 = 1$, a contradiction. ∎

It will be shown later that groups satisfying (1) to (5) of Theorem A have the property $DS(3)$.

Now we assume $\Omega_1(G) \leqslant Z(G)$. We start with two particular cases.

Case 2.6. *Let $a, b \in G$ be such that $|a| = |b| = 4$, $a^2 \neq b^2$ and $a^b = a^{-1}$. Assume $G = \langle a, b\rangle(C_G(a) \cap C_G(b))$ and that (3) does not hold. Then (4) of Theorem A holds.*

Proof: First we remark that $y^2 \in \langle a^2b^2\rangle$ for every $y \in C_G(a) \cap C_G(b)$. In fact let $y \in C_G(a) \cap C_G(b)$. Then $yb \notin C_G(a)$ and, by Lemma 2.3, either $(yb)^2 = a^2$ or $(yb)^2 = b^2$. If $(yb)^2 = a^2$, then $y^2b^2 = a^2$ and $y^2 = a^2b^2$, while if $y^2b^2 = (yb)^2 = b^2$, then $y^2 = 1$.

Now we prove that $C_G(a) \cap C_G(b)$ is abelian. Let $y_1, y_2 \in C_G(a) \cap C_G(b)$ and, by contradiction, assume $[y_1, y_2] \neq 1$. Then $|y_1| = |y_2| = 4$, since $\Omega_1(G) \leqslant Z(G)$. The elements ay_1, by_1 do not permute, and $(ay_1)^2 = a^2y_1^2 \neq b^2y_1^2 = (by_1)^2$. Then Lemma 2.2 applies and $y_2 \in C_G(ay_1) \cup C_G(by_2)$ or $y_2^2 = (ay_1)^2 = a^2y_1^2$ or $y_2^2 = (by_1)^2 = b^2y_1^2$. But $y_2 \notin C_G(y_1)$ and $y_2^2 = y_1^2$, a contradiction. Therefore $C_G(a) \cap C_G(b)$ is an abelian group of exponent 4 and with $\mho_1(C_G(a) \cap C_G(b)) = \langle a^2b^2\rangle$. Thus $C_G(a) \cap C_G(b) = D \times \langle c\rangle$, where $\exp D = 2$ and $c^2 = a^2b^2$, and (4) of Theorem A holds. ∎

Case 2.7. *Assume that there exist $a, b, c, d \in G$ such that $|a| = |b| = |c| = |d| = 4$, and*

(1) $\langle a, c\rangle = \langle a\rangle \times \langle c\rangle$,
(2) $a^b = a^{-1}$, $c^b = c^{-1}$, $b^2 \neq a^2, c^2$,
(3) $a^d = a$, $b^d = b$, $c^d = c^{-1}$.

Then $c^2 = d^2 = a^2b^2$, and (5) of Theorem A holds.

Proof: First we prove that $c^2 = d^2 = a^2b^2$. From $bd \notin C_G(a)$, it follows that either $(bd)^2 = a^2$ or $(bd)^2 = b^2$, by Lemma 2.3. If $(bd)^2 = b^2$ then we have $d^2 = 1$ which is a contradiction. Then $b^2d^2 = a^2$ and $d^2 = a^2b^2$. Similarly if $c^2 \neq d^2$, from $bc \notin C_G(c) \cup C_G(d)$ it follows, by Lemma 2.2, that either $(bc)^2 = c^2$ or $(bc)^2 = d^2$. But $(bc)^2 = b^2$, and so either $b^2 = c^2$ or $b^2 = d^2$, a contradiction.

Therefore $c^2 = d^2 = a^2b^2$.

Now write $D = C_G(a) \cap C_G(c)$. Then we have D normal in G and $|G/D| \leqslant 4$, since $\langle a\rangle$, $\langle c\rangle$ are normal in G by Lemma 2.4. Moreover $G = D\langle b, d\rangle$ because $b, d, bd \notin C_G(a) \cap C_G(c)$.

We prove that $g^2 \in \{1, a^2, b^2, a^2b^2\}$ for every $g \in D$. From that it follows that one of g, ga, gca, gc has order 2 and $D = \Omega_1(D)\langle a, c\rangle$: thus $D = \langle a\rangle \times \langle c\rangle \times Y$ where $Y \leqslant \Omega_1(D) \leqslant Z(G)$, and G has the required structure.

