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Abstract

Let U be an affine smooth curve defined over an algebraically closed field of positive characteristic. The Abhyankar conjecture (proved by Raynaud and Harbater in 1994) describes the set of finite quotients of Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(U)$. In this paper, we consider a purely inseparable analogue of this problem, formulated in terms of Nori's profinite fundamental group scheme $\pi^N(U)$, and give a partial answer to it.

1. Introduction

1.1 Nori's fundamental group scheme

Let X be an algebraic variety over a field k. In [SGA1], Grothendieck defined the *étale fundamental group* $\pi_1^{\text{ét}}(X)$ as a generalization of the fundamental group of a topological space. It is the smallest profinite group classifying G-coverings over X, where G is a finite group. In the case where k is an algebraically closed field of characteristic zero with $k \hookrightarrow \mathbb{C}$, it is known that $\pi_1^{\text{ét}}(X)$ is isomorphic to the profinite completion of the topological one $\pi_1^{\text{top}}(X(\mathbb{C}))$. For example, as the complex line $\mathbb{A}^1(\mathbb{C})$ is simply connected, we have $\pi_1^{\text{ét}}(\mathbb{A}_{\mathbb{C}}^1) = 0$. On the other hand, in the case where k is of positive characteristic $p > 0$, the situation is quite different. This can be seen even in the case where $X = \mathbb{A}_k^1$ is the affine line. Indeed, it is known that $\dim_{\mathbb{F}_p} \text{Hom}(\pi_1^{\text{ét}}(\mathbb{A}_k^1), \mathbb{F}_p) = \infty$. On the other hand, if $X = U$ is a smooth affine curve, the *Abhyankar conjecture* [Abh57], proved by Raynaud and Harbater [Har94, Ray94], gives us another estimate of the difference between $\pi_1^{\text{ét}}(U)$ and the topological one of a Riemann surface of the same type. The conjecture describes the set

$$\pi_A^{\text{ét}}(U) \stackrel{\text{def}}{=} \{\text{finite quotients of } \pi_1^{\text{ét}}(U)\} \tag{1.1}$$

for any smooth affine curve U. Here, more precisely, $\pi_A^{\text{ét}}(U)$ is the set of isomorphism classes of finite groups which appears as a finite quotient of $\pi_1^{\text{ét}}(U)$. For example, it says for any integers $n > 0$ and $r > 0$, there exists a surjective homomorphism $\pi_1^{\text{ét}}(\mathbb{A}_k^1) \twoheadrightarrow \text{SL}_n(\mathbb{F}_{p^r})$ (cf. [Kam86, Nor94, Ser92]).

In [Nor76], [Nor82, ch. II], Nori defined the *fundamental group scheme* $\pi^N(X)$ as a generalization of Grothendieck's étale geometric fundamental group of an algebraic variety X over a field k (cf. [SGA1]). It is a profinite k-group scheme classifying G-torsors over X with G a finite k-group scheme. In [Nor76], he first constructed it under the assumption that X is proper over k by using the theory of Tannakian categories. On the other hand, in [Nor82], he also proved that such a profinite one $\pi^N(X)$ exists also for arbitrary (not necessarily proper) X

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without relying on a Tannakian construction (cf. § 2.1). Note that, if k is an algebraically closed field, then Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X)$ classifies all finite étale torsors over X and is universal for this property. Hence, there exists a homomorphism of $\pi^N(X)$ into the pro-constant k-group scheme, denoted by $\pi^{\text{ét}}(X)$, associated with $\pi_1^{\text{ét}}(X)$. In fact $\pi^{\text{ét}}(X)$ gives the maximal pro-étale quotient of $\pi^N(X)$, called the *étale fundamental group scheme*. In the case where k is of characteristic zero, then the surjective homomorphism $\pi^N(X) \twoheadrightarrow \pi^{\text{ét}}(X)$ is in fact an isomorphism. This is valid because under the assumption that k is an algebraically closed field of characteristic zero, any finite G-torsor $P \to X$ with G a finite k-group scheme is nothing but a $G(k)$-covering over X. On the other hand, in the case where k is of positive characteristic $p > 0$, then $\pi^N(X)$ is strictly larger than $\pi^{\text{ét}}(X)$, in general. Indeed, finite *local* ('purely inseparable') torsors make a contribution to occur the difference between these fundamental group schemes. Here a finite k-group scheme G is said to be *local* if it is connected, i.e., if G^0 denotes the connected component of the identity $1 \in G$, then $G^0 = G$. For example, $G = \alpha_p$, or μ_p.

1.2 Main results

In the present paper, we will attempt to estimate the difference between $\pi^N(U)$ and $\pi^{\text{ét}}(U)$ for a smooth affine curve U defined over an algebraically closed field k of positive characteristic $p > 0$ from the viewpoint of the inverse Galois problem. We will study a purely inseparable analogue of the Abhyankar conjecture for affine curves U (cf. [Abh57, Abh92]; see also § 1.3), i.e., we will try to describe the set

$$\pi_A^{\text{loc}}(U) \overset{\text{def}}{=} \{\text{finite local quotients of } \pi^N(U)\}.$$

More precisely, $\pi_A^{\text{loc}}(U)$ is the set of isomorphism classes of finite local k-group schemes which appears as a finite quotient of $\pi^N(U)$.

Now let us explain the contents of the present paper. In § 2, we will briefly review the definition of Nori's profinite fundamental group scheme and the maximal linearly reductive quotient of it. In § 3, we will see the maximal local linearly reductive quotient of $\pi^N(U)$ provides a necessary condition for a finite local k-group scheme G to belong to the set $\pi_A^{\text{loc}}(U)$ (cf. Proposition 3.1). Now let us explain this. Let X be a smooth compactification of U. Let $n \overset{\text{def}}{=} \#(X \backslash U)$. Note that the affineness assumption of U implies $n > 0$. Let γ be the p-rank of the Jacobian variety of X, i.e., $\gamma = \dim_{\mathbb{F}_p} \text{Pic}_X^0[p](k)$. We will see that for any finite local k-group scheme G, if $G \in \pi_A^{\text{loc}}(U)$, then the character group $\mathbb{X}(G) \overset{\text{def}}{=} \text{Hom}(G, \mathbb{G}_m)$ must be embeddable as a subgroup into $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma + n - 1}$. Then we can ask whether or not the converse is true (cf. Question 3.3).

Question 1.1. Let U be a smooth affine curve and G a finite local k-group scheme. If there exists an injective homomorphism $\mathbb{X}(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma + n - 1}$, then does G belong to the set $\pi_A^{\text{loc}}(U)$?

The main purpose of the present paper is to give a partial affirmative answer to Question 1.1. The main result is the following theorem (cf. Proposition 3.4; Corollaries 4.19; 4.15).

THEOREM 1.2.

(1) *For any smooth affine curve U over k and any finite local nilpotent k-group scheme G, if there exists an injective homomorphism $\mathbb{X}(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma + n - 1}$, then there exists a surjective homomorphism $\pi^N(U) \twoheadrightarrow G$.*

(2) *Let Σ be a semi-simple simply connected algebraic group over k. Then for any integer $r > 0$, there exists a surjective homomorphism*

$$\pi^N(\mathbb{A}_k^1) \twoheadrightarrow \Sigma_{(r)}$$

of $\pi^N(\mathbb{A}_k^1)$ onto the rth Frobenius kernel $\Sigma_{(r)} \overset{\mathrm{def}}{=} \mathrm{Ker}(F^{(r)} : \Sigma^{(-r)} \to \Sigma)$.

(3) *Assume $p = 2$. Then for any integer $r > 0$, there exists a surjective homomorphism*

$$\pi^N(\mathbb{G}_m) \twoheadrightarrow \mathrm{GL}_{2(r)}$$

of $\pi^N(\mathbb{G}_m)$ onto the rth Frobenius kernel $\mathrm{GL}_{2(r)}$ of GL_2.

Here, for each integer $r > 0$, $\Sigma^{(-r)}$ denotes the $(-r)$th Frobenius twist of Σ and $F^{(r)} : \Sigma^{(-r)} \to \Sigma$ is the rth relative Frobenius morphism, which is a homomorphism of algebraic groups. Furthermore, its kernel $\Sigma_{(r)} = \mathrm{Ker}(F^{(r)})$ is a finite local k-group scheme.

Remark 1.3. (1) If Σ is a semi-simple simply connected algebraic group over k, it turns out that $\mathbb{X}(\Sigma_{(r)}) = 1$ (cf. Remark 3.7(3)). Therefore, Theorem 1.2(2) gives an affirmative answer to Question 1.1 for the affine line $\mathbb{A}_k^1$ and for the Frobenius kernels $\Sigma_{(r)}$ $(r > 0)$ of a semi-simple simply connected algebraic group Σ.

(2) Since $\mathbb{X}(\mathrm{GL}_{2(r)}) = \mathbb{Z}/2^r\mathbb{Z} \subset \mathbb{Q}_2/\mathbb{Z}_2$ if $p = 2$, Theorem 1.2(3) gives an affirmative answer to Question 1.1 for the multiplicative group $U = \mathbb{G}_m$ and for the Frobenius kernels $\mathrm{GL}_{2(r)}$ $(r > 0)$ of GL_2 in the case where k is of characteristic $p = 2$.

(3) Note that each homomorphism $\pi^N(U) \to G$ of $\pi^N(U)$ into a finite k-group scheme G corresponds bijectively to a (fpqc) G-torsor $P \to U$ (cf. Proposition 2.2). To prove Theorem 1.2(2), we will show there exists a k-morphism $f : \mathbb{A}_k^1 \to \Sigma$ such that the resulting $\Sigma_{(r)}$-torsor $f^*\Sigma^{(-r)} \to \mathbb{A}_k^1$ realizes a surjective homomorphism $\pi^N(\mathbb{A}_k^1) \twoheadrightarrow \Sigma_{(r)}$. To prove the existence of such a morphism f, we will first reduce the problem to the case where $r = 1$ (cf. Lemma 4.2). Next we will deduce the existence of such an f from a Bertini type theorem for height one torsors (cf. Theorem 4.17).

(4) In particular, Theorem 1.2(1) (or Theorem 1.2(2)) implies that if $p = 2$, then $\mathrm{SL}_{2(1)}$ appears as a finite quotient of $\pi^N(\mathbb{A}_k^1)$ (cf. Example 3.6). On the other hand, it turns out that any $\mathrm{SL}_{2(1)}$-torsor must be of the form $f^*\mathrm{SL}_2^{(-1)} \to \mathbb{A}_k^1$ for some k-morphism $f : \mathbb{A}_k^1 \to \mathrm{SL}_2$. In this case, we can explicitly describe the subset of $\mathrm{Mor}_k(\mathbb{A}_k^1, \mathrm{SL}_2)$ consisting of k-morphisms f such that the torsor $f^*\mathrm{SL}_2^{(-1)}$ realizes a surjective homomorphism $\pi^N(\mathbb{A}_k^1) \twoheadrightarrow \mathrm{SL}_{2(1)}$ (cf. Corollary 4.13).

Remark 1.4. Considering recent developments of the theory of '*tame stacks*' (cf. [AOV08, Mar12, Gil12]), the author expects that the notion of linearly reductiveness might provide us with a good analogy between the answer of Question 1.1 and the Abhyankar conjecture (cf. Theorem 1.5) and that Question 1.1 might be affirmative for any smooth affine curve U and for any finite local k-group scheme G. See also Remark 1.8.

1.3 The Abhyankar conjecture

All the ideas of our arguments in the present paper come from Serre's work [Ser90] (the method of embedding problems) or Nori's one [Kam86, Nor94] in the pursuit of the Abhyankar conjecture for the affine line. Hence, we would like to briefly review the conjecture.

First we will recall the precise statement of it (in a weak form). Let k be an algebraically closed field of positive characteristic $p > 0$ and X a smooth projective curve over k of genus $g \geqslant 0$.

