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Darboux Transformations for the KP Hierarchy in the Segal-Wilson Setting

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Abstract. In this paper it is shown that inclusions inside the Segal-Wilson Grassmannian give rise to Darboux transformations between the solutions of the KP hierarchy corresponding to these planes. We present a closed form of the operators that procure the transformation and express them in the related geometric data. Further the associated transformation on the level of τ-functions is given.

1 The KP Hierarchy

The KP hierarchy consists of a tower of nonlinear differential equations in infinitely many variables $\{t_n \mid n \geq 1\}$. It is named after the simplest nontrivial equation in this tower, the Kadomtsev-Petviashvili equation:

$$\frac{3}{4}\frac{\partial^2 u}{\partial t_2^2} = \frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_3} - 3u\frac{\partial u}{\partial t_1} - \frac{1}{4}\frac{\partial^3 u}{\partial t_1^3}\right), \tag{1.1}$$

which is clearly a two dimensional generalization of the KdV equation

$$\frac{\partial u}{\partial t_3} = 3u\frac{\partial u}{\partial t_1} + \frac{1}{4}\frac{\partial^3 u}{\partial t_1^3}. \tag{1.2}$$

It is used as a model to describe surface waves, see [8]. The idea to treat simultaneously the whole tower and not just a single equation, goes back to the Sato-school, see *e.g.* [4]. We consider solutions of these equations that belong to a commutative ring of functions R, which is stable under the operators $\partial_n = \frac{\partial}{\partial t_n}$. The compact form in which one usually presents the equations of the hierarchy, is the so-called Lax form. This is an equality between operators in the privileged derivation $\partial = \partial_1$ of a specific nature. For notational convenience we sometimes write x instead of t_1 and $\frac{\partial}{\partial x}$ instead of ∂. This simple way to present the equations requires that one extends the ring $R[\partial] = \{\sum_{i=0}^n a_i\partial^i \mid a_i \in R\}$ and adds suitable integral operators to the ring. Then it becomes possible to take the inverse and roots of certain differential operators. *E.g.* the square root $\mathcal{L}^{\frac{1}{2}}$ of the Schrödinger operators $\mathcal{L} = \partial^2 + 2u$ is well-defined in this extension. Thus one arrives at the ring $R[\partial, \partial^{-1})$ of pseudodifferential operators with coefficients in R. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i\partial^i, \quad a_i \in R \text{ for all } i,$$

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that are added in an obvious way and multiplied according to

$$\partial^j \circ a\partial^i = \sum_{k=0}^{\infty} \binom{j}{k} \partial^k(a)\partial^{i+j-k}.$$

Each operator $P = \sum p_j \partial^j$ decomposes as $P = P_+ + P_-$ with $P_+ = \sum_{j \geq 0} p_j \partial^j$ its differential operator part and $P_- = \sum_{j<0} p_j \partial^j$ its integral operator part. We denote by $\operatorname{Res}_\partial P = p_{-1}$ the *residue* of P. An operator $L \in R[\partial, \partial^{-1})$ of the form

$$L = \partial + \sum_{j<0} \ell_j \partial^j, \quad \ell_j \in R \text{ for all } j < 0 \tag{1.3}$$

carries the name *Lax operator*. We call a Lax operator a *solution of the* KP *hierarchy* if and only if it satisfies the system of equations

$$\partial_n(L) = \sum_{j<0} \partial_n(\ell_j)\partial^j = [(L^n)_+, L], \quad n \geq 1. \tag{1.4}$$

They are called the *Lax equations* for L. As such they are a generalization of the so-called Lax equations of the m-th Gelfand-Dickey hierarchy, which is the following system of equations for a differential operator $\mathcal{L} = \partial^m + \sum_{i \leq m-2} l_i \partial^i$ in $R[\partial]$,

$$\partial_n(\mathcal{L}) = [(\mathcal{L}^{\frac{n}{m}})_+, \mathcal{L}], \quad n \geq 1. \tag{1.5}$$

E.g. for $m = 2$ this operator $\mathcal{L}$ will be the Schrödinger operator $\partial^2 + 2u$ and the case $n = 3$ of the Lax equations (1.5) is then equivalent to the property that u is a solution of the KdV equation. Because of this fact, the second Gelfand-Dickey hierarchy is mostly called the KdV hierarchy. A similar situation occurs in the KP-case. First one shows that the Lax equations of the KP hierarchy are equivalent to the following infinite set of conditions for the Lax operator L:

$$\partial_n(L^m)_+ - \partial_m(L^n)_+ = [(L^n)_+, (L^m)_+], \quad m, n \geq 1. \tag{1.6}$$

The case $n = 3$ and $m = 2$ of the system (1.6) implies then that the coefficient ℓ_{-1} of L is a solution of the KP-equation. The equations in (1.6) carry the name *zero curvature equations* or *Zakharov-Shabat equations*.

The equations of the KP hierarchy possess a rich collection of solutions besides the trivial one $L = \partial$. In [14] it was shown how to construct solutions starting from a Grassmann manifold of a Hilbert space. We will refer to this set of solutions of the KP hierarchy as the Segal-Wilson class. The coefficients of the Lax operators thus constructed turned out to be meromorhic functions on a group of commuting flows acting on the manifold and that is the algebra R we will be dealing with here. Besides the construction of this extensive class of solutions, Segal and Wilson also gave a geometric characterization of the solutions in it corresponding to the m-th Gelfand-Dickey hierarchy.

Note that, if L is a Lax operator in $R[\partial, \partial^{-1})$, then for all monic P and Q in $R[\partial]$ the operators PLP^{-1} and $Q^{-1}LQ$ are again Lax operators. Both type of transformations already occurred in the work of Darboux. He considered namely for Schrödinger operators $\mathcal{L} = \partial^2 + 2u$ the following transformation: take a non-zero ϕ such that

$$\mathcal{L}(\phi) = \partial^2(\phi) + 2u\phi = 0$$

and consider then the new Schrödinger operator

$$\tilde{\mathcal{L}} = \partial^2 + 2\tilde{u}, \quad \text{with } \tilde{u} = -u - \left(\frac{\partial(\phi)}{\phi}\right)^2.$$

One easily verifies that $\tilde{\mathcal{L}}$ and $\mathcal{L}$ decompose as follows:

$$\mathcal{L} = \left(\partial + \frac{\partial(\phi)}{\phi}\right)\left(\partial - \frac{\partial(\phi)}{\phi}\right) \quad \text{and} \quad \tilde{\mathcal{L}} = \left(\partial - \frac{\partial(\phi)}{\phi}\right)\left(\partial + \frac{\partial(\phi)}{\phi}\right).$$

Since $\phi^{-1}\partial\phi = \partial + \partial(\phi)\phi^{-1}$, we see that $\tilde{\mathcal{L}}$ is the result of conjugation with the inverse of $\phi^{-1}\partial\phi$:

$$\tilde{\mathcal{L}} = (\phi^{-1}\partial\phi)^{-1}\mathcal{L}(\phi^{-1}\partial\phi).$$

This result is compatible with the KdV equation in the following sense: if $q := \partial(\phi) \cdot \phi^{-1}$ satisfies

$$\partial_3(q) = \frac{1}{4}\partial^3(q) - \frac{3}{2}q^2\partial(q),$$

and u satisfies the KdV equation, then also $\tilde{u}$ satisfies the KdV equation. This brings us in a natural way to the central question we want to address in the present paper.

1.1 Main Question

Given a solution L in $R[\partial, \partial^{-1})$ of the KP hierarchy in the Segal-Wilson class, determine operators P and Q in $R[\partial]$ such that

$$L_P = PLP^{-1} \quad \text{and} \quad L_Q = Q^{-1}LQ$$

belong again to the Segal-Wilson class and describe these transformations geometrically in the context of the Grassmannian.

Both the transformation $L \to L_P$ as the transformation $L \to L_Q$ are called *Darboux transformations* of order respectively the degree of P and minus the degree of Q. Darboux transformations can be used *e.g.* to characterize subsystems of the hierarchy, see *e.g.* [6] and [7]. Let $\mathcal{L}$ in $R[\partial]$ be a solution of the m-th Gelfand-Dickey hierarchy, with $L := \mathcal{L}^{\frac{1}{m}}$ in the Segal-Wilson class. Then a natural sequel to the above question is to determine those operators P and Q in $R[\partial]$ such that

$$\mathcal{L}_P = P\mathcal{L}P^{-1} \quad \text{and} \quad \mathcal{L}_Q = Q^{-1}\mathcal{L}Q$$

are again solutions of the m-th Gelfand-Dickey hierarchy corresponding to Lax operators in the Segal-Wilson class and give a geometric description of these transformations. We will discuss this question too here.

As the name zero curvature equations already suggests the equations (1.4) and (1.6) can be seen as the compatibility conditions of a linear system. Consider namely

$$L\psi = z\psi \quad \text{and} \quad \partial_n(\psi) = (L^n)_+(\psi). \tag{1.7}$$

If one applies ∂_n to both sides of the first equation in (1.7), substitutes the second one and performs the following manipulations on the equation

$$\partial_n(L\psi) = \partial_n(L)\psi + L\partial_n(\psi) = \{\partial_n(L) + L(L^n)_+\}\psi$$
$$\partial_n(z\psi) = z\partial_n(\psi) = z(L^n)_+(\psi) = (L^n_+)(z\psi) = (L^n_+)L\psi,$$

then one gets

$$\{\partial_n(L) + [L, L^n_+]\}\psi = 0. \tag{1.8}$$

Thus we see that, if these manipulations make sense and if we may leave out ψ in the equation (1.8), then the equations (1.7) imply that L is a solution of the KP hierarchy. As we will see in the next section, the proper framework for all this consists of the free $R[\partial, \partial^{-1})$-module of oscillating functions.

2 Wavefunctions and Dual Wavefunctions

For a proper understanding of the form of the elements in the $R[\partial, \partial^{-1})$-module that we will introduce here, it is best to look first for the trivial solution $L = \partial$ of equation (1.4). In that case (1.7) becomes

$$\partial\psi = z\psi \quad \text{and} \quad \partial_n\psi = z^n\psi \quad \text{for all } n \geq 1.$$

Hence, the function $\psi_0(t) = \exp(\sum_{i\geq 1} t_i z^i)$ is a solution. The space M of so-called *oscillating functions* is a space for which we can make sense out of (1.7). As for their dependence of z, the elements of M are formal products of a factor that is meromorphic around $z = \infty$ and the essential singularity around $z = \infty$ corresponding to the trivial solution. More concretely, it is defined as

$$M = \left\{ \Big(\sum_{j\leq N} a_j z^j \Big) e^{\sum t_i z^i} \;\Big|\; a_i \in R, \text{ for all } i \right\}.$$

Expressing $e^{\sum t_i z^i}$ in terms of the elementary Schur functions, *i.e.*,

$$e^{\sum t_i z^i} = \sum_{k=0}^{\infty} p_k(t) z^k$$

one notices that the product

$$\Big(\sum_{j \leq N} a_j z^j\Big) e^{\sum t_i z^i} = \sum_{\ell \in \mathbb{Z}} \Big(\sum_{k=0}^{\infty} a_{\ell-k} p_k\Big) z^\ell$$

is still formal. To make sense of this expression as an infinite series in z and z^{-1}, the coefficients $\sum_{k=0}^{\infty} a_{\ell-k} p_k$ for all $\ell \in \mathbb{Z}$ have to be well-defined functions of $t = (t_n)$. In that light it is natural to have a context such that for each element t

$$\sum_{j \leq N} |a_j(t)|^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} |p_k(t)|^2 < \infty.$$

Such a context has been given in [14] and will be recalled in Section 3. The space M becomes a $R[\partial, \partial^{-1})$-module by the natural extension of the actions

$$b\Big\{\Big(\sum_j a_j z^j\Big) e^{\sum t_i z^i}\Big\} = \Big(\sum_j b a_j z^j\Big) e^{\sum t_i z^i}$$

$$\partial\Big\{\Big(\sum_j a_j z^j\Big) e^{\sum t_i z^i}\Big\} = \Big(\sum_j \partial(a_j) z^j + \sum_j a_j z^{j+1}\Big) e^{\sum t_i z^i}.$$

It is even a free $R[\partial, \partial^{-1})$-module, since we have

$$\Big(\sum p_j \partial^j\Big) e^{\sum t_i z^i} = \Big(\sum p_j z^j\Big) e^{\sum t_i z^i}.$$

For an element $\psi = P\psi_0 := (\sum_{j \leq k} p_j \partial^j)\psi_0$ with p_k non zero, we consider the equations (1.7). Note that in our leading example of the ring R this implies that p_k resp. P is invertible in R resp. $R[\partial, \partial^{-1})$. By applying the fact that M is a free $R[\partial, \partial^{-1})$-module one sees that the equations in (1.7) are equivalent respectively to the operator identities

$$LP = P\partial \quad \text{and} \quad \partial_n(P) + P\partial^n = (L^n)_+ P = L^n P - (L^n)_- P. \tag{2.1}$$

Substituting the first in the second equation and comparing the leading coefficients on both sides gives you directly that $\partial_n(p_k) = 0$ for all n. In our main example this implies that $p_k \in \mathbb{C}$ and in that light it is reasonable to gauge the leading coefficient of P equal to 1. Those standarized elements carry a name: an element ψ in M is called an *oscillating function of type* z^ℓ, if it has the form

$$\psi(z) = \psi(t, z) = \Big\{z^\ell + \sum_{j<\ell} \alpha_j z^j\Big\} e^{\sum t_i z^i}. \tag{2.2}$$

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Since the operator P is now invertible, one sees that the Lax operator is given by $L = P\partial P^{-1}$. By multiplying the second equations in (2.1) with P^{-1} from the right, one sees that the equations (1.7) for $L = P\partial P^{-1}$ are equivalent to the so-called Sato-Wilson equations for P:

$$\partial_n(P)P^{-1} = -(P\partial^n P^{-1})_-. \tag{2.3}$$

Due to the fact that M is a free $R[\partial, \partial^{-1})$-module, we can say now that all manipulations to arrive from the linearization (1.7) to the Lax equations (1.4) are correct inside M. Hence each oscillating function of type z^ℓ that satisfies (1.7) gives you a solution of (1.4). Such a function is then called a *wavefunction* of the KP hierarchy. Note that for an oscillating function ψ of type z^ℓ the second conditions can be weakened. It suffices that we have for all $n \geq 1$ an operator B_n in $R[\partial]$ such that

$$\partial_n(\psi) = B_n\psi. \tag{2.4}$$

If one takes namely the differential operator part of the corresponding operator equation, then one gets directly that $B_n = (L^n)_+$.

