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Strongly Projective Graphs

Benoit Larose

Abstract. We introduce the notion of strongly projective graph, and characterise these graphs in terms of their neighbourhood poset. We describe certain exponential graphs associated to complete graphs and odd cycles. We extend and generalise a result of Greenwell and Lovász [6]: if a connected graph G does not admit a homomorphism to K, where K is an odd cycle or a complete graph on at least 3 vertices, then the graph $G \times K^s$ admits, up to automorphisms of K, exactly s homomorphisms to K.

1 Introduction

In the following, all graphs are assumed to be finite and undirected, and unless otherwise stated all graphs under consideration are without loops. For basic terminology and notation we shall follow [7]. If G and H are graphs, a *homomorphism* from G to H is an edge-preserving map from the vertex-set of G to the vertex-set of H, *i.e.* a map $f\colon G \to H$ such that $f(g)f(g')$ is an edge of H whenever gg' is an edge of G. The (categorical) *product* $G \times H$ of two graphs has vertex set $G \times H$ and two vertices (g,h) and (g',h') are adjacent if gg' and hh' are edges of G and H respectively. In other words, this is the largest set of edges on $G \times H$ such that the projections onto G and onto H are graph homomorphisms. For $n \geq 1$ we let G^n denote the product of G with itself n times. A graph is a *core* if it has no proper retracts, *i.e.* if every homomorphism of G to itself is an automorphism.

The behaviour of graph homomorphisms with respect to the categorical product appears to be quite complex. Consider only the (yet unsolved) Hedetniemi conjecture [9] which states that the chromatic number of the product of two graphs G and H is the minimum of their respective chromatic numbers, *i.e.*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

An *n-ary operation* on a graph G is a homomorphism from G^n to G. An operation f is said to be *idempotent* if it satisfies $f(x,x,\dots,x) = x$ for all x. Call a graph G *projective* (*idempotent trivial*) if the only idempotent operations on G are the projections. Our interest in these graphs is threefold. First, Hedetniemi's conjecture may be rephrased as follows. Call a core graph K *irreducible* if, whenever K is a retract of a product $G \times H$ then it is a retract of G or H. Then the conjecture may be stated as *Every complete graph is irreducible.* Few irreducible graphs are known, but it is shown in [15] that a vertex-transitive core graph which is projective is *weakly irreducible*, *i.e.* if it is a retract of a product of *connected* graphs, then it is a retract of one of the factors (this was first shown in the case of the complete graphs by Duffus, Sands

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and Woodrow [4].) Second, in recent work by Nešetřil and Zhu [18], it was shown that projective core graphs are the natural generalisation of the complete graphs in extending a result of Müller on unique colourability; more precisely, they show that if G is a projective core graph, then there exist graphs of large girth that admit, up to automorphisms of G, only a prescribed set of homomorphisms into G. Finally, a key ingredient in the study of the relationship between products and homomorphisms is the notion of *exponential graph* [17, 5]: Let G and K be two graphs. Define K^G as follows: the vertices are all the functions from G to K, and two such functions f and g are adjacent if they satisfy the following condition: if x and y are adjacent in G then $f(x)$ and $g(y)$ are adjacent in K. In other words, we define the 'right' edge structure on the set K^G to obtain the natural bijection between the sets $\mathrm{Hom}(G \times H, K)$ and $\mathrm{Hom}(H, K^G)$. There is a natural correspondence between graphs and posets under which order-preserving operations correspond to functions in exponential graphs that have at least one neighbour (see below). This correspondence can be used to investigate the structure of certain exponential graphs, namely those of the form K^G where $G = K^s$ for some $s \geq 1$. For example it was shown in [14] that if the poset associated to the graph G is projective, then G must also be projective. In fact, quite a bit more can be said if we modify the notion of projective graph as follows.

Let G be any graph and let $s \geq 1$. For convenience, we shall adopt the following notation in the remainder of the paper: let $E_s(G)$ denote the graph G^{G^s} and let $I_s(G)$ denote the subgraph of $E_s(G)$ consisting of the idempotent functions, *i.e.* those f that satisfy $f(x, \dots, x) = x$ for all x. Call a graph G *strongly projective* if, for every $s \geq 2$, the only $f \in I_s(G)$ with at least one neighbour are the projections.

Our first result characterises strongly projective graphs in terms of the neighbourhood poset alluded to earlier (Theorem 2.1). In the last section we describe in detail the graphs $E_s(K)$ for $s \geq 2$ where K is complete or an odd cycle (Theorems 3.2, 3.6). Theorem 3.2 actually proves a special case of a conjecture by Duffus, Sands and Woodrow [4] (see [11]). We deduce from this an analog for odd cycles of a result of Greenwell and Lovász [6] on unique colourability (Corollary 3.7).