Let $\emptyset \neq U \subset X$ be a nonempty open subset with $n \stackrel{\text{def}}{=} \#(X \backslash U) \geqslant 0$. Let $\Gamma_{g,n}$ be the group defined by

$$\Gamma_{g,n} \stackrel{\text{def}}{=} \left\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid \prod_{i=1}^{g} [a_i, b_i] \gamma_1 \cdots \gamma_n = 1 \right\rangle$$

and $\widehat{\Gamma}_{g,n}$ by its profinite completion. Note that if $n > 0$, then $\Gamma_{g,n}$ is a free group F_{2g+n-1} of rank $2g + n - 1$. A classical result due to Grothendieck [SGA1] implies:

$$\pi_1^{\text{ét}}(U)^{(p')} \simeq \widehat{\Gamma}_{g,n}^{(p')}.$$

Here $(-)^{(p')}$ means the maximal pro-prime to p quotient. In particular, if $n > 0$, i.e., U is affine, then $\pi_1^{\text{ét}}(U)^{(p')} \simeq \widehat{F}_{2g+n-1}^{(p')}$. Hence, in this case, if a finite group G appears as a finite quotient of $\pi_1^{\text{ét}}(U)$, then $G^{(p')}$ must be generated by at most $2g + n - 1$ elements. Here, the group $G^{(p')}$ is the quotient of G by the subgroup $p(G)$ generated by all the p-Sylow subgroups of G. The Abhyankar conjecture claims that the converse is also true.

THEOREM 1.5 (The Abhyankar conjecture in a weak form [Ray94, Har94]). *Assume that U is affine. Let G be an arbitrary finite group. Then $G \in \pi_A^{\text{ét}}(U)$ (cf. (1.1)) if and only if $G^{(p')}$ can be generated by at most $2g + n - 1$ elements.*

Raynaud proved the theorem for the case where $U = \mathbb{A}_k^1$ [Ray94]. Soon after, Harbater obtained the theorem for the general case [Har94].

Remark 1.6. (1) Theorem 1.5 implies that $\pi_A^{\text{ét}}(U)$ is determined by the topological one $\pi_1^{\text{top}}(U_0)$ of a Riemann surface U_0 of the same type (g, n) as U. An affirmative answer to our question (Question 1.1) says that the set of finite local quotients of $\pi^N(U)$ might be determined by the étale one $\pi_1^{\text{ét}}(U)$. Note that the number $\gamma + n - 1$ can be reconstructed from $\pi_1^{\text{ét}}(U)$ group-theoretically (cf. [Tam99]).

(2) The assumption that U is affine is essential. This is because if $U = X$ is projective, then $\pi_1^{\text{ét}}(X)$ is topologically finitely generated and the isomorphism class of $\pi_1^{\text{ét}}(X)$ can be completely determined by the set $\pi_A^{\text{ét}}(X)$ of finite quotients of it (cf. [FJ08, Proposition 5.4]). On the other hand, it is known that $\pi_1^{\text{ét}}(X)$ itself has much information about the moduli of X (cf. [Tam99, Tam04]) if the genus of X is $g \geqslant 2$. Therefore, such a simple answer as above (Theorem 1.5) cannot be expected and the corresponding problem is much more challenging (cf. [PS00]).

In the particular case $U = \mathbb{A}_k^1$ is the affine line, the Abhyankar conjecture states that a finite group G belongs to the set $\pi_A^{\text{ét}}(\mathbb{A}_k^1)$ if and only if $G^{(p')} = 1$. The latter condition is equivalent to the one that G can be generated by p-Sylow subgroups of it, and such a group is called a *quasi-p-group*. Obviously, any p-group is a quasi-p-group. In [Abh57, Abh92], Abhyankar found an explicit equation defining a finite étale Galois covering over $\mathbb{A}_k^1$ whose corresponding homomorphism $\pi_1^{\text{ét}}(\mathbb{A}_k^1) \to G$ is surjective for various nontrivial quasi-p-groups G. In [Ser90], Serre approached the conjecture for the affine line $\mathbb{A}_k^1$ by the method of embedding problems. As a result, he proved that the conjecture is ture for any solvable quasi-p-group. This result gives a first reduction step in the proof due to Raynaud [Ray94]. On the other hand, although his result was not used in Raynaud's proof, Nori provided many examples of quasi-p-groups appearing as a finite quotient of $\pi_1^{\text{ét}}(\mathbb{A}_k^1)$, which gave another evidence of the Abhyankar conjetcure for $\mathbb{A}_k^1$ (cf. [Kam86, Nor94]). Let Σ be a semi-simple simply connected algebraic group (for example, $\Sigma = \text{SL}_n$) over $\mathbb{F}_{p^r}$.

Let $L : \Sigma \to \Sigma$; $g \mapsto g^{-1} F^r(g)$ be the Lang map, where F is the absolute Frobenius morphism of Σ. Note that L gives a Galois covering with group $\Sigma(\mathbb{F}_{p^r})$. Nori showed the existence of a closed immersion $\iota : \mathbb{A}^1_{\mathbb{F}_{p^r}} \hookrightarrow \Sigma$ such that $\iota^* \Sigma$ is geometrically connected, whence the Galois covering $(\iota^* \Sigma)_k \to \mathbb{A}^1_k$ corresponds to a surjective homomorphism $\pi^{\text{ét}}_1(\mathbb{A}^1_k) \twoheadrightarrow \Sigma(\mathbb{F}_{p^r})$. For a brief survey of all the above results, see, for example, [HOPS14, § 3].

Remark 1.7. (1) Our result for finite local nilpotent group schemes (Theorem 1.2(1) (cf. Proposition 3.4)) is motivated by Serre's result on solvable quasi-p-groups [Ser90]. However, we can extend his method only in the nilpotent case.

(2) Theorem 1.2(2) can be considered as a purely inseparable analogue of Nori's result [Kam86, Nor94]. To prove this, we will rely on a Bertini type theorem (Theorem 4.17). Hence, we cannot give an explicit equation defining a saturated $\Sigma_{(1)}$-torsor over $\mathbb{A}^1_k$ for general Σ. In the case where $p = 2$ and $\Sigma = \mathrm{SL}_2$, we can clarify this situation in not a conceptual but an explicit way.

(3) In the classical conjecture, in particular, in the proof due to Raynaud or Harbater, the rigid analytic or formal patching method and the theory of stable curves provide us with strong tools to solve the problem. The author is not sure if these methods can be applicable in our situation.

Remark 1.8. The full statement of the classical Abhyankar conjecture (proved by Harbater) states that as a covering realizing a finite quotient $G \in \pi^{\text{ét}}_A(U)$, one can take a G-covering tamely ramified except for one point $x_0 \in X \backslash U$ (cf. [Har94, Conjecture 1.2]). So, it might be natural to ask for an analogous problem. To formulate it, we need the notion of *tamely ramified torsors*, a similar notion to tamely remified coverings.

The notion of *tameness* of an action of a group scheme G on a scheme X was introduced by Chinburg *et al.* [CEPT96]. On the other hand, Abramovich *et al.* gave another formulation in terms of *tame stacks* (cf. [AOV08]). A relation between these two works has been studied by Marques [Mar12]. Moreover, in [Bor09], Borne defined the fundamental group scheme which classifies tamely ramified torsors by using the Tannakian category of parabolic bundles. To obtain a good analogue of the notion of tame coverings, one needs to consider an extension of a G-torsor P over an open subset U of X with $D = X \backslash U$ a normal crossing divisor to an X-scheme Q together with an action of G. In [Gil12], Gillibert discussed this point. Recently, in [Zal16], Zalamansky formulated a ramification theory in a purely inseparable setting in terms of ramification divisors. The author is not sure if there exist any relations between Zalamansky's formulation and the previous ones.

In view of the above recent developments of the theory of tamely ramified torsors, one can formulate a naive analogue of the strong Abhyankar conjecture [Har94, Conjecture 1.2] in an obvious way. However, the author has no evidence of it. So, in the present paper, we will concentrate on an analogue of the weak Abhyankar conjecture.

Notation

In this paper, k always means a perfect field; an *algebraic variety* over k means a geometrically connected and reduced scheme separated of finite type over k; a *curve* over k is an algebraic variety over k of dimension one; an *algebraic group* over k means a group object of the category of algebraic varieties over k. Note that, automatically, any algebraic group over k is smooth.

Let k be a perfect field of positive characteristic $p > 0$ and X an algebraic variety over k. We denote by F_X, or simply F, the absolute Frobenius morphism $F = F_X : X \to X$. For each integer $n \in \mathbb{Z}$, we denote by $X^{(n)}$ the nth *Frobenius twist* of X.

$$
\begin{array}{ccc}
X^{(n)} & \longrightarrow & X \\
\downarrow & \square & \downarrow \\
\operatorname{Spec} k & \xrightarrow[\simeq]{F^n} & \operatorname{Spec} k
\end{array}
$$

If $n > 0$, the morphism $F^n : X \to X$ then factors uniquely through $X^{(n)}$. The resulting morphism, denoted by $F^{(n)} : X \to X^{(n)}$, is the nth relative Frobenius morphism.

We denote by Vec_k the category of finite dimensional vector spaces over k. For an affine k-group scheme G, we denote by $\operatorname{Rep}_k(G)$ the category of finite dimensional left k-linear representations of G. For each $(V, \rho) \in \operatorname{Rep}_k(G)$, we denote by V^G the G-invariant subspace of V, i.e., $V^G = \{v \in V \mid \rho(v) = v \otimes 1\}$.

2. Fundamental group scheme

2.1 Profinite fundamental group scheme

In this subsection, we will briefly recall the definition of *Nori's profinite fundamental group scheme* [Nor76, Nor82].

Let X be an algebraic variety over k together with a rational point $x \in X(k)$. We define the category $N(X, x)$ as follows. The objects of $N(X, x)$ are all the triples (P, G, p) where:

- G: a finite k-group scheme;
- an (fpqc) G-torsor $\pi : P \to X$;
- $p \in P(k)$: a rational point with $\pi(p) = x$.

Let $(P, G, p), (Q, H, q) \in N(X, x)$ be arbitrary two objects. Then a morphism $(P, G, p) \to (Q, H, q)$ is a pair (f, ϕ) of an X-morphism $f : P \to Q$ and a k-homomorphism $\phi : G \to H$ making the following diagram commute.

$$
\begin{array}{ccc}
P \times G & \longrightarrow & P \\
{\scriptstyle (f,\phi)} \downarrow & & \downarrow {\scriptstyle f} \\
Q \times H & \longrightarrow & Q
\end{array}
$$

Here, the above two horizontal morphisms are the ones defining the actions of torsors. By these objects and morphisms, $N(X, x)$ becomes a category. In [Nor82], Nori proved that the category $N(X, x)$ is a cofiltered category and, in particular that the projective limit

$$
\varprojlim_{(P,G,p) \in N(X,x)} (P, G, p)
$$

exists.

DEFINITION 2.1. The projective limit of underlying group schemes

$$
\pi^N(X, x) \stackrel{\text{def}}{=} \varprojlim_{(P,G,p) \in N(X,x)} G
$$

is called the *profinite fundamental group scheme*, or shortly the *fundamental group scheme*, of (X, x).

From the definition, the following is immediate.

PROPOSITION 2.2. *The fundamental group scheme $\pi^N(X, x)$ is a profinite k-group scheme such that for any finite k-group scheme G, the map*

$$\mathrm{Hom}_k(\pi^N(X, x), G) \to \mathrm{Tors}(G, (X, x))$$
$$\phi \mapsto (X_x^N, x^N) \times^{\pi^N(X,x)} {}^\phi G$$

is bijective. Here

$$(X_x^N, x^N) \overset{\text{def}}{=} \varprojlim_{(P,G,p)\in N(X,x)} (P, p)$$

and $\mathrm{Tors}(G, (X, x))$ is the set of isomorphism classes of pointed G-torsors over (X, x). Moreover, the torsor $(X_x^N, x^N) \times^{\pi^N(X,x)} {}^\phi G$ associated with a homomorphism ϕ is defined as the quotient of the product $(X_x^N, x^N) \times G$ by the diagonal action of $\pi^N(X, x)$:

$$(x, g) \cdot \gamma \overset{\text{def}}{=} (xg^{-1}, g\phi(\gamma))$$

for $(x, g) \in X_x^N \times G$ and $\gamma \in \pi^N(X, x)$.

DEFINITION 2.3. *A G-torsor $(P, p) \to (X, x)$ is said to be* saturated *if the corresponding homomorphism $\pi^N(X, x) \to G$ is surjective.*

Remark 2.4. If k is algebraically closed, then for any finite étale k-group scheme G is the constant group scheme associated with the finite group $G(k)$ and a G-torsor over X is nothing but a $G(k)$-covering over X. Therefore, from the universality of Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$, there exists a k-homomorphism of $\pi^N(X, x)$ to the pro-constant group scheme associated with $\pi_1^{\text{ét}}(X, \overline{x})$. In fact, this homomorphism is surjective. Furthermore, if k is of characteristic zero, then it is an isomorphism. For details, see [EHS08, Remark 2.10].