Now that we have reduced the construction of solutions of the KP hierarchy to the question of finding wavefunctions of this hierarchy and since all the solutions in the Segal-Wilson class are of this type, it is natural to modify the main question to this linearized situation and then we get:

2.1 Linearized Version of the Main Question

Given a wavefunction ψ of the KP hierarchy in the Segal-Wilson class, determine operators P and Q in $R[\partial]$ such that

$$\psi_P = P.\psi \quad \text{and} \quad \psi_Q = Q^{-1}.\psi$$

are again wavefunctions of the KP hierarchy belonging to the Segal-Wilson class and describe these transformations geometrically in the context of the Grassmanian.

We dwell for a moment still on the question, which wavefunctions yield the same solution of the KP hierarchy. Assume $\tilde{\psi} = \tilde{P}\psi_0$ and $\psi = P\psi_0$ are wavefunctions of type z^ℓ such that $L = P\partial P^{-1} = \tilde{P}\partial\tilde{P}^{-1}$. Then we can write

$$\tilde{P} = PR(\partial) := P\Big(1 + \sum_{i\geq 1} r_i\partial^{-i}\Big), \quad \text{where } \partial(r_i) = 0 \text{ for all } i \geq 1. \tag{2.5}$$

The Sato-Wilson equations for $\tilde{P}$ combined with those for P give you then that for all $n \geq 1$, $\partial_n\big(R(\partial)\big) = 0$. So, in the case of the meromorphic functions on the group of flows, the coefficients of $R(\partial)$ all belong to $\mathbb{C}$.

Besides the space of oscillating functions M it is also convenient to have at one's disposal its "adjoint" space M^* consisting of all formal products

$$\Big\{\sum_{j\leq N} a_j z^j\Big\} e^{-\sum t_i z^i}, \quad \text{with } a_j \in R \text{ for all } j.$$

The ring $R[\partial, \partial^{-1})$ acts as expected on M^* by the natural extension of the actions

$$b\Big\{\Big(\sum_j a_j z^j\Big)e^{-\sum t_i z^i}\Big\} = \Big(\sum_j b a_j z^j\Big)e^{-\sum t_i z^i}$$
$$\partial\Big\{\Big(\sum_j a_j z^j\Big)e^{-\sum t_i z^i}\Big\} = \Big(\sum_j \partial(a_j) z^j - \sum_j a_j z^{j+1}\Big)e^{-\sum t_i z^i}.$$

This is also a free $R[\partial, \partial^{-1})$-module, since we have

$$\Big(\sum p_j(-\partial)^j\Big)e^{-\sum t_i z^i} = \Big(\sum p_j z^j\Big)e^{-\sum t_i z^i}.$$

On $R[\partial, \partial^{-1})$ we have an anti-algebra morphism called *taking the adjoint*. The adjoint of $P = \sum p_i \partial^i$ is given by

$$\begin{aligned} P^* &= \sum_i (-\partial)^i p_i = \sum_i (-1)^i \sum_{k=0}^{\infty} \binom{i}{k} \partial^k(p_i)\partial^{i-k} \\ &= \sum_\ell \Big\{\sum_{k=0}^{\infty} (-1)^{\ell+k} \binom{\ell+k}{k} \partial^k(p_{\ell+k})\Big\}\partial^\ell. \end{aligned}$$

If $\psi = P(t,\partial)e^{\sum t_i z^i}$ is an oscillating function of type z^ℓ, then we call the element $\psi^* = \big(P(t,\partial)^*\big)^{-1} e^{-\sum t_i z^i}$ in M^* the adjoint of ψ. If moreover $\psi \in M$ is a wavefunction for the KP hierarchy, then its adjoint ψ^* is also called a dual wavefunction for the KP hierarchy and it satisfies a similar set of linear equations, *viz.*,

$$(2.6) \qquad L^*\psi^* = z\psi^* \quad \text{and} \quad \partial_n(\psi^*) = -(L^n)^*_+(\psi^*),$$

where $L^* = (P\partial P^{-1})^* = (P^*)^{-1}(-\partial)P^*$. The first of these equations is a direct consequence of the definition for

$$(P^*)^{-1}(-\partial)P^*(P^*)^{-1}e^{-\sum t_i z^i} = (P^*)^{-1}(-\partial)e^{-\sum t_i z^i} = z(P^*)^{-1}e^{-\sum t_i z^i} = z\psi^*.$$

The second follows from the Sato-Wilson equations for P. By taking the adjoint of relation (2.3) we get

$$\begin{aligned} \big(\partial_n(P)P^{-1}\big)^* &= P^{*-1}\partial_n(P^*) = -(P\partial^n P^{-1})^*_- \\ &= -\big(P^{*-1}(-\partial)^n P^*\big)_- = -\big((L^*)^n\big)_-. \end{aligned}$$

Since $\partial_n(P^{*-1}) = -P^{*-1}\partial_n(P^*)P^{*-1}$, these equations combine to give

$$(2.7) \qquad \partial_n(P^{*-1})P^* = \big((L^*)^n\big)_-.$$

As we have that

$$\begin{aligned}\partial_n(\psi^*) &= \partial_n(P^{*-1}e^{-\sum t_i z^i}) = \partial_n(P^{*-1})e^{-\sum t_i z^i} - z^n P^{*-1}e^{-\sum t_i z^i} \\ &= \big((L^*)^n\big)_-\psi^* - (L^*)^n\psi^* = -\big((L^*)^n\big)_+\psi^*,\end{aligned}$$

which is exactly the second equation of (2.6). Reversely, if the adjoint of an oscillating function of type z^ℓ satisfies the equations in (2.6), then ψ is a wavefunction of the KP hierarchy. The equations in (2.6) namely imply equation (2.7) and by taking the adjoint of it, we get the Sato-Wilson equations for P. The linearization conditions in (2.6) can be weakened, similar to (2.4), and this leads to the following description of dual wavefunctions.

Lemma 2.1 *Let ψ^* be the adjoint of an oscillating function ψ of type z^ℓ. Then ψ^* is a dual wavefunction of the* KP *hierarchy if and only if there is for all $n \geq 1$ an operator C_n in $R[\partial]$ such that $\partial_n(\psi^*) = C_n\psi^*$.*

3 The Segal-Wilson Grassmannian

First we recall here the necessary ingredients of the analytic approach from [14] to construct wavefunctions of the KP hierarchy. Segal and Wilson consider the Hilbert space

$$H = \Big\{\sum_{n\in\mathbb{Z}} a_n z^n \,\Big|\, a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\Big\},$$

with decomposition $H = H_+ \oplus H_-$, where

$$H_+ = \Big\{\sum_{n\geq 0} a_n z^n \in H\Big\} \quad \text{and} \quad H_- = \Big\{\sum_{n<0} a_n z^n \in H\Big\}$$

and the inner product $\langle\cdot,\cdot\rangle$ is given by

$$\Big\langle \sum_{n\in\mathbb{Z}} a_n z^n, \sum_{m\in\mathbb{Z}} b_m z^m \Big\rangle = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

To this decomposition is associated the Grassmannian $\mathrm{Gr}(H)$ consisting of all closed subspaces W of H such that the orthogonal projection $p_+\colon W \to H_+$ is Fredholm and the orthogonal projection $p_-\colon W \to H_-$ is Hilbert-Schmidt. The connected components of $\mathrm{Gr}(H)$ are given by

$$\mathrm{Gr}^{(\ell)}(H) = \{W \in \mathrm{Gr}(H) \mid p_+\colon z^{-\ell}W \to H_+ \text{ has index zero}\}.$$

Each of these components is a homogeneous space for the group $\mathrm{Gl}^{(0)}_{\mathrm{res}}(H)$ of all bounded invertible operators $g\colon H \to H$ that decompose with respect to $H = H_+ \oplus H_-$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a and d are Fredholm operators of index zero and b and c are Hilbert-Schmidt. For each $N > 1$, we consider the multiplicative group

$$\Gamma_+(N) = \Big\{ \gamma(t) := \exp\Big(\sum_{i\geq 1} t_i z^i\Big) \;\Big|\; t_i \in \mathbb{C}, \sum_{i\geq 1} |t_i| N^i < \infty \Big\}$$

equipped with the uniform norm. These groups are nested in a natural way and the inductive limit is denoted by Γ_+. The groups $\Gamma_+(N)$ act by multiplication on H and this gives a continuous injection of Γ_+ into $\mathrm{Gl}_{\mathrm{res}}^{(0)}(H)$. For example, the element $q_\zeta = 1 - \frac{z}{\zeta}$ belongs to Γ_+ if $|\zeta| > 1$ and it has the following property that will reappear later on

$$\gamma(t) q_\zeta^{-1} = \exp\Big(\sum_{i\geq 1} t_i z^i\Big) \exp\Big(\sum_{i\geq 1} -\frac{1}{i\zeta^i} z^i\Big) = \gamma\Big(\Big(t_i - \frac{1}{i\zeta^i}\Big)\Big). \tag{3.1}$$

The commuting flows from Γ_+ lead to wavefunctions of the KP hierarchy for which the product in (2.2) is real and not formal.

Each wavefunction that will be constructed, can be expressed in a single function, a so-called τ-function. They require a convenient description of the components $\mathrm{Gr}^{(\ell)}(H)$ of the Grassmannian. Let $\mathfrak{P}_\ell$ be the collection of embeddings $w\colon z^\ell H_+ \to H$ such that with respect to the decomposition $H = (z^\ell H_+) \oplus (z^\ell H_+)^\perp$ the operator w has the form $w = \binom{w_+}{w_-}$, with w_- a Hilbert-Schmidt operator and $w_+ - \mathrm{Id}$ a trace class operator. Then $\mathfrak{P}_\ell$ is in a natural way a fibre bundle over $\mathrm{Gr}^{(\ell)}(H)$ with fiber the group

$$\mathfrak{T}_\ell = \{ t \in \mathrm{Aut}(z^\ell H_+) \mid t - \mathrm{Id} \text{ is of trace class} \}.$$

To lift the action of $\mathrm{GL}_{\mathrm{res}}^{(0)}(H)$ on $\mathrm{Gr}^{(\ell)}(H)$ to one on $\mathfrak{P}_\ell$, one has to pass to an extension Gl of $\mathrm{Gl}_{\mathrm{res}}^{(0)}(H)$. It is defined by

$$\mathrm{Gl} = \Bigg\{ (g,q) \;\Big|\; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g \in \mathrm{GL}_{\mathrm{res}}^{(0)}(H), \\ q \in \mathrm{Aut}(z^\ell H_+), aq^{-1} - \mathrm{Id} \text{ is trace class} \Bigg\}.$$

This group acts by $w \mapsto gwq^{-1}$ on $\mathfrak{P}_\ell$. The elements of Γ_+ embed in a natural way into Gl through

$$\gamma_+ = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (\gamma_+, a).$$

This embedding we assume throughout this paper. For each $w \in \mathfrak{P}_\ell$, we define $\tau_w\colon \mathrm{Gl} \to \mathbb{C}$ by

$$\tau_w\big((g,q)\big) = \det\big((g^{-1}wq)_+\big).$$

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If t belongs to $\mathfrak{T}_\ell$, then there holds $\tau_{w \circ t} = \det(t)\tau_w$. Hence, choosing a different embedding, changes a τ-function corresponding to a plane in $\mathrm{Gr}(H)$ only up to a nonzero constant. Therefore we will often denote the restriction of τ_w to Γ_+ by $\tau_W\big((t_i)\big) = \tau_W(t)$.