Assume by contradiction that there exists $g \in D$ with $g^2 \notin \{1, a^2, b^2, a^2b^2\}$. If g commutes with b, then from $bg \notin C_G(a)$ it follows by Lemma 2.3 that either $(bg)^2 = b^2g^2 = a^2$ or $b^2g^2 = (bg)^2 = b^2$, a contradiction. Similarly if $g \in C_G(d)$, then from $dg \notin C_G(c)$ it follows that either $d^2g^2 = (dg)^2 = c^2$ or $d^2g^2 = (dg)^2 = b^2$, a contradiction. Thus $g \notin C_G(b) \cup C_G(d)$. Now we have $g^2 \neq d^2 = c^2$. By Lemma 2.2, every element $x \in G$ is either in $C_G(g) \cup C_G(d)$ or is such that $x^2 = g^2$ or $x^2 = d^2$. Then $(cb)^2 = g^2$ or $(cb)^2 = d^2$. But $(cb)^2 = b^2$ and we get a contradiction. $\square$

Now we can complete our characterisation of finite groups of exponent 4 in $DS(3)$.

PROOF OF THEOREM A: Let $G \in DS(3)$ be a finite 2-group of exponent 4. Assume G non-abelian and $|\mho_1(G)| > 2$. Also suppose that (3) does not hold. Then every element of order 2 is in the centre of G by Lemma 2.5. Moreover there is an element $x \in G$ with $|x| = 4$ and $\langle x \rangle$ is not normal in G, since G is not a *Dedekind* group. Thus there exists $y \in G$ of order 4 with $y \notin N_G(\langle x \rangle)$. If $(xy)^2 = y^2$, then $y^{-1}xy = x^{-1}$ and $y \in N_G(\langle x \rangle)$, a contradiction. Hence $(xy)^2 \neq y^2$. Moreover $|xy| = 4$, because $xy \notin C_G(x)$. Therefore we have found two elements $a, b \in G$ with $ab \neq ba$, $a^2 \neq b^2$, $|a| = |b| = 4$. By Lemma 2.2 we have either $(ab)^2 = a^2$ or $(ab)^2 = b^2$. Without loss of generality we can assume $(ab)^2 = b^2$. Then $b^{-1}ab = a^{-1}$, and $\langle a \rangle$ is normal in G by Lemma 2.4. Moreover, for every $g \in G - C_G(a)$ we have either $g^2 = a^2$ or $g^2 = b^2$, by Lemma 2.3. Write $C = C_G(a)$. Then C is normal in G and $|G/C| = 2$.

Assume first $C \leqslant C_G(b) \cup a^{-1}C_G(b)$; then $G = \langle a, b \rangle (C \cap C_G(b))$. Now 2.6 applies and (4) holds. Then we can assume that there exists $c \in C - \left(C_G(b) \cup a^{-1}C_G(b)\right)$. Thus $|ac| > 2$ and $a^2 \neq c^2$. We claim that $c^b = c^{-1}$. In fact from $bc \notin C_G(a)$ it follows that either $(bc)^2 = a^2$ or $(bc)^2 = b^2$. If $(bc)^2 = b^2$, then $c^b = c^{-1}$. Assume by contradiction $(bc)^2 = a^2$. If $b^2 = c^2$, then $a^2 = (bc)^2 = b^2c^2[b, c]$, thus $[b, c] = a^2 = [b, a]$ and $[b, ac] = 1$, a contradiction. If $b^2 \neq c^2$, then from $[b, c] \neq 1$ it follows that either $(bc)^2 = b^2$ or $(bc)^2 = c^2$ by Lemma 2.2. But $(bc)^2 = a^2$ and $a^2 \neq b^2, c^2$ and we have a contradiction. Therefore we get $c^b = c^{-1}$. Moreover, replacing c by ca if necessary, we can assume $c^2 \notin \{1, a^2, b^2\}$. Then for every $g \in G - C_G(c)$ either $g^2 = b^2$ or $g^2 = c^2$ by Lemma 2.3.