2.2 The maximal linearly reductive quotient of π^N

Now let us recall the maximal *linearly reductive* quotient $\pi^{\text{lin.red}}(X, x)$ of $\pi^N(X, x)$ (cf. [BV15]).

DEFINITION 2.5 (Cf. [AOV08, §2]). *A finite k-group scheme G is said to be* linearly reductive *if one of the following equivalent conditions is satisfied:*

(a) *the functor $\mathrm{Rep}_k(G) \to \mathrm{Vec}_k$; $V \mapsto V^G$ is exact;*

(b) *the category $\mathrm{Rep}_k(G)$ is semi-simple.*

PROPOSITION 2.6 (Abramovich *et al.* [AOV08, Proposition 2.13]). *A finite k-group scheme G is linearly reductive if and only if for an algebraic closure $\overline{k}$ of k, then $G_{\overline{k}} := G \times_k \overline{k}$ is isomorphic to a semi-direct product $H \ltimes \Delta$ where H is a finite constant $\overline{k}$-group scheme of order prime to the characteristic of k and Δ is a finite diagonalizable $\overline{k}$-group scheme.*

Here a finite group scheme G is said to be *diagonalizable* if it is abelian and its Cartier dual is constant (cf. [Wat79, §2.2]). Proposition 2.6 does not require the assumption that k is perfect.

Remark 2.7. Assume that k is an algebraically closed field of characteristic $p > 0$. From Proposition 2.6, we can deduce that if a finite étale group scheme G is linearly reductive if and only if $p \nmid \#G(k)$ and that a finite local k-group scheme is linearly reductive if and only if $G \overset{\sim}{\to} \mathrm{Diag}(\mathbb{X}(G))$ with $\mathbb{X}(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ a p-group.

Definition 2.8 (Cf. [BV15, § 10]). We denote by $\pi^{\mathrm{lin.red}}(X, x)$ the maximal linearly reductive quotient of $\pi^N(X, x)$.

Remark 2.9. (1) In [BV15], Borne–Vistoli studied the linearly reductive quotient in terms of *fundamental gerbes*. They called it the *tame fundamental gerbe*. Here, the word 'tame' stems from the notion of *tame stacks* (cf. [AOV08]).

(2) If k is of characteristic zero, then any finite k-group scheme is linearly reductive and $\pi^N(X, x) = \pi^{\mathrm{lin.red}}(X, x)$.

From now on assume that k is of positive characteristic $p > 0$. Then a finite k-group scheme G is said to be *local* if it is connected. We denote by $\pi^{\mathrm{loc}}(X, x)$ the maximal local quotient of $\pi^N(X, x)$. The arguments in [Ünv10] then imply that $\pi^{\mathrm{loc}}(X, x)$ does not depend on the choice of a rational point $x \in X(k)$. More precisely, let $L(X)$ be the category of pairs (P, G) where G is a finite local k-group scheme and $P \to X$ is a G-torsor. Then the projective limit

$$\varprojlim_{(P,G)\in L(X)} (P, G)$$

exists and for any $x \in X(k)$, there exists a canonical isomorphism

$$\pi^{\mathrm{loc}}(X, x) \simeq \varprojlim_{(P,G)\in L(X)} G.$$

In particular, for any finite local k-group scheme G, the map in Proposition 2.2 induces a bijection:

$$\mathrm{Hom}_k(\pi^{\mathrm{loc}}(X, x), G) \xrightarrow{\simeq} H^1_{\mathrm{fpqc}}(X, G). \tag{2.1}$$

Hence, we write simply $\pi^{\mathrm{loc}}(X)$ instead of $\pi^{\mathrm{loc}}(X, x)$.

If k is algebraically closed, then the maximal linearly reductive quotient $\pi^{\mathrm{loc}}(X)^{\mathrm{lin.red}}$ is diagonalizable and we have the following:

$$\pi^{\mathrm{loc}}(X)^{\mathrm{lin.red}} = \mathrm{Diag}(\mathbb{X}(\pi^{\mathrm{loc}}(X))).$$

Here $\mathbb{X}(\pi^{\mathrm{loc}}(X)) = \mathrm{Hom}(\pi^{\mathrm{loc}}(X), \mathbb{G}_m)$, the group of characters of $\pi^{\mathrm{loc}}(X)$ and for any abelian group A, we denote by $\mathrm{Diag}(A)$ the diagonalizable k-group scheme associated with A [Wat79, § 2.2]. On the other hand, any homomorphism $\pi^{\mathrm{loc}}(X) \to \mathbb{G}_m$ factors through $\mu_{p^n} = \mathrm{Diag}(\mathbb{Z}/p^n\mathbb{Z})$ for some integer $n > 0$. Hence,

$$\mathbb{X}(\pi^{\mathrm{loc}}(X)) = \varinjlim_{n>0} \mathrm{Hom}_k(\pi^{\mathrm{loc}}(X), \mu_{p^n}) \xrightarrow{\simeq} \varinjlim_{n>0} H^1_{\mathrm{fpqc}}(X, \mu_{p^n}).$$

Here, since all the μ_{p^n} are abelian, the map in Proposition 2.2 induces the last isomorphism. Hence, we have seen that the following holds.

Proposition 2.10. *If k is an algebraically closed field of characteristic $p > 0$, then there exists a canonical isomorphism:*

$$\pi^{\mathrm{loc}}(X)^{\mathrm{lin.red}} \simeq \varprojlim_{n>0} \mathrm{Diag}(H^1_{\mathrm{fpqc}}(X, \mu_{p^n})).$$

Note that $\pi^{\mathrm{loc}}(X)^{\mathrm{lin.red}}$ is nothing but the maximal local linearly reductive quotient of $\pi^N(X, x)$.

3. A purely inseparable analogue of the Abhyankar conjecture

Let k be an algebraically closed field of positive characteristic $p > 0$. Let X be a projective smooth curve over k of genus $g \geqslant 0$. Let U be a nonempty open subset of X with $n \overset{\text{def}}{=} \#(X \setminus U) > 0$. The scheme U is then an affine smooth curve over k. We denote by γ the *p-rank* of the Jacobian variety Pic_X^0 of X, i.e.,

$$\gamma \overset{\text{def}}{=} \dim_{\mathbb{F}_p} \operatorname{Pic}_X^0[p](k).$$

Since X is smooth and projective, the invariant γ coincides with the dimension of the $\mathbb{F}_p$-vector space $\operatorname{Hom}_{\mathbb{Z}}(\pi_1^{\text{ét}}(X)^{\text{ab}}, \mathbb{F}_p)$ (cf. [Bou00]). Moreover, in this case, for any integer $m > 0$, we have

$$H_{\text{fpqc}}^1(X, \mu_{p^m}) \simeq \operatorname{Pic}_X^0[p^m](k) \simeq (\mathbb{Z}/p^m\mathbb{Z})^{\oplus \gamma}. \tag{3.1}$$

Here, for the first equality, see, for example, [Ant11, Proposition 3.2]; for the second equality, see, for example, [Mum08, ch. IV].

Let $\pi_A^{\text{loc}}(U)$ be the set of isomorphism classes of finite local k-group schemes G such that there exists a surjective homomorphism $\pi^{\text{loc}}(U) \twoheadrightarrow G$.

3.1 Question

We first give a necessary condition for a finite local k-group scheme G to belong to the set $\pi_A^{\text{loc}}(U)$.

PROPOSITION 3.1. *For any finite local k-group scheme G, if $G \in \pi_A^{\text{loc}}(U)$, then there exists an injective homomorphism $\mathbb{X}(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma + n - 1}$.*

By virtue of Proposition 2.10, Proposition 3.1 is an immediate consequence of the following.

PROPOSITION 3.2. *For any integer $m > 0$, we have*

$$H_{\text{fpqc}}^1(U, \mu_{p^m}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^{\oplus \gamma + n - 1}.$$

Proof [Cf. [Sta, Tag 03RN, Lemma 53.68.3]]. Let $X \setminus U = \{x_1, \ldots, x_n\}$. Then there exists an isomorphism $\operatorname{Pic}(U) \simeq \operatorname{Pic}(X)/R$ with $R \overset{\text{def}}{=} \langle \mathcal{O}_X(x_i) \, | \, 1 \leqslant i \leqslant n \rangle_{\mathbb{Z}} \subset \operatorname{Pic}(X)$. Therefore,

$$H_{\text{fpqc}}^1(U, \mu_{p^m}) \simeq \{(L, \alpha) \, | \, L \in \operatorname{Pic}(U), \alpha : L^{\otimes p^m} \overset{\sim}{\to} \mathcal{O}_U\}/\simeq$$
$$\simeq \left\{ (\overline{L}, D, \overline{\alpha}) \, \left| \, \begin{matrix} \overline{L} \in \operatorname{Pic}(X), D \in \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n, \\ \overline{\alpha} : \overline{L}^{\otimes p^m} \overset{\sim}{\to} \mathcal{O}_X(D) \end{matrix} \right. \right\} \Big/ \overline{R},$$

where $\overline{R}$ is the group defined by

$$\overline{R} \overset{\text{def}}{=} \{(\mathcal{O}_X(D'), p^m D', \operatorname{id}) \, | \, D' \in \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n\}.$$

We identify the group $H_{\text{fpqc}}^1(U, \mu_{p^m})$ with the second one in the right-hand side of the above equation. We then obtain the following exact sequence:

$$0 \to H_{\text{fpqc}}^1(X, \mu_{p^m}) \to H_{\text{fpqc}}^1(U, \mu_{p^m}) \to \bigoplus_{i=1}^n \mathbb{Z}/p^m\mathbb{Z} \overset{\Sigma}{\to} \mathbb{Z}/p^m\mathbb{Z} \to 0.$$

Here the second map is given by $(\overline{L}, \alpha) \mapsto (\overline{L}, 0, \alpha)$ and the third one by $(\overline{L}, D, \alpha) \mapsto (\overline{a_i})_{i=1}^n$ with $D = \sum_{i=1}^n a_i x_i$. This completes the proof (cf. (3.1)). $\square$

Considering the Abhyankar conjecture (cf. Theorem 1.5), the following question naturally arises.

Question 3.3. Let G be a finite local k-group scheme. If there exists an injective homomorphism $\mathbb{X}(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma + n - 1}$, then does the group scheme G belong to the set $\pi_A^{\text{loc}}(U)$?

3.2 Nilpotent case

Now we will show that, for any finite local nilpotent k-group scheme G, Question 3.3 has an affirmative answer.

PROPOSITION 3.4. *Let G be a finite local nilpotent k-group scheme. Then $G \in \pi_A^{\mathrm{loc}}(U)$ if and only if there exists an injective homomorphism $\mathbb{X}(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus \gamma+n-1}$.*

Proof. First we remark on α_p-torsors over U. Since U is affine,

$$H_{\mathrm{fpqc}}^1(U, \mathbb{G}_a) = H^1(U, \mathcal{O}_U) = 0,$$

we have

$$H_{\mathrm{fpqc}}^1(U, \alpha_p) \simeq \Gamma(U, \mathcal{O}_U)/\Gamma(U, \mathcal{O}_U)^p.$$

On the other hand, since U is an affine smooth integral scheme, there exists a dominant morphism $U \to \mathbb{A}_k^1$, whence

$$H_{\mathrm{fpqc}}^1(U, \alpha_p) \hookleftarrow H_{\mathrm{fpqc}}^1(\mathbb{A}_k^1, \alpha_p) = \bigoplus_{p \nmid n} k \cdot t^n, \tag{3.2}$$

where t is the coordinate of $\mathbb{A}^1$. Furthermore, since α_p is simple, any nonzero element of $H_{\mathrm{fpqc}}^1(U, \alpha_p)$ corresponds to a surjective homomorphism $\pi^{\mathrm{loc}}(U) \twoheadrightarrow \alpha_p$ (cf. (2.1)).