For each $W \in \mathrm{Gr}^{(\ell)}(H)$, let Γ_+^W be given by

$$\Gamma_+^W = \Big\{ \gamma(t) = \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) \in \Gamma_+ \;\Big|\; p_+ \colon \gamma^{-1} z^{-\ell} W \to H_+ \text{ is a bijection} \Big\}.$$

In a similar way as in [14], one shows that Γ_+^W is a nonempty open subset of Γ_+. A crucial property of the elements $\gamma(t)$ in this dense open subset of Γ_+ is that

$$W \cap (z^\ell H_+)^\perp \gamma(t) = \{0\}.$$

Now we take for the moment for R the ring of holomorphic functions on Γ_+^W and for ∂_n the partial derivative w.r.t. the parameter t_n of Γ_+. For $\gamma \in \Gamma_+^W$, let $P_W(t,z)$ be z^ℓ times the inverse image of 1 under the projection $p_+ \colon \gamma^{-1} z^{-\ell} W \to H_+$. Then we associate to W an oscillating function $\psi_W(t,z) = P_W(t,z) e^{\sum_{i=1}^{\infty} t_i z^i}$ of type z^ℓ, which has the properties that it is defined on a dense open subset of Γ_+ and that its boundary value at $|z| = 1$ belongs to W. In particular we have

$$\psi_W = \sum_{n \in \mathbb{Z}} c_n\big(\gamma(t)\big) z^n \quad \text{with} \sum_{n \in \mathbb{Z}} \Big| c_n\big(\gamma(t)\big) \Big|^2 < \infty. \tag{3.2}$$

Moreover, it is known that the range of ψ_W spans a dense subspace of W. By combining the foregoing ingredients, Segal and Wilson showed:

Theorem 3.1 *For each $W \in \mathrm{Gr}^{(\ell)}(H)$, the function $\psi_W(t,z) = P_W(t,z) e^{\sum_{i=1}^{\infty} t_i z^i}$ is a wavefunction for the KP hierarchy and, if $w \in \mathfrak{P}_\ell$ has W as its image, then*

$$P_W(t,\zeta) = \zeta^\ell \frac{\tau_w\big(\gamma(t) q_\zeta\big)}{\tau_w\big(\gamma(t)\big)} = \zeta^\ell \frac{\tau_W(t - [\zeta^{-1}])}{\tau_W(t)},$$

where $[\lambda] = (\lambda, \frac{1}{2}\lambda^2, \frac{1}{3}\lambda^3, \frac{1}{4}\lambda^4, \dots)$. The corresponding solution of the KP *hierarchy is denoted by $L_W := P_W \partial P_W^{-1}$.*

Note that our initial choice of the ring R depended on the plane W. However, the expression of ψ_W in the τ-function shows that one can make a uniform choice by taking R equal to the ring of meromorphic functions on Γ_+. This we assume from now on. Each connected component of $\mathrm{Gr}(H)$ generates by this construction the same set of solutions of the KP hierarchy, so it would suffice, as is done in [14], to consider only $\mathrm{Gr}^{(0)}(H)$. However, in order to reach the goals we have set, we need to consider all components here.

Let W belong to $\mathrm{Gr}^{(l)}(H)$, then $W^\perp$ is a closed subspace of H for which the orthogonal projection $p_- \colon W^\perp \to H_-$ is a Fredholm operator of index $-l$ and such

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that $p_+ \colon W^\perp \to H_+$ is a Hilbert-Schmidt operator. Interchanging the role of H_+ and H_-, we see that $W^\perp$ is a plane in the adjoint Grassmanian $\mathrm{Gr}^*(H)$ consisting of planes U for which $p_-|_U$ is a Fredholm operator and $p_+|_U$ is a Hilbert-Schmidt operator. The connected components of $\mathrm{Gr}^*(H)$ are also homogeneous spaces for the group $\mathrm{Gl}_{\mathrm{res}}^{(0)}(H)$ introduced above. On $\mathrm{Gr}^*(H)$ we consider the commuting flows that are the adjoints of the ones from Γ_+. Namely, for each $N > 1$, we take the group

$$\Gamma_-(N) = \Big\{ g(r) = \exp\Big(\sum_{i\geq 1} r_i z^{-i}\Big) \;\Big|\; r_i \in \mathbb{C}, \sum_{i\geq 1} |r_i| N^i < \infty \Big\}$$

and its inductive limit Γ_- w.r.t. N. For the rest of this paper we assume that the group Γ_- has been embedded into Gl as follows

$$\gamma_- = \begin{pmatrix} p & 0 \\ r & s \end{pmatrix} \mapsto (\gamma_-, p).$$

Like for the space W we consider also for $W^\perp \in \mathrm{Gr}^*(H)$ the part of Γ_- on which we will construct a function of the required form

$$\Gamma_-^{W^\perp} = \{ g(r) \in \Gamma_- \mid p_- \colon W^\perp z^{-\ell} g(r)^{-1} \to H_- \text{ is a bijection} \}.$$

By using the facts that for each $V \in \mathrm{Gr}(H)$ the projection $p_+ \colon V \to H_+$ is a bijection if and only if $p_- \colon V^\perp \to H_-$ is a bijection and that for each bounded invertible linear operator $A \colon H \to H$, $A(V^\perp)^\perp = A^{*^{-1}}(V)$, one sees that

$$g(r) \in \Gamma_-^{W^\perp} \Leftrightarrow \big(g(r)^*\big)^{-1} \in \Gamma_+^W.$$

In particular one sees that $\Gamma_-^{W^\perp}$ is a non-empty open subset of Γ_-. For a $g(r)$ in $\Gamma_-^{W^\perp}$ we know by definition that $p_- \colon W^\perp z^{-\ell} g(r)^{-1} \to H_-$ is a bijection. Now, let $Q_{W^\perp}\big(g(r), z\big)$ be the inverse image of the element z^{-1} under this projection and put

$$\psi_{W^\perp}\big(g(r), z\big) = Q_{W^\perp}\big(g(r), z\big) z^\ell g(r).$$

Then the map $g(r) \mapsto \psi_{W^\perp}\big(g(r), z\big)$ is an analytic function on an open dense part of Γ_- with values in $W^\perp$. This function has the form

$$\begin{aligned} \psi_{W^\perp}\big(g(r), z\big) &= \Big\{ z^{-1} + \sum_{j\geq 0} e_j\big(g(r)\big) z^j \Big\} z^\ell \exp\Big(\sum r_i z^{-i}\Big) \\ &= \Big\{ \sum_{k\geq \ell-1} b_k\big(g(r)\big) z^k \Big\} g(r). \end{aligned}$$

Similar as for ψ_W one shows that $b_j \in R^*$, where R^* denotes the ring of meromorphic functions on Γ_-. Hence, if we take $g(r) = \gamma(-t)^*$, with $\gamma(t) \in \Gamma_+^W$, then we get on the boundary $|z| = 1$ the following

$$\begin{aligned}\overline{z\psi_{W^\perp}\big(\gamma(-t)^*, z\big)} &= \Big\{\mathrm{Id} + \sum_{j\geq 0} \overline{e_j\big(\gamma(-t)^*\big)}\, z^{-j-1}\Big\} z^{-\ell}\gamma(-t) \\ &= \Big\{\sum_{k\geq \ell-1} \overline{b}_k\big(\gamma(-t)^*\big)\, z^{-k-1}\Big\}\gamma(-t)\end{aligned}$$

and this is exactly a function of the form we are looking for, namely a dual oscillating function of type $z^{-\ell}$. From the way this function was built, one deduces in the same way as for the wavefunction ψ_W that it satisfies the conditions in Lemma 2.1. Hence it belongs to the class of dual wavefunctions of the KP hierarchy. In fact, there holds:

Theorem 3.2 *The dual wavefunction ψ_W^* of the wavefunction ψ_W of the* KP *hierarchy is equal to $\overline{z\psi_{W^\perp}\big(\gamma(-t)^*, z\big)}$ and its expression in τ-functions is*

$$\psi_W^*(t,\zeta) = \zeta^{-\ell}\frac{\tau_w\big(\gamma(t)q_\zeta^{-1}\big)}{\tau_w\big(\gamma(t)\big)} e^{-\sum_{i=1}^{\infty} t_i\zeta^i} = \zeta^{-\ell}\frac{\tau_W(t+[\zeta^{-1}])}{\tau_W(t)} e^{-\sum_{i=1}^{\infty} t_i\zeta^i}.$$

Proof If $\psi_W = P_W e^{\sum t_i z^i} = \sum a_i\partial^i . e^{\sum t_i z^i}$ and $F = \sum f_k\partial^k = (P_W^*)^{-1}$, then we have by definition that $\psi_W^* = F.e^{-\sum t_i z^i} = \big(\sum (-1)^k f_k z^k\big) e^{-\sum t_i z^i}$. The operator F is the unique pseudodifferential operator of order $-\ell$ that satisfies $P_W F^* = 1$. Now we have that

$$\begin{aligned}P_W F^* &= \sum_{j,k} a_j\partial^j(-\partial)^k f_k = \sum_{j,k}(-1)^k a_j\partial^{j+k} f_k \\ &= \sum_{j,k}(-1)^k a_j \sum_{r=0}^{\infty}\binom{j+k}{r}\partial^r(f_k)\partial^{j+k-r} \\ &= \sum_s\Big\{\sum_j\sum_{r=0}^{\infty}\binom{r+s}{s}(-1)^{r+s+j} a_j\partial^r(f_{r+s-j})\Big\}\partial^s.\end{aligned}$$

Hence the coefficient of ∂^{-n-1}, $n \geq 0$, equals

$$\begin{aligned}&(-1)^{n+1}\sum_j\sum_{r=0}^{\infty}\binom{r-n-1}{r}(-1)^{r+j} a_j\partial^r(f_{r-j-n-1}) \\ &= (-1)^{n+1}\sum_j\sum_{r=0}^{n}\binom{n}{r}(-1)^j a_j\partial^r(f_{r-j-n-1}) \\ &= (-1)^{n+1}\sum_j(-1)^j a_j\sum_{\ell=0}^{n}\binom{n}{\ell}\partial^{n-\ell}(f_{-\ell-j-1}) \\ &= (-1)^n\sum_p(-1)^p a_{-p-1}\sum_{\ell=0}^{n}\binom{n}{\ell}\partial^{n-\ell}(f_{p-\ell}).\end{aligned}$$

Therefore the coefficients of $(P_W^*)^{-1}$ are completely determined by the set of equations

$$\sum_p (-1)^p a_{-p-1} \sum_{\ell=0}^{n} \binom{n}{\ell} \partial^{n-\ell}(f_{p-\ell}) = 0 \quad \text{for all } n \geq 0.$$

From the way $\psi_{W^\perp}$ was constructed, one sees that for all $n \geq 0$, also $(\frac{\partial}{\partial \overline{t_1}})^n(\psi_{W^\perp})$ has boundary values in $W^\perp$. Now we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial \overline{t_1}}\right)^n (\psi_{W^\perp})\big(\gamma(-t)^*\big) \\
&\quad = \left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} \sum_k \left(\frac{\partial}{\partial \overline{t_1}}\right)^{n-\ell} (b_k)(-1)^\ell z^{k-\ell} \right\} \gamma(-t)^* \\
&\quad = \left\{ \sum_p \left(\sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{\partial}{\partial \overline{t_1}}\right)^{n-\ell} (b_{p+\ell})(-1)^\ell z^p \right) \right\} \gamma(-t)^*.
\end{aligned}$$

Since by definition W and $W^\perp$ are orthogonal, we get for all $n \geq 0$ and all $\gamma(s)$ and $\gamma(t) \in \Gamma_+^W$ that

$$\left\langle \psi_W\big(\gamma(t), z\big), \frac{\partial}{\partial \overline{t_1}}^n \Big(\psi_{W^\perp}\big(\gamma(-s)^*, z\big) \Big) \right\rangle = 0. \tag{3.3}$$

If we substitute in the equations (3.3), the expressions for ψ_W and $\frac{\partial}{\partial \overline{t_1}}^n(\psi_{W^\perp})$ and we take $\gamma(t) = \gamma(s)$, then we end up with

$$\sum_p a_p \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \partial^{n-\ell}(\overline{b}_{p+\ell}) = 0.$$

This shows that $(-1)^p \overline{b}_{-p-1}$ satisfies the equations characterizing the coefficients of $(P_W^*)^{-1}$ and hence proves the first claim of the theorem. Finally the expression in the τ-functions was derived in [5]. This concludes the proof of the theorem. ■

Thanks to this theorem we can say now that also the dual wavefunction, as a boundary value, is of the same category as the wavefunction, *i.e.*,

$$\psi_W^* = \sum_{n \in \mathbb{Z}} d_n\big(\gamma(s)\big) z^n \quad \text{with } \sum_{n \in \mathbb{Z}} \left| d_n\big(\gamma(s)\big) \right|^2 < \infty. \tag{3.4}$$

One has in the present context a convergent interpretation of the well-known bilinear identity between a wavefunction ψ and its adjoint

$$\operatorname{Res}_z \psi(t, z)\psi^*(s, z) = 0. \tag{3.5}$$

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For, with the notations from (3.2) and (3.4), the product $\psi_W(t,z)(\psi_W)^*(s,z)$ becomes the following well-defined series in z and z^{-1}

$$\sum_p \Big(\sum_n c_n\big(\gamma(t)\big)\, d_{-n+p}\big(\gamma(s)\big) \Big)\, z^p.$$

The equation (3.3) with $n = 0$ tells you that for all $\gamma(t)$ and all $\gamma(s)$ in Γ_+^W

$$\sum_{n\in\mathbb{Z}} c_n\big(\gamma(t)\big)\, d_{-n-1}\big(\gamma(s)\big) = 0, \tag{3.6}$$

and the left hand side of this equation is exactly the coefficient of z^{-1} in the product $\psi_W(t,z)(\psi_W)^*(s,z)$. This proves equation (3.5) in the present context.