Now we have $G = C\langle b \rangle$. If $d^b = d^{-1}$ for every $d \in C$, then C is abelian and (3) holds. Assume that there exists $d \in C$ such that $d^b \neq d^{-1}$. Then $(db)^2 \neq b^2$ and $(db)^2 = a^2$ since $db \notin C_G(a)$. Similarly from $db \notin C_G(c)$ it follows that either $(db)^2 = c^2$ or $(db)^2 = b^2$. But b^2, c^2 are different from a^2. Hence $db \in C_G(c)$ and $c^d = c^{-1}$. Analogously we get that either $d^2 = c^2$ or $d^2 = b^2$ (since $[c, d] \neq 1$).

First assume $d^2 = c^2$. Consider the elements ac, db. Then we have $[ac, db] = [a, b] \neq 1$, $(ac)^2 = a^2c^2 \neq (db)^2 = a^2$. Hence by Lemma 2.2 $cd \in C_G(ac) \cup C_G(db)$ or $(cd)^2 = (ac)^2 = a^2c^2$ or $(cd)^2 = a^2$. But $(cd)^2 = d^2$, and then the only possibility is $[cd, db] = [d, b] = 1$. Now the elements a, b, c, d satisfy the hypothesis of 2.7 and (5) holds.

Now assume $d^2 = b^2$. Then from $a^2 = (bd)^2 = b^2d^2[d, b]$ it follows that $[d, b] = a^2 = [a, b]$. Hence $[ad, b] = 1$. Thus, with $d' = ad$, we have $a^b = a^{-1}$, $a^c = a$, $a^{d'} = a$, $c^{d'} = c^d = c^{-1}$, $b^{d'} = b$ and the elements a, b, c, d' satisfy the hypothesis of 2.7 and again (5) holds.

Conversely assume that one of (1), (2), (3), (4), (5) holds.

If (1) or (2) or (3) holds, trivially $G \in DS(3)$.

Now assume that (4) holds. Then $\Omega_1(G) \leqslant Z(G)$ and $\mho_1(G) = \{1, a^2, b^2, c^2\}$. Furthermore every element $x \in G$ can be written in the form $x = da^\alpha b^\beta c^\gamma$, where $d \in \Omega_1(G)$, $\alpha, \beta, \gamma \equiv 0, 1 \pmod 2$. Then we have $(a^\alpha bc^\gamma)^2 = a^\alpha bc^\gamma a^\alpha bc^\gamma = b^2 c^{2\gamma} \in \{a^2, b^2\}$ and $(a^\alpha c^\gamma)^2 = c^2$ if and only if $\alpha \equiv 0 \pmod 2), \gamma \equiv 1 \pmod 2$. Hence if $x^2 = c^2$, then $x \in Z(G)$. Now let $X = \{x, y, z\} \subseteq G$. If $X \cap \Omega_1(G) \neq \emptyset$, then $X \cap Z(G) \neq \emptyset$ and $|X^2| < 9$. Also $|X^2| < 9$ if $|\{x^2, y^2, z^2\}| < 3$. Now assume x^2, y^2, z^2 pairwise different and not 1. Then an element of X, say x, is such that $x^2 = c^2$ and $x \in Z(G)$ by the previous remark. Thus $X \cap Z(G) \neq \emptyset$ and $|X^2| < 9$.

Now assume that G satisfies (5). Again we have $\Omega_1(G) \leqslant Z(G)$ and $\mho_1(G) = \{1, a^2, b^2, c^2\}$. Moreover every element g of G can be written as $g = fa^\alpha b^\beta c^\gamma d^\delta$ where $\alpha, \beta, \gamma, \delta \equiv 0, 1 \pmod 2$. As before it is easy to see that $g^2 = a^2$ if and only if either $\beta \equiv 1 \pmod 2$, $\gamma \equiv 0 \pmod 2$, $\delta \equiv 1 \pmod 2$, that is, $g \in \{fbd, fabd / f \in \Omega_1(G)\}$ or $\beta \equiv 0 \pmod 2$, $\alpha \equiv 1 \pmod 2$, $\gamma \equiv \delta \equiv 0 \pmod 2$, that is, $g \in \{fa / f \in \Omega_1(G)\}$. However $g^2 = c^2$ or $g^2 = 1$ if and only if $\alpha \equiv \beta \equiv 0 \pmod 2$. Hence the elements of G whose square is a^2 commute with the elements whose square is c^2. Now it is easy to verify that $G \in DS(3)$. ∎

From the results of [**1**] and from Theorem A it follows easily that if G is in $DS(3)$, then G is abelian, or $G = A\langle x \rangle$ where A is an abelian subgroup of index 2 and $a^x = a^{-1}$ for every $a \in A$, or G is a 2-group of class less than or equal to 2, exponent 4, and with $|\mho_1(G)| \leqslant 4$.