Now let us prove the proposition. It suffices to show the 'if' part. We prove this by induction on the order $\dim_k k[G] = p^r$ ($r > 0$). From the assumption, G is obtained by central extensions of α_p or μ_p. If $\dim_k k[G] = p$, then $G = \alpha_p$, or $= \mu_p$ and the statement is immediate from (3.2), or from the assumption. Since G is a nontrivial nilpotent group scheme, the center $Z(G)$ is nontrivial. Let $H \subset Z(G)$ be a subgroup scheme of order p. Then we get a central extension of finite k-group schemes:

$$1 \to H \to G \to G/H \to 1. \tag{3.3}$$

Since $\dim_k k[G/H] < \dim_k k[G]$ and $\mathbb{X}(G/H) \subseteq \mathbb{X}(G)$, by induction hypothesis, there exists a surjective homomorphism $\overline{\phi} : \pi^{\mathrm{loc}}(U) \to G/H$. Since U is affine, $H_{\mathrm{fpqc}}^q(U, \mathbb{G}_a) = 0$ if $q \neq 0$, we have $H_{\mathrm{fpqc}}^2(U, \alpha_p) = 0$. On the other hand, $H_{\mathrm{fpqc}}^1(U, \mathbb{G}_m) = \mathrm{Pic}(U)$ is divisible (cf. [Sta, Tag 03RN, Proof of Lemma 53.68.3]) and $H_{\mathrm{fpqc}}^2(U, \mathbb{G}_m) = \mathrm{Br}(U) = 0$, we then also have $H_{\mathrm{fpqc}}^2(U, \mu_p) = 0$. Therefore, we find that $H_{\mathrm{fpqc}}^2(U, H) = 0$ and the exactness of (3.3), noticing that $H_{\mathrm{fpqc}}^0(U, G/H) = 0$, implies that the resulting sequence

$$0 \to H_{\mathrm{fpqc}}^1(U, H) \to H_{\mathrm{fpqc}}^1(U, G) \to H_{\mathrm{fpqc}}^1(U, G/H) \to 0$$

is an exact sequence of pointed sets (cf. [Gir71, p. 284, Remarque 4.2.10]). Therefore, the isomorphism (2.1) implies that there exists a lift $\phi : \pi^{\mathrm{loc}}(U) \to G$ of $\overline{\phi}$ and we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \pi^{\mathrm{loc}}(U) & \overset{\overline{\phi}}{\longrightarrow} & G/H & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0
\end{array}
$$

where $K \overset{\mathrm{def}}{=} \mathrm{Ker}(\overline{\phi})$. If f is nontrivial, then it is surjective, whence so is ϕ. Thus, from now on assume $f = 0$. In this case, the homomorphism ϕ factors through G/H. Thus, the central extension (3.3) is trivial, i.e., $G = H \times (G/H)$. We claim that

$$H_{\mathrm{fpqc}}^1(U, H) \supsetneq \mathrm{Ker}(\mathrm{Hom}_k(\pi^{\mathrm{loc}}(U), H) \to \mathrm{Hom}_k(K, H)). \tag{3.4}$$

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If the claim (3.4) is true, then one takes a k-homomorphism $g : \pi^{\mathrm{loc}}(U) \to H$ so that $g|_K \neq 0$ and the one $(g, \bar{\phi}) : \pi^{\mathrm{loc}}(U) \to G = H \times (G/H)$ is surjective. Thus, it remains to show the claim (3.4). Notice that

$$\mathrm{Hom}_k(G/H, H) = \mathrm{Ker}(\mathrm{Hom}_k(\pi^{\mathrm{loc}}(U), H) \to \mathrm{Hom}_k(K, H)).$$

Thus, if $H \simeq \alpha_p$, then the claim (3.4) follows from $\dim_k H^1_{\mathrm{fpqc}}(U, \alpha_p) = \infty$. If $H \simeq \mu_p$, then the claim (3.4) follows from the following inequality:

$$\dim_{\mathbb{F}_p} \mathrm{Hom}_k(G/H, \mu_p) < \dim_{\mathbb{F}_p} \mathrm{Hom}_k(G, \mu_p) \leqslant \gamma + n - 1 = \dim_{\mathbb{F}_p} H^1_{\mathrm{fpqc}}(U, \mu_p).$$

Here for the first inequality, we use $G \simeq \mu_p \times (G/H)$. This completes the proof. $\qquad \square$

COROLLARY 3.5. *Every finite local unipotent k-group scheme appears as a finite quotient of $\pi^{\mathrm{loc}}(U)$.*

Example 3.6. Assume k is of characteristic $p = 2$. In this case, the first Frobenius kernel

$$\mathrm{SL}_{2(1)} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(F^{(1)} : \mathrm{SL}_2^{(-1)} \to \mathrm{SL}_2)$$

of the algebraic group SL_2 is nilpotent. Indeed, noticing that

$$\mathrm{SL}_{2(1)}(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a, b, c, d \in A, \; ad - bc = 1 \\ a^2 = d^2 = 1, \; b^2 = c^2 = 0 \end{array} \right\}$$

for any k-algebra A, the maps

$$\mathrm{SL}_{2(1)}(A) \to (\alpha_2 \times \alpha_2)(A); \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ab, cd), \quad A : k\text{-algebra}$$

then form a k-homomorphism $\mathrm{SL}_{2(1)} \to \alpha_2 \times \alpha_2$, which makes the following sequence

$$1 \to \mu_2 \to \mathrm{SL}_{2(1)} \to \alpha_2 \times \alpha_2 \to 1 \qquad (3.5)$$

a nonsplit central extension (cf. [Wat79, ch. 0, Exercise 3]). In particular, $\mathrm{SL}_{2(1)}$ is nilpotent. From the facts that $\mathbb{X}(\mu_2) = \mathbb{Z}/2\mathbb{Z}$ and that any homomorphism $\mathrm{SL}_{2(1)} \to \mathbb{G}_m$ factors through μ_2, the nonsplitness of (3.5) deduces the condition that $\mathbb{X}(\mathrm{SL}_{2(1)}) = \mathbb{X}(\alpha_2 \times \alpha_2) = 1$. Therefore, by applying Proposition 3.4, we can conclude that there exists a surjective homomorphism $\pi^{\mathrm{loc}}(\mathbb{A}_k^1) \twoheadrightarrow \mathrm{SL}_{2(1)}$.

Remark 3.7. (1) In the particular case where $U = \mathbb{A}_k^1$, that Question 3.3 is affirmative is equivalent to the assertion that any finite local k-group scheme G with $\mathbb{X}(G) = 1$ appears as a quotient of $\pi^{\mathrm{loc}}(\mathbb{A}_k^1)$. For example, for any integers $n, r > 0$, the rth Frobenius kernel $\mathrm{SL}_{n(r)} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(F^{(r)} : \mathrm{SL}_n^{(-r)} \to \mathrm{SL}_n)$ of SL_n gives such a one, i.e., $\mathbb{X}(\mathrm{SL}_{n(r)}) = 1$ (cf. (3) below).

(2) In general, if a finite k-group scheme G is generated by all the unipotent subgroup schemes, then G has no characters, i.e., $\mathbb{X}(G) = 1$. The author expects that the converse might be true, namely, that $\mathbb{X}(G) = 1$ if and only if G is generated by all the unipotent subgroup schemes.

(3) Let us see another example of finite local k-group scheme G with $\mathbb{X}(G) = 1$. Let Σ be a semi-simple simply connected algebraic group over k. Then for any integer $r > 0$, the rth Frobenius kernel

$$\Sigma_{(r)} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Sigma^{(-r)} \xrightarrow{F^{(r)}} \Sigma)$$

has no nontrivial characters, i.e., $\mathbb{X}(\Sigma_{(r)}) = 1$ (indeed, since Σ is semi-simple simply connected, any character $\Sigma_{(r)} \to \mathbb{G}_m$ comes from some character of Σ [Jan03, Part II, chs 3, 3.15, Proposition and Remarks 2)]. However, since Σ is semi-simple, there exist no nontrivial characters of Σ [Jan03, Part II, chs 1, 1.18(3)]). Therefore, if Question 3.3 is affirmative for the affine line $\mathbb{A}_k^1$, then the group scheme $\Sigma_{(r)}$ must appear as a finite quotient of $\pi^{\mathrm{loc}}(\mathbb{A}_k^1)$. We will prove this fact is actually true (cf. Corollary 4.19).

(4) Moreover, if Σ is a semi-simple simply connected algebraic group over k, then one can prove that the first Frobenius kernel $\Sigma_{(1)}$ is generated by all the unipotent subgroup schemes of it. Indeed, fix a maximal torus $T < \Sigma$. Let R be the root system. Choose a positive system $R^+ \subset R$ and denote by S the corresponding set of simple roots ([Jan03, Part II, chs 1, 1.5]). We denote by U^+ (respectively U^-) the unipotent radical corresponding to the positive roots (respectively the negative roots). By [Jan03, Part II, chs 3, 3.2 Lemma], it suffices to show that $T_{(1)} \subset \langle U_{(1)}^\pm \rangle$. Since Σ is simply connected, we have

$$(\alpha^\vee)_{\alpha \in S} : \prod_{\alpha \in S} \mathbb{G}_m \xrightarrow{\simeq} T$$

(cf. [Jan03, Part II, chs 1, 1.6(4)]), where the $\alpha^\vee$ are dual roots. Hence, we are reduced to showing that $\alpha^\vee(\mu_p) \subset \langle U_{(1)}^\pm \rangle$ for any $\alpha \in S$. For this, we may assume that $\Sigma = \mathrm{SL}_2$. In this case, we have $\dim k[\mathrm{SL}_{2(1)}] = p^3$ and $\dim k[U_{(1)}^\pm] = p$, whence $p^2 \leqslant \dim k[\langle U_{(1)}^\pm \rangle] \leqslant p^3$. For the equality $\mathrm{SL}_{2(1)} = \langle U_{(1)}^\pm \rangle$, it suffices to show $\dim k[\langle U_{(1)}^\pm \rangle] = p^3$. However, again by [Jan03, Part II, chs 3, 3.2, Lemma], there exists a surjective k-algebra homomorphism $k[\mathrm{SL}_{2(1)}] \twoheadrightarrow k[U_{(1)}^+ \times U_{(1)}^-]$. This factors through $k[\langle U_{(1)}^\pm \rangle]$ and the resulting algebra map $\phi : k[\langle U_{(1)}^\pm \rangle] \to k[U_{(1)}^+ \times U_{(1)}^-]$ is then surjective. The map ϕ is not isomorphism because $U_{(1)}^+ \times U_{(1)}^-$ is not a subgroup scheme of $\mathrm{SL}_{2(1)}$ but $\langle U_{(1)}^\pm \rangle$ is. Therefore, we have $\dim k[\langle U_{(1)}^\pm \rangle] > \dim k[U_{(1)}^+ \times U_{(1)}^-] = p^2$. Then we must have $\dim k[\langle U_{(1)}^\pm \rangle] = p^3$, which implies that $\mathrm{SL}_{2(1)} = \langle U_{(1)}^\pm \rangle$. This completes the proof.

4. Main results

4.1 Torsors coming from Frobenius endmorphisms of an affine algebraic group

Let k be an algebraically closed field of characteristic $p > 0$ and U a smooth affine curve over k. Let Σ be an affine algebraic group over k. Note that for each integer $r > 0$, the rth relative Frobenius morphism $F^{(r)} : \Sigma^{(-r)} \to \Sigma$ gives a saturated $\Sigma_{(r)} \overset{\mathrm{def}}{=} \mathrm{Ker}(F^{(r)})$-torsor over Σ. Here, recall that the saturatedness means the corresponding homomorphism $\pi^N(\Sigma) \to \Sigma_{(r)}$ is surjective (cf. Definition 2.3). In §4, motivated by Question 3.3, we will consider the following question.

Question 4.1. Fix an integer $r > 0$.

(1) Does there exist any k-morphism $f : U \to \Sigma$ so that the $\Sigma_{(r)}$-torsor $f^*\Sigma^{(-r)} \to U$ defined by the following cartesian diagram is saturated?

$$
\begin{array}{ccc}
f^*\Sigma^{(-r)} & \longrightarrow & \Sigma^{(-r)} \\
\downarrow & \square & \downarrow {\scriptstyle F^{(r)}} \\
U & \xrightarrow{\quad f \quad} & \Sigma
\end{array}
$$

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(2) Furthermore, if it exists, for which k-morphism $f : U \to \Sigma$, is the resulting $\Sigma_{(r)}$-torsor $f^*\Sigma^{(-r)} \to U$ saturated?

Let us begin with showing that one can reduce the problem to the case $r = 1$.