4 Trivial Transformations of Wavefunctions

The class of transformations we will discuss in this subsection, arises naturally, when one raises the question which Darboux transformations render the same solution of the KP hierarchy. These transformations do not change the Lax operator L and and are therefore called *trivial*. For, let $\psi = P\psi_0$ be a wavefunction of the KP hierarchy and let P_1 and P_2 be operators in $R[\partial]$ such that $P_1\psi = P_2\psi$ is another wavefunction of the KP hierarchy. Then we know from Section 2 that there holds

$$P_1 P = P_2 P R(\partial), \quad \text{with } R(\partial) := \partial^k\Big(1 + \sum_{j=1}^{\infty} r_j \partial^{-j}\Big) \text{ with } r_j \in \mathbb{C}.$$

Since these trivial operators $R(\partial)$ are in the center of $R[\partial, \partial^{-1}]$, their action on wavefunctions consists of multiplying a wavefunction with a function of the form

$$R(z) = z^k\Big(1 + \sum_{j=1}^{\infty} r_j z^{-j}\Big) \quad \text{with } r_j \in \mathbb{C}. \tag{4.1}$$

Thus, if $\psi(t,z) = P(t,\partial)e^{\sum_i t_i z^i}$ is a wavefunction of type z^ℓ, then

$$\begin{aligned}\psi_R(t,z) = R(z)\psi(t,z) &= P(t,\partial)R(\partial)e^{\sum_i t_i z^i} \\ &= z^k\Big(1 + \sum_{j=1}^{\infty} r_j z^{-j}\Big)\psi(t,z)\end{aligned}$$

is the new wavefunction of type $z^{k+\ell}$. Its adjoint wavefunction is of the form

$$\psi_R^*(t,z) = P(t,\partial)^{*-1}R(\partial)^{*-1}e^{-\sum_i t_i z^i} = R(z)^{-1}\psi^*(t,z).$$

Since all r_j's are constants, this does not change the form of L, but it clearly changes the tau-function. Rewrite $R(z)$ as in (4.1) as follows:

$$R(z) = z^k e^{-\sum_{j=1}^{\infty} \frac{q_j}{j} z^{-j}}, \tag{4.2}$$

then

$$\begin{aligned}\psi_R(t,z) &= z^k e^{-\sum_{j=1}^{\infty}\frac{q_j}{j}z^{-j}}\psi(t,z)\\ &= z^{k+\ell}\frac{\tau(t-[z^{-1}])}{\tau(t)}e^{-\sum_{j=1}^{\infty}\frac{q_j}{j}z^{-j}}e^{\sum_{i=1}^{\infty}t_iz^i}\\ &= z^{k+\ell}\frac{\tau(t-[z^{-1}])e^{\sum_{j=1}^{\infty}q_j(t_j-\frac{z^{-j}}{j})}}{\tau(t)e^{\sum_{j=1}^{\infty}q_jt_j}}e^{\sum_{i=1}^{\infty}t_iz^i},\end{aligned}$$

which suggests that $\tau_R(t) := \tau(t)e^{\sum_{j=1}^{\infty}q_jt_j}$ is a to ψ_R corresponding tau-function. We will show this in the remainder of this subsection. First of all, one notices that the multiplication by z^k does not change the tau-function. Hence we may assume that $k=0$ and we consider the action of

$$\rho = \exp\Bigl(-\sum_{j=1}^{\infty}\frac{q_j}{j}z^{-j}\Bigr) \in \Gamma_-(N)$$

on $\mathfrak{P}_\ell$. The operator ρ decomposes with respect to $H = z^\ell H_+ \oplus (z^\ell H_+)^\perp$ as $\rho = \left(\begin{smallmatrix}\alpha & 0\\ \beta & \delta\end{smallmatrix}\right)$. Hence its action on $w = \left(\begin{smallmatrix}w_+\\ w_-\end{smallmatrix}\right) \in \mathfrak{P}_\ell$ is

$$\rho\circ w\circ\alpha^{-1} = \begin{pmatrix}\alpha w_+\alpha^{-1}\\ (\beta w_+ + \delta w_-)\alpha^{-1}\end{pmatrix}.$$

Let $\gamma^{-1} = \exp(-\sum_i t_iz^i) \in \Gamma_+(N)$ decompose with respect to $H = z^\ell H_+ \oplus (z^\ell H_+)^\perp$ as $\gamma^{-1} = \left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right)$. Then the tau-function corresponding to ρW, where $W = \mathrm{Im}(w)$, $w \in \mathfrak{P}_\ell$, is by definition

$$\tau_{\rho\circ w\circ\alpha^{-1}}(\gamma) = \det\bigl((\gamma^{-1}\rho w\alpha^{-1}a^{-1})_+\bigr).$$

Since γ^{-1} and ρ commute we have the relation

$$\alpha a = a\alpha + b\beta \quad \text{or equivalently} \quad \alpha a\alpha^{-1}a^{-1} = I + b\beta\alpha^{-1}a^{-1},$$

so that we see that the operator $\alpha a\alpha^{-1}a^{-1}$ is of the form "*identity* + *trace-class*" and hence has a determinant. Thus we get for the tau-function of ρW that

$$\begin{aligned}\tau_{\rho\circ w\circ\alpha^{-1}}(\gamma) &= \det\bigl((\gamma^{-1}\rho wa^{-1}\alpha^{-1})_+\bigr)\,\det(\alpha a\alpha^{-1}a^{-1})\\ &= \det\bigl((\rho\gamma^{-1}wa^{-1}\alpha^{-1})_+\bigr)\,\det(\alpha a\alpha^{-1}a^{-1})\\ &= \tau_w(\gamma)\det(\alpha a\alpha^{-1}a^{-1}).\end{aligned}$$

Hence we merely have to show that

$$B(\alpha,a) = \det(\alpha a\alpha^{-1}a^{-1})$$

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is the required function. This we will prove in a few steps. First one notices that if $\rho_1 = \left(\begin{smallmatrix}\alpha_1 & 0\\ \beta_1 & \delta_1\end{smallmatrix}\right)$ and $\rho_2 = \left(\begin{smallmatrix}\alpha_2 & 0\\ \beta_2 & \delta_2\end{smallmatrix}\right)$ are in $\Gamma_-(N)$, then

$$\begin{aligned} B(\alpha_1\alpha_2, a) &= \det(\alpha_1\alpha_2 a\alpha_2^{-1}\alpha_1^{-1}a^{-1}) = \det(\alpha_2 a\alpha_2^{-1}a^{-1}a\alpha_1^{-1}a^{-1}\alpha_1)\\ &= \det(\alpha_2 a\alpha_2^{-1}a^{-1})\det(a\alpha_1^{-1}a^{-1}\alpha_1)\\ &= B(\alpha_2, a)B(\alpha_1, a). \end{aligned}$$

Likewise one shows for $\gamma_i^{-1} = \left(\begin{smallmatrix}a_i & b_i\\ 0 & d_i\end{smallmatrix}\right) \in \Gamma_+(N)$ that

$$B(\alpha, a_1a_2) = B(\alpha, a_1)B(\alpha, a_2).$$

Hence if we put $\rho_i = \exp(-\frac{q_i}{i}z^{-i}) = \left(\begin{smallmatrix}\alpha_i & 0\\ \beta_i & \delta_i\end{smallmatrix}\right)$, then $\rho = \lim_{N\to\infty}\rho_1\rho_2\cdots\rho_N$. The form B is continuous in α, so that we get

$$B(\alpha, a) = \prod_{i=1}^{\infty} B(\alpha_i, a)$$

and thus we merely have to prove the formula for ρ_i. If

$$\gamma_i^{-1} = \exp(-t_iz^i) = \begin{pmatrix} a_i & b_i\\ 0 & d_i\end{pmatrix} \in \Gamma_+(N),$$

then $\gamma = \lim_{N\to\infty}\gamma_1\gamma_2\cdots\gamma_N$. As B is also continuous and multiplicative in a, we end up with the formula

$$B(\alpha, a) = \prod_{i,j\geq 1} B(\alpha_i, a_j).$$

Therefore we only have to prove the formula for $\rho = \rho_i$ and $\gamma = \gamma_j$. Clearly the map $q_i \to B(\alpha_i, a_j)$ defines a continuous morphism $\chi_i^{(j)}$ from $\mathbb{C}$ to $\mathbb{C}^\times$ and therefore it has the form

$$\chi_i^{(j)}(q_i) = e^{\beta_{ij}q_i},$$

for some $\beta_{ij} \in \mathbb{C}$, depending on γ_j. Likewise the map $t_j \to B(\alpha_i, a_j)$ defines a continuous morphism $\phi_j^{(i)}$ from $\mathbb{C}$ to $\mathbb{C}^\times$ that can be written as

$$\phi_j^{(i)}(t_j) = e^{\alpha_{ij}t_j},$$

for some $\alpha_{ij} \in \mathbb{C}$, depending on ρ_j. Hence we can say that for all $i \geq 1$ and $j \geq 1$

$$B(\alpha_i, a_j) = e^{a_{ij}t_iq_j} \quad \text{with } a_{ij} \in \mathbb{C}.$$

The $\{a_{ij}\}$ can easily be determined by computing the lowest non-trivial term in $B(\alpha_i, a_j)$. Since

$$b_j = \begin{pmatrix} \vdots & & & & & \\ 0 & & & & 0 & \\ -t_j & \ddots & & & & \\ 0 & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & -t_j & 0 & \cdots \end{pmatrix} \quad \text{and}$$

$$\beta_j = \begin{pmatrix} \cdots & 0 & -\frac{q_i}{i} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & -\frac{q_i}{i} \\ & 0 & & & & 0 \\ & & & & & \vdots \end{pmatrix},$$

$$\begin{aligned} B(\alpha_i, a_j) &= \det(I + \alpha_i^{-1} a_j^{-1} b_j \beta_i) \\ &= 1 + \sum_{n \geq 1} \operatorname{Tr}\left(\Lambda^n(\alpha_i^{-1} a_j^{-1} b_j \beta_i)\right) \end{aligned}$$

we see that only $\operatorname{Tr}(\alpha_i^{-1} a_j^{-1} b_j \beta_i)$ contributes to the lowest non-trivial term. For the same reason the lowest non-trivial term of $\operatorname{Tr}(\alpha_i^{-1} a_j^{-1} b_j \beta_i)$ is equal to that of $\operatorname{Tr}(b_j \beta_i)$. A straightforward calculation gives the required result, *viz.*, $a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$. We state the results of this computation in a proposition.

Proposition 4.1 *Let ρ be the element $\exp(\sum_{j \geq 1} -\frac{q_j}{j} z^{-j})$ in $\Gamma_-(N)$. Then a tau-function corresponding to the plane ρW is given by*

$$\tau_{\rho W}(t) = \tau_{\rho \circ w \circ \alpha^{-1}}(\gamma) = \exp\left(\sum_{j \geq 1} \frac{q_j}{j} t_j\right) \tau_w(\gamma) = \exp\left(\sum_{j \geq 1} \frac{q_j}{j} t_j\right) \tau_W(t).$$

5 Elementary Darboux Transformations and their Geometric Description

In this section we will consider the Darboux transformations that correspond to the first order differential operators P and Q occurring in the question (2.1). Since they will be the building blocks of the general Darboux transformations we call them the *elementary Darboux transformations*. For some literature on Darboux transformations we refer the reader to [1], [2], [9], [10] and [11].