Hence any finite group in $DS(3)$ is soluble. But groups in $DS(k)$ for some k are finite-by-abelian-by-finite, so then any group in $DS(3)$ is soluble.

Also we remark that from Theorem A it follows that if G is a non-abelian finite group in $DS(3)$ then $Z(G)$ has exponent at most 4, and has exponent 2 if (3) holds.

These will be our starting points in the next section.

3. Arbitrary groups in $DS(3)$

We prove the following

Theorem C. *Let G be a group. Then $G \in DS(3)$ if and only if one of (1), (2), (3), (4), (5) of Theorem A holds.*

Proof: Let $G \in DS(3)$. Then G is soluble and finite-by-abelian-by-finite, by an unpublished result of P. Neumann ([**6**], see [**3**] for a proof).

Assume first that G is finitely generated. Then G is polycyclic and thus residually finite, by a theorem of Hirsch (see for example [**7**, 5.4.17, p.149]). Suppose G is non-abelian. Then there exists a normal subgroup N of G of finite index with G/N non-abelian. If for every $x \in G$ we have $x^2H \in Z(G/H)$ for any H normal in G, $H \leqslant N$, of finite index, then G is nilpotent of class 2 and $x^2 \in Z(G)$. In this case G is periodic. For, if y is a torsion-free element of G, we have $y^2 \in Z(G)$. Thus if M is normal in G with G/M finite, $M \leqslant N$ and y^4, $y^8 \notin M$, we get $y^2M \in Z(G/M)$, G/M non-abelian, y^2M of order different from 2, 4, a contradiction by the remarks at the end of Section 2. Then G is a periodic finitely generated nilpotent group, so G is finite and has the required structure.

Now assume that there exist N normal in G of finite index and an element $xN \in G/N$ with $x^2N \notin Z(G/N)$. Then $G/N = A/N\langle xN\rangle$ with $a^xN = a^{-1}N$, and A/N abelian and $x^2 \in A$ by Theorem A. For every normal subgroup $M \leqslant N$ of finite index we have $x^2M \notin Z(G/M)$, hence $G/M = B/M\langle yM\rangle$, B/M abelian, $y^2M \in B/M$, $b^yM = b^{-1}M$ for every $b \in B$. If either $A \nleqslant B$, or $B \leqslant A$, from $G/M = A/M\, B/M$, with A/M, B/M abelian of index 2, it follows that $g^2M \in A/M \cap B/M \leqslant Z(G/M)$ for every $g \in G$, a contradiction.

Thus $A = B$, and $xM = byM$, for some $b \in B$. Hence $G/M = A/M\langle xM\rangle$ with A/M abelian and $aa^x \in M$ for every $a \in A$. This holds for any normal M of finite index, $M \leqslant N$ with the same element x. Then since G is residually finite we get $G = A\langle x\rangle$, A abelian of index 2, $a^x = a^{-1}$ for every $a \in A$, as required.