LEMMA 4.2. *Let Σ be an affine algebraic group over k and $f : U \to \Sigma$ a k-morphism. If $f^*\Sigma^{(-1)}$ is saturated, then for any integer $r > 1$, $f^*\Sigma^{(-r)}$ is also saturated.*

Proof. We will show this by induction on $r \geqslant 1$. We will denote by $\phi^{(-r)} : \pi^{\mathrm{loc}}(U) \to \Sigma_{(r)}$ the homomorphism corresponding to the torsor $f^*\Sigma^{(-r)}$. Assume $\phi^{(-r)}$ is surjective. Let us show that $\phi^{(-r-1)}$ is also surjective. Since $F^{(r)*}f^*\Sigma^{(-r)}$ is a trivial torsor, the composition

$$\pi^{\mathrm{loc}}(U^{(-r)}) \xrightarrow{F^{(r)}_*} \pi^{\mathrm{loc}}(U) \overset{\phi^{(-r)}}{\twoheadrightarrow} \Sigma_{(r)}$$

is trivial. We then obtain the following commutative diagram:

$$(4.1)$$

and the map $\phi^{(-r-1)} \circ F^{(r)}_*$ factors through $\Sigma^{(-r)}_{(1)}$. We denote by ψ the resulting homomorphism $\pi^{\mathrm{loc}}(U^{(-r)}) \to \Sigma^{(-r)}_{(1)}$. We are then reduced to showing the surjectivity of ψ. The commutativity of the diagram (4.1) implies that $\mathrm{Ind}^{\Sigma_{(r+1)}}_{\Sigma^{(-r)}_{(1)}}(Q) \simeq F^{(r)*}f^*\Sigma^{(-r-1)}$, where Q is the torsor over $U^{(-r)}$ corresponding to the morphism ψ. On the other hand, by considering the tautological commutative diagram

we can find that

$$\mathrm{Ind}^{\Sigma_{(r+1)}}_{\Sigma^{(-r)}_{(1)}}(\Sigma^{(-r-1)} \xrightarrow{F^{(1)}} \Sigma^{(-r)}) = F^{(r)*}(\Sigma^{(-r-1)} \xrightarrow{F^{(r+1)}} \Sigma).$$

Therefore, by the construction, Q is nothing but the torsor defined by the cartesian diagram

where $f^{(-r)}$ is just the rth Frobenius twist of f.

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Then, the saturatedness of the torsor $f^*\Sigma^{(1)} \to U$ indicates the saturatedness of the torsor $Q \to U^{(-r)}$, or equivalently, the surjectivity of ψ. This completes the proof. $\qquad\square$
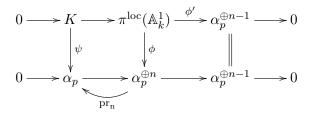
Next let us see a basic example. The following proposition gives a complete answer to Question 4.1 for the pairs $(U, \Sigma) = (\mathbb{A}_k^1, \mathbb{G}_a^{\oplus n})$ $(n \geqslant 1)$.

PROPOSITION 4.3. *Assume that k is of characteristic $p > 0$ and $n > 0$ an integer. Let*

$$\underline{f} = (f_i(t)) \in k[t]^{\oplus n} = \mathrm{Mor}_k(\mathbb{A}_k^1, \mathbb{G}_a^{\oplus n})$$

be a k-morphism $\mathbb{A}_k^1 \to \mathbb{G}_a^{\oplus n}$. We define the $\alpha_p^{\oplus n}$-torsor $P_{\underline{f}}$ over $\mathbb{A}_k^1$ by the pulling back of the relative Frobenius morphism $F^{(1)} : \mathbb{G}_a^{(-1) \oplus n} \to \mathbb{G}_a^{\oplus n}$, i.e., $P_{\underline{f}} \stackrel{\mathrm{def}}{=} \underline{f}^* \mathbb{G}_a^{(-1) \oplus n}$. Then $P_{\underline{f}}$ is saturated if and only if the images $\overline{f_i(t)}\,(1 \leqslant i \leqslant n)$ in $H^1_{\mathrm{fpqc}}(\mathbb{A}_k^1, \alpha_p) = k[t]/k[t^p]$ are linearly independent over k.

Proof. We will show this by induction on $n > 0$. In the case where $n = 1$, then since α_p is simple, the assertion is obvious. From now on assume that $n > 1$ and that $\dim_k \langle \overline{f_i(t)} \,|\, 1 \leqslant i \leqslant n-1 \rangle_k = n - 1$. Put $\underline{f}' := (f_1(t), \ldots, f_{n-1}(t))$. Then $P_{\underline{f}'}$ is an $\alpha_p^{\oplus n-1}$-torsor over $\mathbb{A}_k^1$. We denote by ϕ and ϕ' the homomorphism $\pi^{\mathrm{loc}}(\mathbb{A}_k^1) \to \alpha_p^{\oplus n}$ corresponding to $P_{\underline{f}}$ and the one $\pi^{\mathrm{loc}}(\mathbb{A}_k^1) \to \alpha_p^{\oplus n-1}$ to $P_{\underline{f}'}$, respectively. From the assumption, ϕ' is surjection. Let $K \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\phi')$. We then obtain the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \pi^{\mathrm{loc}}(\mathbb{A}_k^1) & \stackrel{\phi'}{\longrightarrow} & \alpha_p^{\oplus n-1} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & \alpha_p & \longrightarrow & \alpha_p^{\oplus n} & \longrightarrow & \alpha_p^{\oplus n-1} & \longrightarrow & 0 \\
& & & \underset{\mathrm{pr_n}}{\longleftarrow} & & & & &
\end{array}
$$

Then we have

$$
\begin{aligned}
\phi : \text{surjective} &\iff \psi \neq 1 \\
&\iff \mathrm{pr_n} \circ \phi \in \mathrm{Hom}(\pi^{\mathrm{loc}}(\mathbb{A}_k^1), \alpha_p) \setminus \phi'^* \mathrm{Hom}(\alpha_p^{\oplus n-1}, \alpha_p) \\
&\iff \overline{f_n(t)} \in (k[t]/k[t^p]) \setminus \langle \overline{f_1(t)}, \ldots, \overline{f_{n-1}(t)} \rangle_k \\
&\iff \dim_k \langle \overline{f_i(t)} \,|\, 1 \leqslant i \leqslant n \rangle_k = n.
\end{aligned}
$$

This completes the proof. $\qquad\square$

4.2 Explicit equations defining saturated $\mathrm{SL}_{2(1)}$-torsors in the characteristic $p = 2$ case

We will continue to use the same notation as in §4.1. As we have seen in Example 3.6, in the case where k is of characteristic $p = 2$, there exists a saturated $\mathrm{SL}_{2(1)}$-torsor $P \to \mathbb{A}_k^1$ (cf. Definition 2.3). On the other hand, since $H^1_{\mathrm{fpqc}}(\mathbb{A}_k^1, \mathrm{GL}_2) = 0$ and $H^0_{\mathrm{fpqc}}(\mathbb{A}_k^1, \mathrm{GL}_2) \to H^0_{\mathrm{fpqc}}(\mathbb{A}_k^1, \mathbb{G}_m) = k^*$ is surjective, we have $H^1_{\mathrm{fpqc}}(\mathbb{A}_k^1, \mathrm{SL}_2) = 0$, whence

$$H^1_{\mathrm{fpqc}}(\mathbb{A}_k^1, \mathrm{SL}_{2(1)}) \simeq \mathrm{SL}_2(k[t^2]) \backslash \mathrm{SL}_2(k[t]).$$

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Therefore, such a torsor $P \to \mathbb{A}_k^1$ must be obtained by the pulling back of the relative Frobenius morphism $F^{(1)} : \mathrm{SL}_2^{(-1)} \to \mathrm{SL}_2$ along some k-morphism $f : \mathbb{A}_k^1 \to \mathrm{SL}_2$.

$$
\begin{array}{ccc}
P & \longrightarrow & \mathrm{SL}_2^{(-1)} \\
\downarrow & \square & \downarrow {\scriptstyle F^{(1)}} \\
\mathbb{A}_k^1 & \xrightarrow{\ f\ } & \mathrm{SL}_2
\end{array}
$$

Hence, by Proposition 3.4 and Example 3.6, combining with Lemma 4.2, we can obtain an affirmative answer to Question 4.1(1) for the pair $(U, \Sigma) = (\mathbb{A}_k^1, \mathrm{SL}_2)$ in the case where $p = 2$. As a consequence of it, we have the following.

COROLLARY 4.4. *Assume $p = 2$. Then there exists a surjective homomorphism*

$$
\pi^{\mathrm{loc}}(\mathbb{A}_k^1) \twoheadrightarrow \varprojlim_{r > 0} \mathrm{SL}_{2(r)}.
$$

In particular, for any integer $r > 0$, there exists a surjective homomorphism $\pi^{\mathrm{loc}}(\mathbb{A}_k^1) \twoheadrightarrow \mathrm{SL}_{2(r)}$.

Next, we will consider Question 4.1(2) to the pair $(U, \Sigma) = (\mathbb{A}_k^1, \mathrm{SL}_2)$ in the case where k is of characteristic $p = 2$ and will give an answer (cf. Corollary 4.13).

Recall that the saturatedness of a finite étale torsor $P \to U$ depends only on the underlying scheme of it. In fact, it is saturated if and only if it is (geometrically) connected (cf. [Zha13, Lemma 2.3]). One of the difficulties of our problem is that the saturatedness of a local torsor, in contrast to the étale case, depends also on the multiplicative structure of the underlying group scheme. The following simple example indicates such a situation.

Example 4.5. Let l be a prime number with $l \neq p$. We define the k-morphism $f : \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ by $a \mapsto (a, a^l)$. We define the one $g : \mathbb{G}_m \to \mathbb{G}_a \times \mathbb{G}_m$ as the composition $\mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \times \mathbb{G}_m \subset \mathbb{G}_a \times \mathbb{G}_m$. Then the underlying schemes of the torsors $f^*(\mathbb{G}_m^{(-1)} \times \mathbb{G}_m^{(-1)})$ and $g^*(\mathbb{G}_a^{(-1)} \times \mathbb{G}_m^{(-1)})$ are isomorphic to each other. However, the former is not saturated, but the latter is.

Hence, it seems to be difficult to obtain a concise characterization of the saturatedness of finite local torsors purely in terms of the category $N(X, x)$ (cf. § 2). To avoid this problem, we will rely on a Tannakian interpretation of $\pi^N(U)$. For definitions and basic notions of Tannakian categories, see [DM82, Del90]. We will use the category of *generalized stratified bundles*, introduced by Esnault and Hogadi [EH12]. For the full definition of it, see [EH12] (for the definition of stratified bundles in the usual sense, see, for example, [Gie75, doS07]). By virtue of Lemma 4.2, the category $\mathrm{Strat}(U, 1)$ of 1-*stratified bundles* is large enough for our purpose.

DEFINITION 4.6. Let X be a smooth algebraic variety over a perfect field k of characteristic $p > 0$. A 1-*stratified bundle* on X is a sequence $\{E^{(i)}\}_{i=0}^\infty$ of coherent sheaves $E^{(i)}$ over $X^{(i)}$ together with isomorphisms $\sigma^{(i)} : E^{(i)} \xrightarrow{\simeq} F^{(1)*}E^{(i+1)}$ for $i \geqslant 1$ and

$$
\sigma^{(0)} : F^{(1)*}E^{(0)} \xrightarrow{\simeq} F^{(2)*}E^{(1)}. \tag{4.2}
$$

Let $E = \{E^{(i)}, \sigma^{(i)}\}, E' = \{E'^{(i)}, \sigma'^{(i)}\}$ be arbitrary two 1-stratified bundles. A *homomorphism* of E into E' is a sequence of $\mathcal{O}_{X^{(i)}}$-linear homomorphisms $\phi^{(i)} : E^{(i)} \to E'^{(i)}$ $(i \geqslant 0)$ satisfying

$$
\begin{aligned}
\sigma'^{(i)} \circ \phi^{(i)} &= F^{(1)*}\phi^{(i+1)} \circ \sigma^{(i)}, \quad i \geqslant 1; \\
\sigma'^{(0)} \circ F^{(1)*}\phi^{(0)} &= F^{(2)*}\phi^{(1)} \circ \sigma^{(0)}.
\end{aligned}
$$

The homomorphisms of 1-stratified bundles satisfy the composition rule and one obtains the category $\mathrm{Strat}(X,1)$ of 1-stratified bundles on X.