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Let ψ_W be a wavefunction of type z^ℓ corresponding to the plane W in $\mathrm{Gr}(H)$ and let L_W be the corresponding solution of the KP hierarchy. For nonzero elements $q, r \in R$, one considers the first order differential operators $q\partial q^{-1}$ and $r^{-1}\partial^{-1}r$ that are of the same form as the operators considered by Darboux. Clearly the functions $\psi_q := q\partial q^{-1}\psi_W$ and $\psi_r := r^{-1}\partial^{-1}r\psi_W$ are oscillating functions of type $z^{\ell+1}$ respectively $z^{\ell-1}$. The question we want to discuss here, is: when are ψ_q and ψ_r again wavefunctions in the Segal-Wilson class? In that case conjugating L_W with $q\partial q^{-1}$ resp. $r^{-1}\partial^{-1}r$ will be a Darboux transformation of this class of solutions. We describe first the geometric consequences of this fact and we present the relation between q and the wavefunction ψ_W and the one between r and the dual wavefunction ψ_W^* that it implies.

Proposition 5.1 *Let the functions ψ_q and ψ_r be the wavefunctions of the planes W_q and W_r. Then we have the following codimension 1 inclusions:*

$$W_q \subset W, \quad W^\perp \subset W_q^\perp, \quad W \subset W_r, \quad \text{and} \quad W_r^\perp \subset W^\perp.$$

Take any nonzero $s(z) \in W \cap W_q^\perp$, then $W_q = \{w \in W \mid \langle w, s(z)\rangle = 0\}$. If we introduce the function $q_{s,W} = \langle \psi_W(t,z), s(z)\rangle$, then this defines a nonzero element in R and we have the relation

$$\frac{\partial(q)}{q} = \frac{\partial \log q}{\partial x} = \frac{\partial \log\langle \psi_W(t,z), s(z)\rangle}{\partial x} = \frac{\partial(q_{s,W})}{q_{s,W}}.$$

Now, if one takes a nonzero $t(z) \in W^\perp \cap W_r$, then $W_r^\perp = \{w \in W^\perp \mid \langle t(z), w\rangle = 0\}$. Moreover, if we define $r_{t,W} := \langle z\psi_W^*(t,z), \overline{t(z)}\rangle$, then this gives a nonzero element in R and there holds

$$\frac{\partial(r)}{r} = \frac{\partial \log r}{\partial x} = \frac{\partial \log\langle z\psi_W^*(t,z), \overline{t(z)}\rangle}{\partial x} = \frac{\partial(r_{t,W})}{r_{t,W}}.$$

Proof The inclusions between the spaces W and W_q follow from the relation $\psi_q = q\partial q^{-1}\psi_W$ and the fact that the values of a wavefunction corresponding to an element of $\mathrm{Gr}(H)$ are lying dense in that space. Since for a suitable γ in Γ_+ the orthogonal projections of $\gamma^{-1}W$ on $z^\ell H_+$ resp. $\gamma^{-1}W_q$ on $z^{\ell+1}H_+$ are bijections and the one of $z^\ell H_+$ on $z^{\ell+1}H_+$ has a one dimensional kernel, one obtains the codimension one inclusion $W_q \subset W$. The second inclusion follows from this one by taking the orthogonal complement. For a nonzero $s(z) \in W \cap W_q^\perp$, the codimension one result and the choice of $s(z)$ imply that $W_q = \{w \in W \mid \langle w, s(z)\rangle = 0\}$. Next consider the element $q_{s,W} := \langle \psi_W(t,z), s(z)\rangle$ in R. It is nonzero since the boundary values of ψ_W are lying dense in W. By definition there holds

$$\langle \psi_q(t,z), s(z)\rangle = \langle q\partial q^{-1}\psi_W(t,z), s(z)\rangle = \left(\partial - \frac{\partial(q)}{q}\right)(q_{s,W}) = 0$$

and this yields the desired relation between q, $s(z)$ and ψ_W. For the inclusions between the spaces W and W_r we consider the dual wavefunctions $\psi_W^* = P_W^{*-1}e^{-\sum t_i z^i}$

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and $\psi_r^* = -r\partial r^{-1}\psi_W^*$. Since the complex conjugate $\overline{z\psi_W^*(t,z)}$ of $z\psi_W^*(t,z)$ corresponds to the space $W^\perp$, a similar reasoning as before shows the codimension 1 inclusion:

$$W_r^\perp := \text{the closure of Span}\, \{\overline{z\psi_r^*(t,z)}\} \subset W^\perp.$$

Taking the orthogonal complement renders the remaining inclusion. For a nonzero $t(z) \in W^\perp \cap W_r$, the fact that W has codimension one in W_r, implies that $\overline{W_r^\perp} = \{w \in W^\perp \mid \langle t(z), w\rangle = 0\}$. Now, one defines $r_{t,W} := \langle z\psi_W^*(t,z), \overline{t(z)}\rangle$ and this is a nonzero element of R, since the boundary values of $\psi_{W^\perp}$ are dense in $W^\perp$. As the boundary values of $\psi_{W_r}^*$ belong to $\overline{W_r^\perp}$ we have by definition that

$$0 = \langle z\psi_{W_r}^*(t,z), \overline{t(z)}\rangle = \left\langle z(-r\partial r^{-1})\big(\psi_W^*(t,z)\big), \overline{t(z)}\right\rangle = \left(-\partial + \frac{\partial(r)}{r}\right)(r_{t,W})$$

and this gives the stated relation between r, $t(z)$ and ψ_W^*. This concludes the proof of the proposition. ■

To get an idea of sufficient conditions on q and r that lead to the proper Darboux transformations, we have a look at the functions $q_{s,W}$ and $r_{t,W}$ that occur in Proposition 5.1. They satisfy for all $n = 1, 2, \ldots$

$$\partial_n(q_{s,W}) = \langle \partial_n\psi_W, s(z)\rangle = \langle (L^n)_+\psi_W, s(z)\rangle = (L^n)_+(q), \tag{5.1}$$

$$\partial_n(r_{t,W}) = \langle \partial_n z\psi_W^*, \overline{t(z)}\rangle = \langle -(L^n)_+^* z\psi_W^*, \overline{t(z)}\rangle = -(L^n)_+^*(r_{t,W}). \tag{5.2}$$

We will show next that these relations are also sufficient to procure that ψ_q and ψ_r belong again to the Segal-Wilson class. Consider namely the operators P_q and P_r defined by

$$P_q := q\partial q^{-1}P_W \quad \text{and} \quad P_r := r^{-1}\partial^{-1}rP_W. \tag{5.3}$$

It was shown in [6] that the operators P_q and the P_r satisfy the Sato-Wilson equations. If $\psi = \psi_W$ belongs to the Segal-Wilson class, then all derivatives of ψ resp. ψ^* w.r.t. one of the parameters t_n have boundary values in W resp. $\overline{zW^\perp}$. This holds in particular for ψ_q resp. ψ_r^*. Hence ψ_q is a wavefunction of the Segal-Wilson class and ψ_r^* is the dual of such a wavefunction. We thus have proven part (a) of the following:

Theorem 5.1 *Let ψ be a wavefunction of type z^ℓ and let ψ^* be its dual wavefunction. Then the following holds:*

(a) Let q and r be nonzero elements in R that satisfy

$$\partial_n(q) = (L^n)_+(q), \tag{5.4}$$

$$\partial_n(r) = -(L^n)_+^*(r) \quad \textit{for all } n = 1, 2, \ldots. \tag{5.5}$$

Then $q\partial q^{-1}\psi$ (resp. $r^{-1}\partial^{-1}r\psi$) is a wavefunction of type $z^{\ell+1}$ (resp. $z^{\ell-1}$) and $-q^{-1}\partial^{-1}q\psi^*$ (resp. $-r\partial r^{-1}\psi^*$) are the corresponding dual wavefunctions. In particular, if ψ is of the Segal-Wilson class, then the same holds for ψ_q and ψ_r.

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(b) If both q_1 and q_2 (resp. r_1 and r_2) satisfy (5.4) (resp. (5.5)) such that

$$q_1\partial q_1^{-1} = q_2\partial q_2^{-1} \quad (\textit{resp. } r_1^{-1}\partial r_1 = r_2^{-1}\partial r_2),$$

then $q_1 = \lambda q_2$ (resp. $r_1 = \lambda r_2$) for a certain $\lambda \in \mathbb{C}^*$.

Proof To prove part (b), we note that the conditions $q_1\partial q_1^{-1} = q_2\partial q_2^{-1}$ (resp. $r_1^{-1}\partial r_1 = r_2^{-1}\partial r_2$), simply mean that

$$\frac{\partial(q_1)}{q_1} = \frac{\partial(q_2)}{q_2} \quad \text{and} \quad \frac{\partial(r_1)}{r_1} = \frac{\partial(r_2)}{r_2}.$$

The first calculation shows then

$$\partial\left(\frac{q_1}{q_2}\right) = \frac{q_1}{q_2}\frac{\partial(q_1)}{q_1} - \frac{q_1}{q_2}\frac{\partial(q_2)}{q_2} = 0 \quad \text{resp.} \quad \partial\left(\frac{r_1}{r_2}\right) = \frac{r_1}{r_2}\frac{\partial(r_1)}{r_1} - \frac{r_1}{r_2}\frac{\partial(r_2)}{r_2} = 0.$$

Thus we see that for all $k \geq 1$, $\partial^k(\frac{q_1}{q_2}) = 0$ resp. $\partial^k(\frac{r_1}{r_2}) = 0$ and that there holds: $\partial^k(q_1) = \partial^k(\frac{q_1}{q_2}q_2) = \frac{q_1}{q_2}\partial^k(q_2)$ and likewise with q replaced by r. Now let $B_\ell = (L^\ell)_+$ and $C_\ell = -(L^\ell)^*_+$, then we have for all $n \geq 1$ that $q_1 q_2^{-1}$ is independent of the parameter t_n

$$\begin{aligned}
\partial_n\left(\frac{q_1}{q_2}\right) &= \frac{B_n(q_1)}{q_2} - \frac{q_1}{q_2^2}B_n(q_2)\\
&= \frac{1}{q_2}B_n\left(\frac{q_1}{q_2}q_2\right) - \frac{q_1}{q_2^2}B_n(q_2)\\
&= \frac{q_1}{q_2^2}B_n(q_2) - \frac{q_1}{q_2^2}B_n(q_2)\\
&= 0
\end{aligned}$$

and likewise for $r_1 r_2^{-1}$ one gets that $\partial_n(\frac{r_1}{r_2}) = 0$. So we conclude that $q_1 = \lambda q_2$ and $r_1 = \mu r_2$ for a certain λ and $\mu \in \mathbb{C}^*$. This finishes the proof of Theorem 5.1. ■

We say that a $q \in R$ is an *eigenfunction* of the Lax operator L if it satisfies (5.4). An element $r \in R$ that satisfies (5.5) is called an *adjoint eigenfunction* of the Lax operator L. Clearly operators $q\partial q^{-1}$ and $r^{-1}\partial r$ can yield Darboux transformations without q and/or r having to satisfy the conditions (5.4) or (5.5). One can namely always multiply them with a function that is independent of t_1 without affecting the operators $q\partial q^{-1}$ and $r^{-1}\partial r$. The theorem tells you that you can always find $\tilde{q}$ and $\tilde{r}$ that are eigenfunctions of L and yield the same operators.

The Darboux operators of Theorem 5.1 are clearly invertible. The eigenfunctions corresponding to the inverse transformations are given by the following useful lemma.

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Lemma 5.1 *Let $q, r \in R$, then q (resp. r) satisfies (5.4) (resp. (5.5)) if and only if q^{-1} (resp. r^{-1}) satisfies*

$$\partial_n(q^{-1}) = -(q\partial q^{-1}L^n q\partial^{-1}q^{-1})_+^*(q^{-1}),$$
$$\partial_n(r^{-1}) = (r^{-1}\partial^{-1}rL^n r^{-1}\partial r)_+(r^{-1}), \quad \textit{respectively}.$$

Proof We will use the following elementary properties. Let f belong to R and Q to $R[\partial, \partial^{-1})$, then

(a) $\mathrm{Res}_\partial(Qf) = \mathrm{Res}_\partial(fQ) = f\,\mathrm{Res}_\partial(Q)$,
(b) $\mathrm{Res}_\partial(Qf\partial^{-1}) = Q_+(f)$,
(c) $\mathrm{Res}_\partial(\partial^{-1}fQ) = Q_+^*(f)$.