Now assume that G is an arbitrary group. Suppose G is non-abelian and $|\mho_1(G)| > 2$. Then there exists a finitely generated subgroup Y of G which is non-abelian and with $|\mho_1(Y)| > 2$. First assume that G does not have a subgroup of type (4) nor (5). Then for every finitely generated subgroup $F \geqslant Y$ of G we have $F = A_F\langle g\rangle$, where A_F is abelian, $g^2 \in A_F$ and $a^g = a^{-1}$ for every $a \in A_F$. Therefore every finitely generated subgroup of G has an abelian subgroup of index at most 2. Then, using Proposition 1.K.2 of [5, p.55], it is easy to show that G has an abelian subgroup A of index 2 and $G = A\langle x\rangle$. Assume by contradiction that there exists $a \in A$ with $a^x \neq a^{-1}$, $a^x \neq a$. Then for every finitely generated subgroup $X \geqslant \langle a, x, F\rangle$ we have $X = B\langle y\rangle$, with B abelian of index 2 and $b^y = b^{-1}$ for every $b \in B$. If either $B \leqslant X \cap A$ or $B \geqslant X \cap A$, then $B = X \cap A$ and we have $a \in B$, $x \notin B$ and $x = cy$, with $c \in B$; then $a^x = a^{cy} = a^y = a^{-1}$, a contradiction. Thus there exists $z \in B - (X \cap A)$ and we have $z = dx$ with $d \in A$. Then $B \cap X \cap A \leqslant C_G(dx) \cap C_G(d) \leqslant C_G(x)$ and from $|X : B \cap X \cap A| \leqslant 4$ we get $|X : C_G(x)| \leqslant 4$. Furthermore there exists $s \in (A \cap X) - B$ and we have $b^s = b^{-1} = b$, for every $b \in B \cap A \cap X$; hence $A \cap B \cap X$ has exponent 2 and is in $Z(X)$. We have proved that X is a finite 2-group nilpotent of class 2 and exponent at most 4 and $|X : C_G(x)| \leqslant 4$. This holds for any $X \geqslant \langle Y, a, x\rangle$. We get easily that A has exponent at most 4 and $|G : C_G(x)| \leqslant 4$. Hence we have $G = Z(G)F$ with F finite, $F \geqslant \langle a, x\rangle$ and $Z(G)$ of exponent 2 by the remark at the end of Section 2. Then $G = T \times F$ with F finite and $\exp T = 2$. By Theorem A we have a contradiction. Then $A = A_1 \cup C_A(x)$, with $A_1 = \{a \in A / a^x = a^{-1}\}$ a subgroup of A, and $A = A_1$ since G is not abelian. Therefore (3) holds.

Now assume that there exists a finite subgroup H of G satisfying (4) or (5) of Theorem A. Then H is not abelian, $|\mho_1(H)| = 4$ and H does not have the structure in (3). Then any finitely generated subgroup of G containing H satisfies either (4) or (5). It follows easily that $|\mho_1(G)| = 4$ and $\Omega_1(G) \leqslant Z(G)$. First assume $H = \langle a, b, c, d\rangle$, $|a| = |b| = |c| = |d| = 4$, $a^b = a^{-1}$, $c^b = c^{-1}$, $c^d = c^{-1}$, $[a, c] = [a, d] = [b, d] = 1$. Then we have $\langle a\rangle$ and $\langle c\rangle$ normal in G. Write $D = C_G(a) \cap C_G(c)$. Then we have D normal in G and $G = D\langle b, d\rangle$. Furthermore for any $g \in D$, $|g| = 4$, we have $g^2 = a^2$, $g^2 = c^2$ or $g^2 = b^2 = (ac)^2$. Then ag, cg or gac has order 2. Then $D \leqslant \Omega_1(G)\langle a, c\rangle$ and $G = \Omega_1(G)\langle a, b, c, d\rangle = Y \times \langle a, b, c, d\rangle$ with Y of exponent 2. Therefore (5) holds. Similarly if $G \geqslant \langle a, b, c\rangle$ and $\langle a, b, c\rangle$ satisfies (4) of Theorem A, then we get $G = Y \times \langle a, b, c\rangle$, with $\exp Y = 2$, and (4) holds. $\square$

References

[1] J.G. Berkovich, G.A. Freiman and C.E. Praeger, 'Small squaring and cubing properties of finite groups', *Bull. Austral. Math. Soc.* **44** (1991), 429–450.

[2] G.A. Freiman, 'On two and three-element subsets of groups', *Aequationes Math.* **22** (1981), 140–152.

[3] M. Herzog, P. Longobardi and M. Maj, 'On a combinatorial problem in group theory', submitted.

[4] B. Huppert, *Endliche Gruppen I* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[5] O.H. Kegel and B.A.F. Wehrfritz, *Locally finite groups* (North-Holland Publishing Company, Amsterdam, London, 1973).

[6] P.M. Neumann, 'A combinatorial problem in group theory', Canberra 1989, private communication.

[7] D.J.S. Robinson, *A course in the theory of groups* (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

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