THEOREM 4.7 (Esnault–Hogadi [EH12]). *The category $\mathrm{Strat}(X,1)$ of 1-stratified bundles is a k-linear abelian rigid tensor category, and if one takes a k-rational point $x \in X(k)$, then the functor $\omega_x : \mathrm{Strat}(X,1) \to \mathrm{Vec}_k; \{E^{(i)}, \sigma^{(i)}\} \mapsto x^*E^{(0)}$ defines a neutral fiber functor. Furthermore, the maximal profinite quotient of its Tannakian fundamental group $\pi_1(\mathrm{Strat}(X,1), \omega_x)$ coincides with the image of $F^{(1)} : \pi^N(X,x) \to \pi^N(X,x)^{(1)}$.*

In particular, if G is a finite local k-group scheme of height one, i.e., $G_{(1)} = G$, then any homomorphism $\phi : \pi^{\mathrm{loc}}(X) \to G$ factors through the maximal pro-finite local quotient $\pi_1(\mathrm{Strat}(X,1), \omega_x)^{\mathrm{prof.\,loc}}$. Denote by ψ the resulting homomorphism

$$\psi : \pi_1(\mathrm{Strat}(X,1), \omega_x)^{\mathrm{prof.\,loc}} \to G.$$

Let us consider the composition

$$h_\phi : \mathrm{Rep}_k(G) \xrightarrow{\psi^*} \mathrm{Rep}_k(\pi_1(\mathrm{Strat}(X,1), \omega_x)^{\mathrm{prof.\,loc}}) \subset \mathrm{Strat}(X,1). \tag{4.3}$$

Then, from the standard Tannakian argument (cf. [Nor82, ch. II, Proposition 3]), we have the following.

LEMMA 4.8. *The homomorphism ϕ is surjective if and only if*

$$\dim_k \mathrm{Hom}_{\mathrm{Strat}(X,1)}(\mathbb{I}, h_\phi(k[G], \rho_{\mathrm{reg}})) = 1.$$

Here $\mathbb{I} \stackrel{\mathrm{def}}{=} \{\mathcal{O}_{X^{(i)}}, \mathrm{id}\}$ is a unit object of $\mathrm{Strat}(X,1)$.

Remark 4.9. Since $h_\phi(k[G], \rho_{\mathrm{reg}})$ is an algebra object in $\mathrm{Strat}(X,1)$, the 1-stratified bundle $h_\phi(k[G], \rho_{\mathrm{reg}})$ admits a morphism from the unit object $\mathbb{I} \to h_\phi(k[G], \rho_{\mathrm{reg}})$ corresponding to the unit element of the algebra. Thus, the dimension $\dim_k \mathrm{Hom}_{\mathrm{Strat}(X,1)}(\mathbb{I}, h_\phi(k[G], \rho_{\mathrm{reg}}))$ is always greater than or equal to 1.

If $\pi : P \to X$ is the G-torsor corresponding to the homomorphism $\phi : \pi^{\mathrm{loc}}(X) \to G$. Then one can describe the 1-stratified bundle $h_\phi(k[G], \rho_{\mathrm{reg}})$ as follows (cf. [EH12, Construction 4.1]). We define $\{E^{(i)}\}_{i=1}^\infty$ by

$$E^{(i)} \stackrel{\mathrm{def}}{=} \mathcal{O}_{X^{(i)}} \otimes k[G] = \mathcal{O}_{X^{(i)}}^{\oplus \dim k[G]}, \quad i \geqslant 1;$$
$$E^{(0)} \stackrel{\mathrm{def}}{=} (\pi_* \mathcal{O}_P \otimes k[G])^G.$$

Here G acts on $\pi_* \mathcal{O}_P \otimes k[G]$ by

$$(P \times G) \times G \ni ((p,h), g) \mapsto (p \cdot g^{-1}, h \cdot g) \in P \times G.$$

We define $\{\sigma^{(i)}\}_{i=0}^\infty$ by $\sigma^{(i)} \stackrel{\mathrm{def}}{=} \mathrm{id}$ for $i \geqslant 1$ and $\sigma^{(0)}$ the canonical trivialization morphism:

$$F^{(1)*}((\pi_* \mathcal{O}_P \otimes k[G])^G) \xrightarrow{\simeq} F^{(2)*}E^{(1)} = \mathcal{O}_{X^{(-1)}} \otimes k[G]. \tag{4.4}$$

Here, since G is of height one, the torsor $F^{(1)*}P$ is trivial and admits a canonical section $F^{(1)*}P \supset (F^{(1)*}P)_{\mathrm{red}} = X^{(-1)}$. The isomorphism $\sigma^{(0)}$ is the one corresponding to this section. We then have $h_\phi(k[G], \rho_{\mathrm{reg}}) = \{E^{(i)}, \sigma^{(i)}\}$. Note that $(k[G], \rho_{\mathrm{reg}})$ is a G-torsor object in $\mathrm{Rep}_k(G)$ and the functor h_ϕ is a tensor functor, hence $h_\phi(k[G], \rho_{\mathrm{reg}})$ gives a G-torsor object in $\mathrm{Strat}(X,1)$. We also denote $P^{\text{1-strat}} \stackrel{\mathrm{def}}{=} h_\phi(k[G], \rho_{\mathrm{reg}})$. Lemma 4.8 implies that P is saturated if and only if $P^{\text{1-strat}}$ is connected as an algebra object in $\mathrm{Strat}(X,1)$.

Example 4.10. Let $m > 0$ be an integer and $f : \mathbb{G}_m \to \mathbb{G}_m$ a k-morphism defined by $a \mapsto a^m$. Let $P^{1\text{-strat}} = \{E^{(i)}, \sigma^{(i)}\}$ with $P \stackrel{\text{def}}{=} f^* \mathbb{G}_m^{(-1)}$ a $\mu_p = \operatorname{Spec} k[z]/(z^p - 1)$-torsor. Then

$$E^{(0)} = \left(\frac{k[x, x^{-1}, y]}{(y^p - x^m)} \otimes \frac{k[z]}{(z^p - 1)} \right)^{\mu_p} = \bigoplus_{j=0}^{p-1} k[x, x^{-1}] \cdot (\overline{yz})^j.$$

Therefore, the representation matrix $A \in \operatorname{GL}_p(k[x^{(-1)\pm 1}])$ of $\sigma^{(0)}$ with respect to the basis $\{\overline{yz^j} \,|\, 0 \leqslant j \leqslant p - 1\}$ and $\{\overline{z}^j \,|\, 0 \leqslant j \leqslant p - 1\}$ is given by the diagonal matrix:

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & x^{(-1)m} & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & x^{(-1)m(p-1)} \end{pmatrix}.$$

A direct computation then recovers the following well-understood result:

$$\operatorname{Hom}_{\operatorname{Strat}(\mathbb{G}_m, 1)}(\mathbb{I}, P^{1\text{-strat}}) \simeq \begin{cases} k & \text{if } p \nmid m, \\ k[\mu_p] & \text{if } p \mid m. \end{cases}$$

Example 4.11. Let $n > 0$ be an integer and $\underline{f} = (f_i(x)) \in k[x]^{\oplus n} = \operatorname{Mor}_k(\mathbb{A}_k^1, \mathbb{G}_a^{\oplus n})$ a k-morphism (cf. Proposition 4.3). Let $P^{1\text{-strat}} = \{E^{(i)}, \sigma^{(i)}\}$ with

$$P \stackrel{\text{def}}{=} \underline{f}^* \mathbb{G}_a^{(-1)\oplus n} = \operatorname{Spec} \frac{k[x, y_i \,(1 \leqslant i \leqslant n)]}{(y_i^p - f_i(x) \,(1 \leqslant i \leqslant n))}$$

an $\alpha_p^{\oplus n} = \operatorname{Spec} k[z_i(1 \leqslant i \leqslant n)]/(z_i^p(1 \leqslant i \leqslant n))$-torsor. Then

$$E^{(0)} = \bigoplus_{0 \leqslant j_1, \ldots, j_n \leqslant p-1} k[x] \cdot (\overline{y_1 + z_1})^{j_1} \cdots (\overline{y_n + z_n})^{j_n}.$$

The representation matrix $A \in \operatorname{GL}_{p^n}(k[x^{(-1)}])$ of $\sigma^{(0)}$ with respect to the basis

$$\{(\overline{y_1 + z_1})^{j_1} \cdots (\overline{y_n + z_n})^{j_n}\}, \quad \{\overline{z_1}^{j_1} \cdots \overline{z_n}^{j_n}\}$$

is unipotent and, by a direct computation, one can verify that the condition that $\{\overline{f_i(x)}\}_{i=1}^n \subset k[x]/k[x^p]$ is linearly independent over k is equivalent to the condition that

$$\dim_k \operatorname{Hom}_{\operatorname{Strat}(\mathbb{A}_k^1, 1)}(\mathbb{I}, P^{1\text{-strat}}) = 1.$$

This then gives another proof of Proposition 4.3.

Now let us prove the main result of this subsection.

THEOREM 4.12. *Assume $p = 2$. Let*

$$\underline{f} = \begin{pmatrix} f_{22}(x)x^{-m} & -f_{12}(x)x^{-m} \\ -f_{21}(x)x^{-m} & f_{11}(x)x^{-m} \end{pmatrix} \in \operatorname{GL}_2(k[x^{\pm 1}]) = \operatorname{Mor}_k(\mathbb{G}_m, \operatorname{GL}_2)$$

be a k-morphism with $f_{ij}(x) \in k[x]$ $(1 \leqslant i, j \leqslant 2)$ and

$$\det \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{pmatrix} = x^m$$

for some $m \in \mathbb{Z}_{\geqslant 0}$. Let $P \stackrel{\text{def}}{=} \underline{f}^* \operatorname{GL}_2^{(-1)}$ be the resulting torsor over $\mathbb{G}_m$.

(1) *In the case $2 \mid m$, assume that one of the following conditions is satisfied:*
- $\dim_k \langle \overline{f_{11}}, \overline{f_{21}} \rangle = \dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = 2$;
- $\dim_k \langle \overline{f_{12}}, \overline{f_{22}} \rangle = \dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = 2$.

   Then we have
$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m, 1)}(\mathbb{I}, P^{\text{1-strat}}) \simeq k[\mu_2].$$

(2) *In the case $2 \nmid m$, assume that one of the following conditions is satisfied:*
- $\dim_k \langle \overline{f_{11}}, \overline{f_{21}} \rangle = 2$, $\dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = \dim_k \langle \overline{f_{11}f_{12}}, \overline{f_{21}f_{22}}, \overline{x^m} \rangle = 3$;
- $\dim_k \langle \overline{f_{12}}, \overline{f_{22}} \rangle = 2$, $\dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = \dim_k \langle \overline{f_{11}f_{12}}, \overline{f_{21}f_{22}}, \overline{x^m} \rangle = 3$

   Then we have
$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m, 1)}(\mathbb{I}, P^{\text{1-strat}}) = k.$$

Here, for each $f \in k[x]$, $\overline{f}$ denotes the image of f in $k[x]/k[x^p]$.

Proof. Let $P^{\text{1-strat}} = \{E^{(i)}, \sigma^{(i)}\}$. Let $\pi : P \to \mathbb{G}_m$ be the structure morphism. We then have

$$\pi_* \mathcal{O}_P = \frac{k[x^{\pm 1}, y_{ij} \ (i, j = 1, 2)]}{(y_{11}^2 - f_{22}(x)x^{-m}, y_{12}^2 - f_{12}(x)x^{-m}, y_{21}^2 - f_{21}(x)x^{-m}, y_{22}^2 - f_{11}(x)x^{-m})}.$$

Let ρ_P (respectively ρ_0) be the coaction $\pi_* \mathcal{O}_P \to \pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}]$ induced by the action $P \times \mathrm{GL}_{2(1)} \to P$ (respectively the one by the trivial action). Let ι be the antipode of $k[\mathrm{GL}_{2(1)}]$. Then the composition

$$\pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}] \xrightarrow{\rho \otimes \mathrm{id}} \pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}] \otimes k[\mathrm{GL}_{2(1)}]$$
$$\xrightarrow{\mathrm{id} \otimes \iota \otimes \mathrm{id}} \pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}] \otimes k[\mathrm{GL}_{2(1)}] \xrightarrow{\mathrm{id} \otimes m} \pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}] \quad (4.5)$$

gives a $k[\mathrm{GL}_{2(1)}]$-comodule isomorphism

$$(\pi_* \mathcal{O}_P, \rho_P) \otimes (k[\mathrm{GL}_{2(1)}], \rho_{\mathrm{reg}}) \xrightarrow{\simeq} (\pi_* \mathcal{O}_P, \rho_0) \otimes (k[\mathrm{GL}_{2(1)}], \rho_{\mathrm{reg}}). \quad (4.6)$$

If one writes $k[\mathrm{GL}_{2(1)}] = k[z_{ij}]/(z_{ij}^2 - \delta_{ij})$, then

$$\begin{pmatrix} \iota(z_{11}) & \iota(z_{12}) \\ \iota(z_{21}) & \iota(z_{22}) \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}^{-1}.$$

Therefore, via the map (4.5), the element

$$e_{ij} \stackrel{\mathrm{def}}{=} \sum_{q=1}^{2} y_{iq} \otimes z_{qj} \in \pi_* \mathcal{O}_P \otimes k[\mathrm{GL}_{2(1)}]$$

is mapped to $y_{ij} \otimes 1$, which belongs to

$$((\pi_* \mathcal{O}_P, \rho_0) \otimes (k[\mathrm{GL}_{2(1)}], \rho_{\mathrm{reg}}))^{\mathrm{GL}_{2(1)}} = k[x^{\pm 1}][y_{ij} \otimes 1 \mid 1 \leqslant i, j \leqslant 2].$$

Hence, by (4.6), we find that

$$E^{(0)} = k[x^{\pm 1}][e_{ij} \mid 1 \leqslant i, j \leqslant 2].$$

Notice that $E^{(i)} \simeq k[x^{\pm 1}]^{\oplus 2^4}$ with free basis $\{e_{11}^{m_{11}} e_{12}^{m_{12}} e_{21}^{m_{21}} e_{22}^{m_{22}} \mid m_{ij} = 0, 1\}$.