Suppose that q satisfies the conditions in (5.4), then

$$\begin{aligned}
\partial_n(q^{-1}) &= -q^{-2}(L^n)_+(q)\\
&= -q^{-2}\,\mathrm{Res}_\partial\big(\partial^{-1}q(L^n)^*\big)\\
&= -\,\mathrm{Res}_\partial\big(q^{-1}\partial^{-1}q(L^n)^*q^{-1}\big)\\
&= -\,\mathrm{Res}_\partial\big(q^{-1}\partial^{-1}q(L^n)^*q^{-1}\partial q q^{-1}\partial^{-1}\big)\\
&= -(q^{-1}\partial^{-1}qL^{n*}q^{-1}\partial q)_+(q^{-1})\\
&= -(q\partial q^{-1}L^n q\partial^{-1}q^{-1})_+^*(q^{-1}).
\end{aligned}$$

The reverse statement is a consequence of the property for r, that can be proven in a similar way. This proves the statements in the lemma. ■

The next step in analyzing the situation in $\mathrm{Gr}(H)$ from Theorem 5.1,

$$W_q \subset W \subset W_r, \quad \dim(W_r/W) = \dim(W/W_q) = 1,$$

is to determine the tau-function corresponding to W_q resp. W_r and their relation to that of W. Let $q(t)\partial q(t)^{-1}$ (resp. $-r(t)\partial r(t)^{-1}$) act on the wavefunction $\psi_W(t,z)$ (resp. adjoint wavefunction $\psi_W^*(t,z)$ and compare this with $\psi_q(t,z)$ (resp. $\psi_r^*(t,z)$). We thus obtain

$$\begin{aligned}
zP_{W_q}(t,z) &= \frac{\partial P_W(t,z)}{\partial x} + \left(z - q^{-1}\frac{\partial q}{\partial x}\right)P_W(t,z)\\
zP_{W_r}^{*-1}(t,-z) &= -\frac{\partial P_W^{*-1}(t,-z)}{\partial x} + \left(z + r^{-1}\frac{\partial r}{\partial x}\right)P_W^{*-1}(t,-z).
\end{aligned} \tag{5.6}$$

Comparing the coefficients of z we find that

$$\frac{\partial \log q}{\partial x} = \frac{\partial \log \tau_{W_q}/\tau_W}{\partial x} \quad \text{and} \quad \frac{\partial \log r}{\partial x} = \frac{\partial \log \tau_{W_r}/\tau_W}{\partial x}. \tag{5.7}$$

This suggests the following relations

$$\tau_{W_q}(t) = \lambda q(t)\tau_W(t) \quad \text{and} \quad \tau_{W_r}(t) = \mu r(t)\tau_W(t),$$

with $\lambda, \mu \in \mathbb{C}^\times$. Since the tau-function of a plane W in $\mathrm{Gr}(H)$ is only determined up to a constant, one may assume that the constants λ and μ are equal to 1.

We will carry out the computations for W_r. Then that for W_q is an easy consequence of it. As above let $t(z) \in H$ be such that

$$r(t) = \langle z\psi_W^*, \overline{t(z)}\rangle = \left\langle t(z), \psi_{W^\perp}\left(\gamma(-\bar{t}), z\right)\right\rangle.$$

If $w\colon z^\ell H_+ \to W$ is in $\mathfrak{P}_\ell$, then we can extend it to an embedding $w_r\colon z^{\ell-1}H_+ \to W_r$ by putting $w_r(z^{\ell-1}) = t(z)$ and $w_r(f) = w(f)$ for $f \in z^\ell H_+$. Then w_r belongs to $\mathfrak{P}_{\ell-1}$. Next we decompose w_r with respect to $H = z^\ell H_+ \oplus \mathbb{C}z^{\ell-1} \oplus (z^{\ell-1}H_+)^\perp$ and w with respect to $H = z^\ell H_+ \oplus (z^\ell H_+)^\perp$, then we get

$$w_r = \begin{pmatrix} w_{++} & w_{+0} \\ w_{0+} & w_{00} \\ w_{-+} & w_{-0} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \quad \text{with } w_+ = w_{++} \text{and } w_- = \begin{pmatrix} w_{0+} \\ w_{-+} \end{pmatrix}.$$

We do the same with the operator γ^{-1} from $\Gamma_+(N)$. This gives you respectively

$$\gamma^{-1} = \begin{pmatrix} a_{++} & a_{+0} & a_{+-} \\ 0 & 1 & a_{0-} \\ 0 & 0 & a_{--} \end{pmatrix} = \begin{pmatrix} a_{++} & b \\ 0 & c \end{pmatrix} \text{ with } b = (a_{+0}\; a_{+-}) \text{ and } c = \begin{pmatrix} 1 & a_{0-} \\ 0 & a_{--} \end{pmatrix}.$$

Before computing τ_{w_r}, we notice that

$$\begin{pmatrix} a_{++} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{++}^{-1} & -a_{++}^{-1}a_{+0} \\ 0 & 1 \end{pmatrix} - I = \begin{pmatrix} 0 & -a_{+0} \\ 0 & 0 \end{pmatrix}$$

is a 1-dimensional operator of zero-trace. Hence we have

$$\det\left(\begin{pmatrix} a_{++} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{++}^{-1} & -a_{++}^{-1}a_{+0} \\ 0 & 1 \end{pmatrix}\right) = 1,$$

so that it suffices to compute the projection of $\gamma^{-1}w_r\left(\begin{smallmatrix} a_{++} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$ onto $z^{\ell-1}H_+$:

$$\begin{pmatrix} a_{++}w_+a_{++}^{-1} + bw_-a_{++}^{-1} & a_{++}w_{+0} + a_{+0}w_{00} + a_{+-}w_{-0} \\ (w_{0+} + a_{0-}w_{-+})a_{++}^{-1} & w_{00} + a_{0-}w_{-0} \end{pmatrix} = \begin{pmatrix} A & \beta \\ \alpha & \delta \end{pmatrix}.$$

As A is equal to $(\gamma^{-1}wa_{++}^{-1})_+$ and γ is chosen such that this operator is invertible, we can decompose $\left(\begin{smallmatrix} A & \beta \\ \alpha & \delta \end{smallmatrix}\right)$ as follows

$$\begin{pmatrix} A & \beta \\ \alpha & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \delta - \alpha A^{-1}\beta \end{pmatrix} \begin{pmatrix} 1 & A^{-1}\beta \\ 0 & 1 \end{pmatrix}.$$

From this we see that

$$\tau_{w_r}(\gamma) = \det\left(\begin{pmatrix} A & \beta \\ \alpha & \delta \end{pmatrix}\right) = \det(A)(\delta - \alpha A^{-1}\beta) = \tau_w(\gamma)(\delta - \alpha A^{-1}\beta).$$

Hence we merely have to show that $r(t) = \delta - \alpha A^{-1}\beta$. We know that $g(r) = \gamma(-t)^*$ with $\gamma(t) \in \Gamma_+^W(N)$ and thus we get

$$r(t) = \left\langle \gamma(t)t(z), \hat{\psi}_{W^\perp}\left(\gamma(-t)^*, z\right) \right\rangle.$$

By definition one has

$$\hat{\psi}_{W^\perp}\left(\gamma(-t)^*, z\right) = z^{\ell-1} + \sum_{k\geq\ell} \psi_k\left(\gamma(-t)^*\right) z^k.$$

Therefore, if $\beta = \sum_{k\geq\ell} \beta_k(t) z^k$, then the expression for $r(t)$ becomes

$$\begin{aligned} r(t) &= \left\langle \delta z^{\ell-1} + \sum_{k\geq\ell} \beta_k(t) z^k, z^{\ell-1} + \sum_{j\geq\ell} \psi_j\left(\gamma(-t)^*\right) z^j \right\rangle \\ &= \delta + \sum_{k\geq\ell} \beta_k(t) \overline{\psi_k\left(\gamma(-t)^*\right)}. \end{aligned}$$

From the computation of $\hat{\psi}_{W^\perp}$ in [5], we see that, if $\alpha A^{-1}(z^{k-1}) = \alpha_k z^{\ell-1}$ for all $k \geq \ell$, then $\psi_k\left(\gamma(-t)^*\right) = -\overline{\alpha_k}$. In other words, there holds

$$r(t) = \delta - \alpha A^{-1}\beta.$$

Next we treat the case $W_q \subset W$ and $\dim(W/W_q) = 1$. Let $s(z) \in W$ be such that $W_q = \{w \in W \mid \langle w, s(z)\rangle = 0\}$ and $q(t) = \langle \psi_W(t,z), s(z)\rangle$. Let $w_0 \in \mathfrak{P}_{\ell+1}$ be such that $w_0(z^{\ell+1}H_+) = W_q$, then $w = \left(w_0 s(z)\right)$ belongs to $\mathfrak{P}_\ell$ and $w(z^\ell H_+) = W$. According to the foregoing result, we have

$$\tau_w(t) = r(t)\tau_{w_0}(t) \quad \text{with } r(t) = \left\langle s(z), \psi_{W_q^\perp}\left(\gamma(-\bar{t})\right) \right\rangle.$$

In particular, one has the relations

$$P_W = r^{-1}\partial^{-1} r P_{W_q} \quad \text{and} \quad P_{W_q} = q\partial q^{-1} P_W,$$

so that we may conclude

$$r^{-1}\partial r = q\partial q^{-1}.$$

According to Lemma 5.1 r^{-1} satisfies

$$\partial_n(r^{-1}) = (r^{-1}\partial^{-1} r P_{W_q} \partial^n P_{W_q}^{-1} r^{-1}\partial r)_+(r^{-1}) = (L_W^n)_+(r^{-1})$$

like q. Hence by Theorem 5.1 we may conclude that $q(t) = \lambda r(t)^{-1}$ with $\lambda \in \mathbb{C}^\times$. We state the foregoing results in the form of a theorem.

Theorem 5.2 *Let $W, W_q, W_r \in \mathrm{Gr}(H)$ be such that $W_q \subset W \subset W_r$ and $\dim(W_r/W) = \dim(W/W_q) = 1$. Then the tau-function of W_r is equal to $\tau_{W_r}(t) = r(t)\tau_W(t)$, with $r(t) = \langle v, \psi_{W^\perp}(\bar{t}, z)\rangle$ for some $v \in W_r \setminus W$ and the tau-function of W_q is equal to $\tau_{W_q}(t) = q(t)\tau_w(t)$, where $q(t) = \langle \psi_W(t,z), s\rangle$ for some $s \in W \cap (W_q)^\perp$. Moreover, the wave functions are related as follows*

$$
\begin{aligned}
\psi_q(t,z) &= z\frac{q(t-[z^{-1}])}{q(t)}\psi_W(t,z), \quad \psi_q^*(t,z) = z^{-1}\frac{q(t+[z^{-1}])}{q(t)}\psi_W^*(t,z),\\
\psi_r(t,z) &= z^{-1}\frac{r(t-[z^{-1}])}{r(t)}\psi_W(t,z), \quad \psi_r^*(t,z) = z\frac{r(t+[z^{-1}])}{r(t)}\psi_W^*(t,z).
\end{aligned} \tag{5.8}
$$

6 Higher Order Darboux Transformations

In Section 5 we computed the Darboux transformations that related the solutions corresponding to a plane $W \in \mathrm{Gr}(H)$ to that of the codimension one planes $W_q \subset W$ and $W \subset W_r$. At the same time we derived the corresponding transformation for the τ-functions. In this subsection we want to describe the Darboux transformation $D(V, W)$ that relates the wavefunction $\psi_W(t,z) = P_W(\partial)e^{\sum t_i z^i}$ for a $W \in \mathrm{Gr}(H)$, to that of a plane $V \in \mathrm{Gr}(H)$ such that V has finite codimension inside W or vice versa. Further we want to describe the transformation on the τ-function level. Of course, if $\psi_V(t,z) = P_V(\partial)e^{\sum t_i z^i}$, then it is clear that $D(V,W) = P_V(\partial)P_W(\partial)^{-1}$, but we want to express $D(V,W)$ in ψ_W and some additional geometric data relating V and W, not knowing ψ_V.

We start with the case that V has codimension n in W. We will construct $D(V,W)$ in this case with a chain of codimension one inclusions

$$V = W_n \subset W_{n-1} \subset \cdots \subset W_1 \subset W_0 = W$$

and the corresponding elementary Darboux transformations. This sequence is constructed inductively from $l = 0$: assuming W_l has been defined, one chooses a nonzero element $w_{l+1} \in W_l \cap V^\perp$ and defines W_{l+1} as $\{w \mid w \in W_l, \langle w, w_{l+1}\rangle = 0\}$. If the functions $\{q_j \mid 1 \le j \le n\}$ in R are defined by $q_j = \langle \psi_{W_{j-1}}, w_j\rangle$, then we know that $\psi_{W_j} = q_j \partial q_j^{-1}\psi_{W_{j-1}}$ and in particular that

$$D(V,W) = \prod_{j=1}^{n} q_j\partial q_j^{-1} = \sum_{k=0}^{n} a_k\partial^k = \partial^n + \sum_{k=0}^{n-1} a_k\partial^k.$$

Clearly the a_j are polynomial expressions in the q_j's and their derivatives and we will derive now its closed form. Since $\langle \psi_V, w_j\rangle = 0$ for all $j = 1, 2, \ldots, n$, we find that the nontrivial coefficients of $D(V,W)$ satisfy the linear system of equations

$$\sum_{k=0}^{n-1} a_k \left\langle \frac{\partial^k \psi_W(t,z)}{\partial x^k}, w_j(z)\right\rangle = -\left\langle \frac{\partial^n \psi_W(t,z)}{\partial x^n}, w_j(z)\right\rangle, \quad \text{for } j = 1, 2, \ldots, n. \tag{6.1}$$

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Let l be an integer, $1 \leq l \leq n$, and let A be a meromorphic function on Γ_+ with values in H. Then $\mathcal{M}(A; w_1, w_2, \dots, w_l)$ is the $l \times l$-matrix whose (k, j)-entry, with $0 \leq k \leq l-1$ and $1 \leq j \leq l$, is given by

$$\mathcal{M}(A; w_1, w_2, \dots, w_l)_{kj} = \langle \partial^k(A), w_j \rangle.$$

The matrix $\mathcal{M}(\psi_W; w_1, w_2, \dots, w_l)$ is invertible. For, a linear combination of its rows leads to an equation $\langle \psi_W, \sum_{j=1}^{l} \lambda_j w_j \rangle = 0$ and as the boundary values of ψ_W are lying dense in W and the $\{w_j\}$ are linear independent, we get that $\lambda_j = 0$ for all j. Therefore one can resolve the a_k from the equations (6.1) with Cramer's rule and we obtain the following expression for them

$$a_k = (-1)^{n-k} \det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)^{-1}\bigr)\, \mathcal{W}_k(\psi_W; w_1, w_2, \dots, w_n),$$

where $\mathcal{W}_k(A; w_1, w_2, \dots, w_n)$ is equal to

$$\det\begin{pmatrix} \langle A, w_1\rangle & \cdots & \langle \frac{\partial^{k-1}A}{\partial x^{k-1}}, w_1\rangle & \langle \frac{\partial^{k+1}A}{\partial x^{k+1}}, w_1\rangle & \cdots & \langle \frac{\partial^{n}A}{\partial x^{n}}, w_1\rangle \\ \langle A, w_2\rangle & \cdots & \langle \frac{\partial^{k-1}A}{\partial x^{k-1}}, w_2\rangle & \langle \frac{\partial^{k+1}A}{\partial x^{k+1}}, w_2\rangle & \cdots & \langle \frac{\partial^{n}A}{\partial x^{n}}, w_2\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle A, w_n\rangle & \cdots & \langle \frac{\partial^{k-1}A}{\partial x^{k-1}}, w_n\rangle & \langle \frac{\partial^{k+1}A}{\partial x^{k+1}}, w_n\rangle & \cdots & \langle \frac{\partial^{n}A}{\partial x^{n}}, w_n\rangle \end{pmatrix}.$$

This is in fact a generalization of a result of Crumm [3]. Note that the coefficients of $D(V, W)$ do not change, if one passes from the basis w_j of $W \cap V^\perp$ to another one.

As for the corresponding transformation of the τ-functions, we know from (5.1) that we have for all j, $0 \leq j \leq n-1$,

$$\tau_{W_{j+1}} = \langle \psi_{W_j}, w_j \rangle \tau_{W_j}.$$

Hence we get for τ_V that it is given by

$$\tau_V = \langle D(W_{n-1}, W)\psi_W, w_n \rangle \tau_{W_{n-1}}.$$

From the foregoing computations we know that $D(W_{n-1}, W) = \sum_{i=0}^{n-1} b_i \partial^i$, with $b_{n-1} = 1$ and

$$b_i = (-)^{n-1-i} \det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_{n-1})^{-1}\bigr)\, \mathcal{W}_i(\psi_W; w_1, w_2, \dots, w_{n-1}).$$

By using the expansion of $\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)\bigr)$ w.r.t. the last row we get that

$$\begin{aligned} \tau_V &= \sum_{i=0}^{n-1} b_i \langle \partial^i \psi_W, w_n \rangle \tau_{W_{n-1}} \\ &= \frac{\Bigl(\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_{n-1})\bigr)\langle \partial^{n-1}\psi_W, w_n\rangle + \sum_{i=0}^{n-2} b_i \mathcal{W}_i(\psi_W; w_1, w_2, \dots, w_{n-1})\langle \partial^i \psi_W, w_n\rangle\Bigr)}{\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_{n-1})\bigr)} \tau_{W_{n-1}} \\ &= \frac{\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)\bigr)}{\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_{n-1})\bigr)} \tau_{W_{n-1}}. \end{aligned}$$

This implies the following relation betweeen τ_V and τ_W

$$\tau_V = \det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)\bigr)\, \tau_W.$$

Consider now for each integer $n \geq 1$ the collection

$$\mathcal{D}_n = \{(V, W) \mid V, W \in \mathrm{Gr}(H), V \subset W \text{ of codimension } n\}.$$

For each pair (V, W) in $\mathcal{D}_n$ we have that

$$L_V = D(V, W) L_W D(V, W)^{-1},$$

so that we have answered the first half of the question raised in (2.1) and we state these results in a theorem.

Theorem 6.1

(a) The variety $\mathcal{D}_n$ describes the Segal-Wilson solutions of the KP *hierarchy that are linked by a Darboux transformation of order n. It is a fiber bundle over* $\mathrm{Gr}(H)$ *with typical fiber consisting of the Grassmann manifold of n-dimensional subspaces of H_+.*

(b) For each pair (V, W) in $\mathcal{D}_n$, let $w_1, \dots, w_n$ be a basis of $W \cap V^{\perp}$. Then a monic differential operator $D(V, W)$ of order n that realizes the Darboux transformation from L_W to L_V is given by

$$D(V, W) = \partial^n + \sum_{i=0}^{n-1} (-1)^{n-i} \frac{\mathcal{W}_i(\psi_W; w_1, w_2, \dots, w_n)}{\det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)\bigr)} \partial^i,$$

(c) With the same notations as in part (b), the τ-functions of the planes V and W, with $(V, W) \in \mathcal{D}_n$, are related by

$$\tau_V = \det\bigl(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_n)\bigr)\, \tau_W.$$

Next we treat the situation that W is a subpace of V of codimension m. We will construct the pseudodifferential operator $D(V, W)$ of order $-m$ with a chain of codimension one inclusions

$$W = U_0 \subset U_1 \subset \cdots \subset U_{m-1} \subset U_m = V$$

and the corresponding elementary transformations. Thereto, one chooses successively elements $\{v_1, \dots, v_m\}$ in $V \cap W^{\perp}$ according to: assuming that U_l, $0 \leq l \leq m-1$, has been defined, we take a nonzero v_{l+1} in $V \cap W^{\perp}$, orthogonal to U_l. Then one defines $U_{l+1} := U_l \oplus \mathbb{C} v_{l+1}$ and the function r_j by $r_j := \langle z\psi^*_{U_{j-1}}, \overline{v_j}\rangle = \langle \psi^*_{U_{j-1}}, v_j^{\dagger}\rangle$, where the convenient notation $v^{\dagger}$ is given by

$$v^{\dagger}(z) := \overline{zv(z)}. \tag{6.2}$$

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Then we know from Section 5 that $\psi_{U_j} = r_j^{-1}\partial^{-1} r_j \psi_{U_{j-1}}$ and thus that

$$D(V,W) = \prod_{j=1}^{m} r_j^{-1}\partial^{-1} r_j.$$

In order to give a closed expression for $D(V,W)$ in ψ_W^* and the $\{v_j\}$, we first consider the operator $\big(D(V,W)^*\big)^{-1}$ that relates the dual wavefunctions ψ_W^* and ψ_V^* by $\psi_V^* = \big(D(V,W)^*\big)^{-1}\psi_W^*$. As $\big(D(V,W)^*\big)^{-1} = (-1)^m \prod_{j=1}^m r_j\partial r_j^{-1}$, we see that it has the form $\sum_{k=0}^m c_k\partial^k$, with $c_m = (-1)^m$. Since the boundary values of ψ_V^* belong to $\overline{zV^\perp}$, we have $\langle \psi_V^*, v_j^\dagger\rangle = 0$ for all j. Hence the $\{c_i \mid 0 \le i \le m-1\}$ satisfy the following system of linear equations

$$\sum_{k=0}^{m-1} c_k \left\langle \frac{\partial^k \psi_W^*(t,z)}{\partial x^k}, v_j^\dagger(z)\right\rangle = (-1)^{m+1}\left\langle \frac{\partial^m \psi_W^*(t,z)}{\partial x^n}, v_j^\dagger(z)\right\rangle. \tag{6.3}$$

A similar reasoning as for the matrix $\mathcal{M}(\psi_W; w_1, w_2, \dots, w_l)$ shows that also the $l\times l$-matrix $\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_l^\dagger)$ is invertible. From Cramer's rule we obtain then that for each k, $0 \le k \le m-1$,

$$c_k = (-1)^k \det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger)^{-1}\big)\, \mathcal{W}_k(\psi_W; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger).$$

This closed expression for the coefficients of $\big(D(V,W)^*\big)^{-1}$ can be used to compute the corresponding transformation of the τ-functions. We know namely from Section 5 that for all $l, 0 \le l \le m-1$, $\tau_{U_{l+1}} = \langle \psi_{U_l}^*, v_{l+1}^\dagger\rangle \tau_{U_l}$. In particular we get that

$$\begin{aligned}
\tau_V &= \sum_{i=0}^{m-1} b_i \langle \partial^i \psi_W^*, v_n^\dagger\rangle \tau_{W_{n-1}} \\
&= \frac{\begin{aligned}&\Big(\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_{m-1}^\dagger)\big)\,\langle \partial^{m-1}\psi_W^*, v_n^\dagger\rangle \\ &\qquad + \sum_{i=0}^{n-2} b_i \mathcal{W}_i(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_{m-1}^\dagger)\langle \partial^i \psi_W^*, v_m^\dagger\rangle\Big)\end{aligned}}{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_{m-1}^\dagger)\big)} \tau_{U_{m-1}} \\
&= \frac{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger)\big)}{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_{m-1}^\dagger)\big)} \tau_{U_{m-1}}.
\end{aligned}$$

Hence we may conclude that

$$\tau_V = \det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger)\big)\, \tau_W.$$

What remains to be done is a more direct formula for $D(V,W)$ than $D(V,W) = (Q^*)^{-1}$, with $Q = \sum_{k=0}^m c_k\partial^k$. Recall that we denote the transposed of a matrix A with A^T. Consider now the following linear system

$$\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger)^T (b_1, \dots, b_m)^T = (0, \dots, 0, 1)^T.$$

We claim now that $Q^{-1} = (-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j$. Thereto we compute

$$Q\Big((-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j \Big) = \Big(Q(-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j \Big)_+ + \Big(Q(-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j \Big)_- .$$

The last term of the right hand side we reduce with the fact that the constant term of $Q\langle \psi_W^*, v_j^\dagger \rangle$ is equal to $Q(\langle \psi_W^*, v_j^\dagger \rangle) = 0$, since $\langle \psi_W^*, v_j^\dagger \rangle \in \operatorname{Ker} Q$, and obtain

$$\Big(Q(-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j \Big)_- = \Big((-1)^m Q\Big(\sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \Big) \partial^{-1} b_j \Big)_- = 0.$$

For the first term of the right hand side we use the fact that for all $f \in R$ we have

$$f\partial^{-1} = -(\partial^{-1} f)^* = \sum_{\ell=0}^{\infty} \Big((-1)^{\ell+1} \frac{\partial^\ell f}{\partial x^\ell} \partial^{-\ell-1} \Big)^* = \sum_{\ell=0}^{\infty} \partial^{-\ell-1} \frac{\partial^\ell f}{\partial x^\ell}$$

and obtain then the desired equality

$$\begin{aligned}
\Big(Q(-1)^m \sum_{j=1}^m \langle \psi_W^*, v_j^\dagger \rangle \partial^{-1} b_j \Big)_+ &= \Big(Q(-1)^m \sum_{j=1}^m \sum_{\ell=0}^{\infty} \partial^{-\ell-1} \frac{\partial^\ell \langle \psi_W^*, v_j^\dagger \rangle}{\partial x^\ell} b_j \Big)_+ \\
&= \Big(Q(-1)^m \sum_{\ell=0}^{m-1} \partial^{-\ell-1} \sum_{j=1}^m \frac{\partial^\ell \langle \psi_W^*, v_j^\dagger \rangle}{\partial x^\ell} b_j \Big)_+ \\
&\quad + \Big(Q(-1)^m \sum_{j=1}^m \sum_{\ell \geq m} \partial^{-\ell-1} \frac{\partial^\ell \langle \psi_W^*, v_j^\dagger \rangle}{\partial x^\ell} b_j \Big)_+ \\
&= \big(Q(-1)^m \partial^{-m} \big)_+ + 0 \\
&= 1.
\end{aligned}$$

For the coefficients b_k we have according to Cramer's rule the closed expression

$$b_k = \frac{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, \dots, v_{k-1}^\dagger, \hat{v_k^\dagger}, v_{k+1}^\dagger, \dots, v_m^\dagger) \big)}{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \dots, v_m^\dagger) \big)}.$$

Thus we get for $D(V, W) = (Q^{-1})^*$ the following expression

$$D(V, W) = -\sum_{j=1}^m b_j \partial^{-1} \langle \psi_W^*, v_j^\dagger \rangle.$$

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For each integer $m \geq 1$, we introduce the space

$$\mathcal{D}_{-m} = \{(V, W) \mid V, W \in \mathrm{Gr}(H), W \subset V \text{ of codimension } m\}.$$

For each pair (V, W) in $\mathcal{D}_n$ we know that $D(V, W)^{-1}$ is a differential operator of order m that relates the solutions of the KP hierarchy L_V and L_W by

$$L_V = D(V, W)L_W D(V, W)^{-1}.$$

Thus we have settled the second half of the question raised in (2.1) and we state these results in a theorem.