Let $A \in \mathrm{GL}_{2^4}(k[x^{(-1)\pm 1}])$ be the representation matrix of the inverse isomorphism $\sigma^{(0)-1}$ with respect to the basis

$$1, \nu_{11}\nu_{21}, \nu_{12}\nu_{22}, \nu_{11}\nu_{12}\nu_{21}\nu_{22},;$$
$$\nu_{11}, \nu_{12}, \nu_{21}, \nu_{22},$$
$$\nu_{11}\nu_{12}, \nu_{11}\nu_{22}, \nu_{12}\nu_{21}, \nu_{21}\nu_{22};$$
$$\nu_{11}\nu_{12}\nu_{21}, \nu_{11}\nu_{12}\nu_{22}, \nu_{11}\nu_{21}\nu_{22}, \nu_{12}\nu_{21}\nu_{22}$$

for $\nu \in \{e, z\}$. One then obtains

$$A = \begin{pmatrix} C & O & O & O \\ & B \otimes E & O & B \otimes D \\ & & B \otimes B & O \\ O & & & B \otimes x^{(-1)m}E \end{pmatrix},$$

with

$$B = \begin{pmatrix} f'_{11}(x^{(-1)}) & f'_{21}(x^{(-1)}) \\ f'_{12}(x^{(-1)}) & f'_{22}(x^{(-1)}) \end{pmatrix}; \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$C = \begin{pmatrix} 1 & f'_{11}(x^{(-1)})f'_{21}(x^{(-1)}) & f'_{12}(x^{(-1)})f'_{22}(x^{(-1)}) & * \\ 0 & x^{(-1)m} & 0 & f'_{12}(x^{(-1)})f'_{22}(x^{(-1)})x^{(-1)m} \\ 0 & 0 & x^{(-1)m} & f'_{11}(x^{(-1)})f'_{21}(x^{(-1)})x^{(-1)m} \\ 0 & 0 & 0 & x^m \end{pmatrix};$$

$$D = \begin{pmatrix} 0 & f'_{12}(x^{(-1)})f'_{22}(x^{(-1)}) \\ f'_{11}(x^{(-1)})f'_{21}(x^{(-1)}) & 0 \end{pmatrix},$$

where $\{f'_{ij}(x^{(-1)})\}$ is defined by

$$f'_{ij}(x^{(-1)})^2 = f_{ij}(x).$$

Here, notice that $x^{(-1)2} = x$ and that $\det B = x^{(-1)m}$. Furthermore, the assumption on $\{f_{ij}(x)\}$ implies the same condition on $\{f'_{ij}(x^{(-1)})\}$.

Now one can reduce the problem to calculating the right-hand side of the following equation:

$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m,1)}(\mathbb{I}, P^{1\text{-strat}}) \simeq \{(\underline{a}, \underline{b}) \in k^{\oplus 2^4} \times k[x^{\pm 1}]^{\oplus 2^4} \mid A \cdot \underline{a} = \underline{b}\}.$$

First we consider the case where $2 \mid m$. In this case, without loss of generality, we may assume that $m = 0$. Note that the condition that $\det(f_{ij}(x)) = 1$ implies that the set

$$\{\overline{f_{11}f_{12}}, \overline{f_{21}f_{22}}\}$$

is also linearly independent over k. Indeed, this follows from the equation:

$$\begin{pmatrix} f_{11}f_{12} \\ f_{21}f_{22} \end{pmatrix} = \begin{pmatrix} f_{12}^2 & f_{11}^2 \\ f_{22}^2 & f_{21}^2 \end{pmatrix} \begin{pmatrix} f_{11}f_{21} \\ f_{12}f_{22} \end{pmatrix}.$$

We will solve the simultaneous equations $A \cdot \underline{a} = \underline{b}$ from the bottom. Then the condition that the set $\{\overline{f'_{11}}, \overline{f'_{21}}\}$, or $\{\overline{f'_{12}}, \overline{f'_{22}}\}$ is linearly independent implies that

$$a_i = 0 \quad (13 \leqslant i \leqslant 16).$$

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Moreover, the conditions that $\{\overline{f'_{11}f'_{12}},\ \overline{f'_{21}f'_{22}}\}$ is linearly independent and that $\det B = f'_{11}f'_{22} - f'_{12}f'_{21} = 1$ imply that

$$a_i = 0 \quad (i = 9, 12),$$
$$a_{10} = a_{11}.$$

By solving the equations

$$\begin{pmatrix} C & O \\ O & B \otimes E \end{pmatrix},$$

again combined with the assumption on $\overline{f'_{ij}}$, we have

$$a_i = 0 \quad (2 \leqslant i \leqslant 8).$$

All the above computations then imply

$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m,1)}(\mathbb{I}, P^{1\text{-strat}}) \simeq k \oplus k \cdot \det(z_{ij}) \simeq k[\mu_2].$$

Finally, let us consider the case where $2 \nmid m$. In this case, the equations in the part $B \otimes x^{(-1)m} E$ cannot imply $a_i = 0$ $(13 \leqslant i \leqslant 16)$. However, by solving the simultaneous equations

$$\begin{pmatrix} O & B \otimes E & O & B \otimes D \end{pmatrix} \underline{a} = \underline{b},$$

we can conclude that

$$a_i = 0 \quad (5 \leqslant i \leqslant 8 \text{ or } 13 \leqslant i \leqslant 16).$$

Next let us solve the part C. Notice that $\{\overline{x^{(-1)m}},\ \overline{f'_{11}f'_{21}x^{(-1)m}}\}$ is linearly independent over k. Thus, we find that $a_4 = 0$. On the other hand, by the assumption that $\{\overline{f'_{11}f'_{21}},\ \overline{f'_{12}f'_{22}}\}$ is linearly independent, we also have $a_2 = a_3 = 0$. Finally let us solve the part $B \otimes B$. Then the condition that $\overline{f'_{11}f'_{21}} \neq 0$ in $k[x^{(-1)\pm1}]/k[x^{(-1)\pm2}]$ implies that $a_{10} = a_{11}$. Then the condition that $\{\overline{f_{11}f_{12}},\ \overline{f_{21}f_{22}},\ \overline{x^m}\}$ is linearly independent over k implies that $a_9 = a_{10} = a_{11} = a_{12} = 0$. Therefore, we can conclude that

$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m,1)}(\mathbb{I}, P^{1\text{-strat}}) = k. \qquad \square$$

As an immediate consequence of Theorem 4.12 (or its proof), we have the following.

COROLLARY 4.13. *Assume $p = 2$. Let*

$$\underline{f} = \begin{pmatrix} f_{22}(x) & f_{12}(x) \\ f_{21}(x) & f_{11}(x) \end{pmatrix} \in \mathrm{SL}_2(k[x]) = \mathrm{Mor}_k(\mathbb{A}^1_k, \mathrm{SL}_2)$$

be a k-morphism. Then the resulting $\mathrm{SL}_{2(1)}$-torsor $\underline{f}^* \mathrm{SL}_2^{(-1)} \to \mathbb{A}^1_k$ is saturated if and only if one of the following conditions is satisfied:

- $\dim_k \langle \overline{f_{11}}, \overline{f_{21}} \rangle = \dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = 2$;
- $\dim_k \langle \overline{f_{12}}, \overline{f_{22}} \rangle = \dim_k \langle \overline{f_{11}f_{21}}, \overline{f_{12}f_{22}} \rangle = 2$.

Proof. By considering the composition

$$\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k \xrightarrow{f} \mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2,$$

we are reduced to the case where $\underline{f} : \mathbb{G}_m \to \mathrm{GL}_2$ with $\det(\underline{f}) = 1$. Thus, the statement follows from the argument in the proof of Theorem 4.12, noticing that the condition

$$\dim_k \mathrm{Hom}_{\mathrm{Strat}(\mathbb{G}_m,1)}(\mathbb{I}, P^{1\text{-strat}}) = 2$$

implies the image of the homomorphism $\pi^{\mathrm{loc}}(\mathbb{G}_m) \to \mathrm{GL}_{2(1)}$ corresponding to the torsor $P = \underline{f}^* \mathrm{GL}_2^{(-1)} \to \mathbb{G}_m$ coincides with $\mathrm{SL}_{2(1)}$. $\qquad\square$

Remark 4.14. For example, the embedding

$$\mathbb{A}_k^1 \ni a \mapsto \begin{pmatrix} 1 + a^2 + a^3 & a \\ a + a^2 & 1 \end{pmatrix} \in \mathrm{SL}_2$$

satisfies the condition in Corollary 4.13.

Combined with Lemma 4.2, Theorem 4.12 also implies the following.

COROLLARY 4.15. *Assume $p = 2$. Then, there exists a surjective homomorphism*

$$\pi^{\mathrm{loc}}(\mathbb{G}_m) \twoheadrightarrow \varprojlim_{r>0} \mathrm{GL}_{2(r)}.$$

In particular, for any $r > 0$, the rth Frobenius kernel $\mathrm{GL}_{2(r)}$ appears as a finite quotient of $\pi^{\mathrm{loc}}(\mathbb{G}_m)$.

Proof. It suffices to find a k-morphism $\underline{f} \in \mathrm{GL}_2(k[x^{\pm 1}]) = \mathrm{Mor}_k(\mathbb{G}_m, \mathrm{GL}_2)$ satisfying the condition of Theorem 4.12. The morphism

$$\mathbb{G}_m \ni a \mapsto \begin{pmatrix} \dfrac{1}{a} + \dfrac{1}{a^4} & \dfrac{1}{a^2} \\ 1 & \dfrac{1}{a} \end{pmatrix} \in \mathrm{GL}_2$$

gives such a one. $\qquad\square$

Remark 4.16. (1) Corollary 4.13, combined with Lemma 4.2, gives an answer to Question 4.1(2) for the pair $(U, \Sigma) = (\mathbb{A}_k^1, \mathrm{SL}_2)$.

(2) Theorem 4.12 and the proof of Corollary 4.15, combined with Lemma 4.2, gives an affirmative answer to Question 4.1(1) for the pair $(U, \Sigma) = (\mathbb{G}_m, \mathrm{GL}_2)$ in the case where k is of characteristic $p = 2$. Furthermore, since $\mathbb{X}(\mathrm{GL}_{2(r)}) = \mathbb{Z}/2^r\mathbb{Z} \subset \mathbb{Q}_2/\mathbb{Z}_2$ (note that the first equality follows from the exactness of $1 \to \mathrm{SL}_{2(r)} \to \mathrm{GL}_{2(r)} \to \mu_{p^r} \to 1$ and $\mathbb{X}(\mathrm{SL}_{2(r)}) = 1$), Corollary 4.15 gives an affirmative answer to Question 3.3 for $\mathbb{G}_m$ and for $\mathrm{GL}_{2(r)}$ $(r > 0)$ in the case where k is of characteristic $p = 2$. However, we have restricted our attention to a special class of k-morphisms $\mathbb{G}_m \to \mathrm{GL}_2$, so it is not enough to give a complete answer to Question 4.1(2).