Theorem 6.2

(a) *The variety $\mathcal{D}_{-m}$ describes the Segal-Wilson solutions of the* KP *hierarchy that are linked by a Darboux transformation of order $-m$. It is a fiber bundle over* $\mathrm{Gr}(H)$ *with typical fiber consisting of the Grassmann manifold of m-dimensional subspaces of H_-.*

(b) *For each pair (V, W) in $\mathcal{D}_{-m}$, let $v_1, \ldots, v_m$ be a basis of $V \cap W^\perp$. Then a monic pseudodifferential operator $D(V, W)$ of order $-m$ that realizes the Darboux transformation from L_W to L_V is given by*

$$D(V, W) = -\sum_{j=1}^{m} \frac{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, \ldots, v_{j-1}^\dagger, \hat{v_j^\dagger}, v_{j+1}^\dagger, \ldots, v_m^\dagger)\big)}{\det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \ldots, v_m^\dagger)\big)} \partial^{-1} \langle \psi_W^*, v_j^\dagger \rangle$$

(c) *With the same notations as in part (b), τ-functions of the planes V and W, with $(V, W) \in \mathcal{D}_{-m}$, are related by*

$$\tau_V = \det\big(\mathcal{M}(\psi_W^*; v_1^\dagger, v_2^\dagger, \ldots, v_m^\dagger)\big)\, \tau_W.$$

7 Darboux Transformations for the Gelfand-Dickey Hierarchy

In this section we want to consider the elementary Darboux transformations for the Gelfand-Dickey hierarchies. It is well-known that these correspond to the n-th reduced KP hierarchy, *i.e.*, $L^n = (L^n)_+$. Since $L^n\psi_W(t, z) = z^n\psi_W(t, z)$ we have a trivial Darboux transformation that maps W into z^nW. Since $L^n = (L^n)_+$, $z^n\psi_W(t, z) = \partial_n\big(\psi_W(t, z)\big)$ and hence $z^nW \subset W$. We now want to determine which vectors $s \in W$ we can choose such that $W_s = \{w \in W \mid \langle w, s\rangle = 0\}$ also satisfies $z^nW_s \subset W_s$.

We will first prove the following Lemma.

Lemma 7.1 *Let $W' \subset W$ such that $z^nW' \subset W$ and*

$$W_s = \{w \in W \mid \langle w, s\rangle = 0\} \quad \textit{and} \quad W'_s = \{w \in W' \mid \langle w, s\rangle = 0\}.$$

Then $z^nW'_s \subset W_s$ if and only if there exists a $\lambda \in \mathbb{C}$ such that

$$(z^{-n} - \lambda)s \in W'^{\perp}. \tag{7.1}$$

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Proof Let $v \in W_s'$ and $z^n v \in W_s$, then

$$\langle z^n v, s(z)\rangle = \langle v, z^{-n} s(z)\rangle = 0,$$

in other words $z^{-n}s(z) \in W_s'^{\perp}$. Hence, since $W_s'^{\perp} = W'^{\perp} \oplus \mathbb{C}s$, there must exist a $\lambda \in \mathbb{C}$ such that $(z^{-n} - \lambda)s \in W'^{\perp}$.

Now assume that (7.1) holds, then for every $v \in W_s'$ one has

$$\langle z^n v, s(z)\rangle = \langle v, z^{-n}s(z)\rangle = \overline{\lambda}\langle v, s(z)\rangle = 0,$$

hence $z^n v \in W_s$. ■

So it suffices to assume that $W = W'$ and that $s(z) \in W$ satisfies (7.1). Let

$$U = W \cap (z^n W)^{\perp}, \tag{7.2}$$

then (see *e.g.* [13]) $W = \bigoplus_{k=0}^{\infty} z^{kn}U$, so we write

$$s(z) = \sum_{k=0}^{\infty} u_k z^{kn} \quad \text{with } u_k \in U.$$

Since $(z^{-n} - \lambda)s(z) \in W^{\perp}$, one obtains that $u_{k+1} = \lambda u_k$ for $k = 0, 1, \dots$ and hence,

$$s(z) = \sum_{k=0}^{\infty} (\lambda z)^{kn} u_0 \quad \text{with } u_0 \in U \text{ and } \lambda \in \mathbb{C}.$$

Since

$$\begin{aligned}\langle s(z), s(z)\rangle &= \Big\langle \sum_{k=0}^{\infty} (\lambda z)^{kn} u_0, \sum_{k=0}^{\infty} (\lambda z)^{kn} u_0 \Big\rangle \\ &= \sum_{k=0}^{\infty} |\lambda|^{2k}\, |u_0|^2,\end{aligned}$$

must be finite, we must choose $|\lambda| < 1$. Notice that if W satisfies $z^n W \subset W$, then $W^{\perp}$ satisfies $z^{-n}W^{\perp} \subset W^{\perp}$. Now let $V = W^{\perp} \cap (z^{-n}W^{\perp})^{\perp}$, then $V = z^{-n}U$ and hence one obtains that $t(z) \in W^{\perp}$ which gives the correct Darboux transformation is

$$t(z) = \sum_{k=0}^{\infty} \left(\frac{\lambda}{z}\right)^{nk} z^{-n}u_0, \quad \text{with } u_0 \in U \text{ and } \lambda \in \mathbb{C}, |\lambda| < 1.$$

We thus have proven the following theorem.

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Theorem 7.1 *Let $W \in \mathrm{Gr}(H)$ satisfy $z^nW \subset W$ and hence $z^{-n}W^{\perp} \subset W^{\perp}$, then $s(z) \in W$ (resp. $t(z) \in W^{\perp}$) defines an eigenfunction $q(t) = \langle \psi_w(t,z), s(z)\rangle$ (resp. adjoined eigenfunction $r(t) = \langle s(z), \overline{z\psi^*_W(t,z)}\rangle$) and $\psi_q(t,z) = q\partial q^{-1}\psi_W(t,z)$ (resp. $\psi_r(t,z) = r^{-1}\partial^{-1}r\psi_W(t,z)$) be a new wavefunction defined by the elementary Darboux transformation that maps W into $W_q = \{w \in W \mid \langle w, s(z)\rangle = 0\}$ (resp. $W_r = W \oplus \mathbb{C}t(z)$). Let $U = W \cap (z^nW)^{\perp}$. Then W_q satisfies $z^nW_q \subset W_q$ (resp. W_r satisfies $z^nW_r \subset W_r$) if and only if*

$$s(z) = \sum_{k=0}^{\infty} (\lambda z)^{kn} u_0 \quad (\textit{resp. } t(z) = \sum_{k=0}^{\infty} \left(\frac{\lambda}{z}\right)^{nk} z^{-n}u_0),$$

with $u_0 \in U$ and $\lambda \in \mathbb{C}$, $|\lambda| < 1$.

Next we want to give the geometrical interpretation (in terms of the Segal-Wilson Grassmannian) of the classical Darboux transformation of an n-th Gelfand-Dickey Lax operator L^n.

The classical Darboux transformation consists of factorizing an n-th order differential operator $Q = RS$, with R and S differential operators of order r, s, respectively, with $r + s = n$, and then exchanging the place of the factors, *i.e.*, defining a new differential operator $Q' = SR$. We assume that $Q = L^n$ is a Gelfand-Dickey operator and we want that Q' is again a Gelfand-Dickey operator. Let $L = P_W\partial P_W^{-1}$, with $W \in \mathrm{Gr}(H)$, since $Q' = SQS^{-1}$, we find that $Q' = L'^n$, with $L' = P_{W'}\partial P_{W'}^{-1}$ and $P_{W'} = SP_W$. Because $Q = RS$ we find that

$$z^nW \subset W' \subset W,$$

hence the vectors $w_1(z), w_2(z), \dots, w_s(z) \in W$, that define the Darboux transformation S, must be perpendicular to z^nW and thus must belong to U, defined by (7.2). We thus have proven:

Theorem 7.2 *Let $W \in \mathrm{Gr}(H)$ satisfy $z^nW \subset W$ and let $L = P_w\partial P_W^{-1}$. The classical Darboux transformation of the Gelfand-Dickey differential operator $Q = L^n$, consists of choosing s ($0 < s < n$) linearly independent vectors $v_j \in U$ (U defined by (7.2)), such that*

$$S = \partial^s + \sum_{i=0}^{s-1} (-1)^{s-i} \frac{\mathcal{W}_i(\psi_W; w_1, w_2, \dots, w_s)}{\det\big(\mathcal{M}(\psi_W; w_1, w_2, \dots, w_s)\big)} \partial^i.$$

Then $Q' = SQS^{-1}$ is again a Gelfand-Dickey differential operator.

References

[1] M. Adler and P. van Moerbeke, *Birkhoff Strata, Bäcklund Transformations and Regularization of Isospectral Operators.* Adv. in Math. **108**(1994), 140–204.

[2] B. Bakalov, E. Horozov and M. Yakimov, *Bäcklund-Darboux transformations in Sato's Grassmannian.* Serdica Math. J. **22**(1996), 571–586.

[3] M. M. Crum, *Associated Sturm-Liouville systems.* Quart. J. Math. **6**(1955), 121–127.

[4] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations.* In: Nonlinear integral systems—classical theory and quantum theory (eds. M. Jimbo and T. Miwa), World Scientific, 1983, 39–120.

[5] G. F. Helminck and G. F. Post, *Geometrical interpretation of the bilinear equations for the KP hierarchy.* Letters Math. Phys. **16**(1988), 359–364.

[6] G. F. Helminck and J. W. van de Leur, *An analytic description of the vector constrained KP hierarchy.* Comm. Math Phys. **193**(1998), 627–641 (solv-int 9706004).

[7] ———, *Constrained and Rational Reductions of the KP hierarchy.* Preprint, to appear in "Supersymmetry and Integrable Models", Springer Lecture Notes in Physics **502**, 1998, 167–182.

[8] B. B. Kadomtsev and V. I. Petviashvilii, *On the stability of solitary waves in weakly dispersing media.* Sov. Phys. Doklady **15**, 539–541.

[9] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons.* Springer Series in Nonlinear Dynamics, Springer, Berlin, 1991.

[10] W. Oevel, *Darboux theorems and Wronskian formulas for integrable systems (I. Constrained KP flows).* Physica **A 195**(1993), 533-576.

[11] W. Oevel and W. Schief, *Darboux theorems and the KP hierarchy.* In: Applications of analytic and geometric Methods in Differential Equations (Proceedings of the NATO Advanced Research Workshop, Exeter, 14–17 July 1992, UK) (ed. P. A. Clarkson), Kluwer Publ., 1993, 193–206.

[12] W. Oevel and W. Strampp, *Wronskian solutions of the constrained Kadomtsev-Petviashvili hierarchy.* Journ. Math. Phys. **37**(1996), 6213–6219.

[13] A. Pressley and G. Segal, *Loop groups.* Oxford Mathematical Monographs, Oxford University Press, Oxford, 1988.

[14] G. Segal and G. Wilson, *Loop groups and equations of KdV type.* Inst. Hautes Études Sci. Publ. Math. **63**(1985), 1–64.

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