4.3 Bertini type theorem for finite local torsors and its application

Finally let us prove a purely inseparable analogue of a Bertini type theorem (cf. [Jou83]). As an application, we will give an affirmative answer to Question 4.1(1) for the pair $(\mathbb{A}_k^1, \Sigma)$ with Σ a semi-simple simply connected algebraic group over k, whence, to Question 3.3 for $\mathbb{A}_k^1$ and for $\Sigma_{(r)}$ $(r > 0)$.

THEOREM 4.17. *Let $n \geqslant 2$ be an integer. Let k be a perfect field of positive characteristic $p > 0$. Let G be a finite local k-group scheme of height one and $\pi : P \to \mathbb{A}_k^n$ be a saturated G-torsor. Then there exists a closed immersion $\iota : \mathbb{A}_k^{n-1} \hookrightarrow \mathbb{A}_k^n$ such that the G-torsor $\iota^* P \to \mathbb{A}_k^{n-1}$ obtained by pulling back P along ι is saturated as well.*

To prove this theorem, we will rely on the Tannakian interpretation (cf. Theorem 4.7) again. Then we are reduced to showing the following lemma of linear algebras.

LEMMA 4.18. *Let $n \geqslant 2$ be an integer. Let k be a perfect field of positive characteristic $p > 0$. Let $V_1, \ldots, V_m \subset k[x_1, \ldots, x_n]/k[x_1^p, \ldots, x_n^p]$ be finite dimensional subspaces. Then there exists a polynomial $g = g(x_1, \ldots, x_{n-1})$ so that the k-linear map*

$$k[x_1, \ldots, x_n]/k[x_1^p, \ldots, x_n^p] \to k[x_1, \ldots, x_{n-1}]/k[x_1^p, \ldots, x_{n-1}^p];$$

$$x_i \mapsto \begin{cases} x_i, & 1 \leqslant i \leqslant n-1, \\ g(x_1, \ldots, x_{n-1}), & i = n, \end{cases}$$

maps all the subspaces V_i injectively into $k[x_1, \ldots, x_{n-1}]/k[x_1^p, \ldots, x_{n-1}^p]$.

Proof. By considering $V := V_1 + \cdots + V_m$, we are reduced to the case where $m = 1$. Let V be a finite dimensional subspace of $k[x_1, \ldots, x_n]/k[x_1^p, \ldots, x_n^p]$. Without loss of generality, we may assume that V is of the form

$$V = \langle x_1^{m_1} \cdots x_n^{m_n} \mid 0 \leqslant m_i \leqslant d; \ p \nmid m_i \text{ for some } i \rangle$$

for a sufficiently large integer $d > 0$. Take an integer $M > d$ with $p \nmid M$ and an integer $N > 0$ with $p^N > d(M + 1)$. Let us define

$$g \stackrel{\text{def}}{=} \prod_{i=1}^{n-1} (x_i^{p^N} + x_i^M).$$

Note that the following subset of $k[x_1, \ldots, x_{n-1}]/k[x_1^p, \ldots, x_{n-1}^p]$ is linearly independent over k:

$$x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \quad (0 \leqslant m_i \leqslant d, m_n = 0; \ p \nmid m_i \text{ for some } 1 \leqslant i \leqslant n-1),$$

$$\prod_{i=1}^{n-1} x_i^{m_i + m_n M} \quad (p \mid m_i \text{ for any } 1 \leqslant i \leqslant n-1, \text{ whence } p \nmid m_n),$$

$$\prod_{i=1}^{n-1} x_i^{m_i + m_n p^N} \quad (0 \leqslant m_i \leqslant d; \ m_n \neq 0; \ p \nmid m_i \text{ for some } 1 \leqslant i \leqslant n-1).$$

One can then conclude that the polynomial g gives a desired polynomial. $\square$

Proof of Theorem 4.17. We will show that there exists a closed immersion $\iota : \mathbb{A}_k^{n-1} \hookrightarrow \mathbb{A}_k^n$ such that

$$\dim_k \operatorname{Hom}_{\operatorname{Strat}(\mathbb{A}_k^{n-1}, 1)}(\mathbb{I}, (\iota^* P)^{1\text{-strat}}) = 1.$$

Note that $(\iota^* P)^{1\text{-strat}} = \iota^*(P^{1\text{-strat}})$ and that, from the assumption, we have

$$\dim_k \operatorname{Hom}_{\operatorname{Strat}(\mathbb{A}_k^n, 1)}(\mathbb{I}, P^{1\text{-strat}}) = 1. \tag{4.7}$$

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Let $P^{\text{1-strat}} = \{E^{(i)}, \sigma^{(i)}\}$. By Serre's conjecture on vector bundles on an affine space (cf. [Lam78]), the vector bundle $E^{(0)}$ is a free $\mathcal{O}_{\mathbb{A}_k^n}$-module. Let

$$A \in \mathrm{GL}_q(k[x_1^{(-1)}, \ldots, x_n^{(-1)}])$$

with $q = \dim_k k[G]$, which is some power of p, be the representation matrix of $\sigma^{(0)-1}$ with respect to some basis. The assumption (4.7) then amounts to saying that the vector space

$$\mathrm{Hom}_{\mathrm{Strat}(\mathbb{A}_k^n, 1)}(\mathbb{I}, P^{\text{1-strat}}) \simeq \{\underline{a} \in k^{\oplus q} \mid A \cdot \underline{a} \in k[x_1, \ldots, x_n]^{\oplus q}\}$$

is of dimension one. Then, by permuting basis if necessary, we may assume that

$$\{\underline{a} \in k^{\oplus q} \mid A \cdot \underline{a} \in k[x_1, \ldots, x_n]^{\oplus q}\} = \left\{ \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \middle| a \in k \right\}.$$

Then, the matrix A has the following form:

$$A = \begin{pmatrix} a_{11} & * & \cdots & * \\ a_{21} & * & \cdots & * \\ \vdots & & \ddots & \\ a_{q1} & * & \cdots & * \end{pmatrix} \quad \text{with } a_{i1} \in k[x_1, \ldots, x_n] \ (1 \leqslant i \leqslant q).$$

Let us write

$$B \overset{\text{def}}{=} \begin{pmatrix} a_{12} & \cdots & a_{1q} \\ & \cdots & \\ a_{q2} & \cdots & a_{qq} \end{pmatrix}.$$

Then the condition that $\dim_k \mathrm{Hom}_{\mathrm{Strat}(\mathbb{A}_k^1, 1)}(\mathbb{I}, P^{\text{1-strat}}) = 1$ is equivalent to the condition that

$$B \cdot \underline{b} \notin k[x_1, \ldots, x_n]^{\oplus q} \quad (0 \neq \underline{b} \in k^{\oplus q-1}).$$

Therefore, we are reduced to showing the following statement: if a matrix

$$B = (b_{ij}) \in \mathrm{Matrix}_{s,t}(k[x_1^{(-1)}, \ldots, x_n^{(-1)}])$$

satisfies the condition

$$B \cdot \underline{b} \notin k[x_1, \ldots, x_n]^{\oplus t} \quad (0 \neq \underline{b} \in k^{\oplus s}),$$

then there exists a polynomial $g = g(x_1^{(-1)}, \ldots, x_{n-1}^{(-1)})$ such that the condition

$$B' \cdot \underline{b} \notin k[x_1, \ldots, x_{n-1}]^{\oplus t} \quad (0 \neq \underline{b} \in k^{\oplus s}),$$

is fulfilled. Here $B' = (b'_{ij})$ is a matrix with $b'_{ij} = b_{ij}(x_1^{(-1)}, \ldots, x_{n-1}^{(-1)}, g)$. Indeed, by applying Lemma 4.18 to the subspaces

$$V_i \overset{\text{def}}{=} \langle \overline{b_{i1}}, \ldots, \overline{b_{is}} \rangle \subset k[x_1^{(-1)}, \ldots, x_n^{(-1)}]/k[x_1, \ldots, x_n] \quad (1 \leqslant i \leqslant t),$$

we can find a polynomial $g = g(x_1^{(-1)}, \ldots, x_{n-1}^{(-1)})$ so that the homomorphism

$$k[x_1^{(-1)}, \ldots, x_n^{(-1)}]/k[x_1, \ldots, x_n] \to k[x_1^{(-1)}, \ldots, x_{n-1}^{(-1)}]/k[x_1, \ldots, x_{n-1}]$$

induced by $x_n^{(-1)} \mapsto g(x_1^{(-1)}, \ldots, x_{n-1}^{(-1)})$ maps all the subspaces V_i injectively into

$$k[x_1^{(-1)}, \ldots, x_{n-1}^{(-1)}]/k[x_1, \ldots, x_{n-1}].$$

Then for any $\underline{b} \in k^{\oplus s}$, the condition that $B' \cdot \underline{b} \in k[x_1, \ldots, x_{n-1}]^{\oplus t}$ implies the condition that $B \cdot \underline{b} \in k[x_1, \ldots, x_n]^{\oplus t}$. This completes the proof. $\qquad \square$

COROLLARY 4.19. *Let k be a perfect field of positive characteristic $p > 0$. Let Σ be a semi-simple simply connected algebraic group over k. Then there exists a surjective homomorphism*

$$\pi^{\mathrm{loc}}(\mathbb{A}^1_k) \twoheadrightarrow \varprojlim_{r>0} \Sigma_{(r)}.$$

In particular, for any $r > 0$, the rth Frobenius kernel $\Sigma_{(r)}$ of Σ appears as a finite quotient of $\pi^{\mathrm{loc}}(\mathbb{A}^1_k)$.

Proof. We will adopt the argument of [Ser92, § 3.2]. Let us find a k-morphism $\mathbb{A}^1_k \to \Sigma$ so that the resulting tower of torsors

$$\cdots \to \Sigma^{(-r)}|_{\mathbb{A}^1_k} \to \cdots \to \Sigma^{(-1)}|_{\mathbb{A}^1_k} \to \mathbb{A}^1_k$$

is saturated. By virtue of Lemma 4.2, it suffices to find a k-morphism $\mathbb{A}^1_k \to \Sigma$ so that $\Sigma^{(-1)}|_{\mathbb{A}^1_k}$ is saturated. Fix a maximal torus $T < \Sigma$ and a set of positive roots. Let $U^+ < \Sigma$ (respectively $U^- < \Sigma$) be the unipotent radical corresponding to the positive roots (respectively the negative roots). Let us define a k-morphism $f : U^+ \times U^- \to \Sigma$ by the composition

$$U^+ \times U^- \hookrightarrow \Sigma \times \Sigma \xrightarrow{m} \Sigma.$$

Here the first map is the natural inclusion and the second one the multiplication of Σ. Then the pulling-back $P \overset{\mathrm{def}}{=} f^*\Sigma$ defines a saturated $\Sigma_{(1)}$-torsor over $U^+ \times U^-$. Indeed, notice that the diagram

commutes. Then the $\Sigma_{(1)}$-torsor $i^*_\pm P \to U^\pm$ is reduced to the saturated $U^\pm_{(1)}$-torsor $U^{\pm(-1)} \to U^\pm$, i.e.,

$$i^*_\pm P \simeq \mathrm{Ind}^{\Sigma_{(1)}}_{U^\pm_{(1)}}(U^{\pm(-1)}) \overset{\mathrm{def}}{=} U^{\pm(-1)} \times^{U^\pm_{(1)}} \Sigma_{(1)}.$$

If we denote by $\phi : \pi^{\mathrm{loc}}(U^+ \times U^-) \to \Sigma_{(1)}$ (respectively $\psi^\pm : \pi^{\mathrm{loc}}(U^\pm) \to U^\pm_{(1)}$) the homomorphism corresponding to the torsor P (respectively $i^*_\pm P$), this amounts to saying that the diagram homomorphism

commutes. Therefore, $\mathrm{Im}(\phi) \supset U^\pm_{(1)}$, whence $\mathrm{Im}(\phi) = \langle U^\pm_{(1)} \rangle = \Sigma_{(1)}$, where, for the last equality, see Remark 3.7(4). Therefore, P is a saturated $\Sigma_{(1)}$-torsor over $U^+ \times U^- \simeq \mathbb{A}^N_k$ for some $N > 1$. Then by applying Theorem 4.17, we can conclude that there exists a closed immersion $\iota : \mathbb{A}^1_k \hookrightarrow U^+ \times U^-$ so that $\iota^*P \to \mathbb{A}^1_k$ is saturated as well. Therefore, the k-morphism $f \circ \iota : \mathbb{A}^1_k \to \Sigma$ gives a desired one. This completes the proof. $\square$

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