2 Strong Projectivity

Let G be a graph. If $a \in G$ we denote the neighbourhood of a by N_a. More generally, if X is a set of vertices let N_X denote the set of all vertices in G that are adjacent to all elements of X, *i.e.*

$$N_X = \bigcap_{x \in X} N_x.$$

We say that G is *ramified* when for all $a, b \in G$, if $N_a \subseteq N_b$ then $N_a = N_b$. Let us now consider the graphs $E_s(G)$. For every $s \geq 1$ this graph contains loops, namely, the homomorphisms from G^s to G. When we say that a vertex of $E_s(G)$ is *isolated* we mean that it is adjacent to no vertex other than (possibly) itself.

If G is a ramified, connected, non-bipartite graph, let $\mathcal{P}_G$ denote the poset of non-empty intersections of neighbourhoods of G ordered by inclusion. We'll drop the subscript G when no confusion is possible. (See [14] for details). Notice that the maximal elements of the poset are the neighbourhoods of G, and that because G is

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ramified, its minimal elements are the one-element subsets of G.

We shall need a few technicalities about posets (the reader may also consult [16]). If P and Q are posets, a map from P to Q is *order-preserving* if $f(x) \leq f(y)$ in Q whenever $x \leq y$ in P. Let Q^P denote the poset of all order-preserving maps from P to Q ordered pointwise, *i.e.* $f \leq g$ if $f(x) \leq g(x)$ for all $x \in P$. For our purposes, define a connected poset Q to be *dismantlable* if Q^P is connected for every poset P. A poset P with at least 2 elements is *ramified* if for all $x < y$ in P there exist a and b in P such that $a < y$ but $a \not\leq x$ and $b > x$ but $b \not\geq y$. It is easy to verify the following: P is ramified if and only if the identity map id: $P \to P$ is comparable only to itself in the poset P^P. More generally, P is ramified iff the projections are isolated in P^{P^s} for all s. It is easy to see that, if G is ramified, connected and non-bipartite then the poset $\mathcal{P}_G$ is ramified and connected.

The *product* of two posets P and Q is defined in the obvious way: $(a, b) \leq (c, d)$ in $P \times Q$ iff $a \leq b$ and $c \leq d$. An *s-ary operation* on a poset P is an order-preserving map $f \colon P^s \to P$. Just as in the case of graphs, we say that a poset is *projective* if the only idempotent operations on P are projections.

We now state our first result.

Theorem 2.1 *Let G be a ramified, connected, non-bipartite graph. Then the following statements are equivalent:*

1. *G is strongly projective.*
2. *G is projective and $I_2(G)$ consists of isolated vertices.*
3. *$\mathcal{P}_G$ is a projective poset.*

The rest of this section will be devoted to the proof of this result. Let G and H be two connected, non-bipartite ramified graphs. For convenience let $P = \mathcal{P}_G$ and let $Q = \mathcal{P}_H$. Let f and g be neighbours in H^G. Then we define a map $\hat{f} \colon P \to Q$ as follows:

$$\hat{f}(X) = \bigcap_{a \in N_X} N_{f(a)}.$$

We claim that this is a well-defined order-preserving map from P to Q. Indeed, it is easy to see that $\hat{f}(X)$ contains $g(X)$ so it is not empty. Suppose that $X \subseteq Y$ in P. If $u \in \hat{f}(X)$ then u is adjacent to $f(a)$ for every $a \in N_X$. But clearly $N_Y \subseteq N_X$ so $u \in \hat{f}(Y)$ and $\hat{f}$ is order-preserving.

It is a simple exercise to see that for any graphs G and H as above, the posets $\mathcal{P}_{G \times H}$ and $\mathcal{P}_G \times \mathcal{P}_H$ are naturally isomorphic, using the fact that $N_{A \times B} = N_A \times N_B$. If $f \in E_s(G)$ has a neighbour g then using this isomorphism we may express $\hat{f}$ as an s-ary operation on P_G:

$$\hat{f}(X_1, \ldots, X_s) = \bigcap_{a_i \in N_{X_i}} N_{f(a_1, \ldots, a_s)}$$

for all $X_i \in P_G$.

Lemma 2.2 *Let G be a ramified, connected, non-bipartite graph and let $f \in E_s(G)$ have at least one neighbour. Then*

1. *The operation $\hat{f}$ is well-defined and order-preserving.*
2. *If f is idempotent then $\hat{f}$ is idempotent.*
3. *If $\hat{f}$ is a projection then f is a projection.*
4. *If f satisfies $f(x_1,\ldots,x_s)\in\{x_1,\ldots,x_s\}$ for all x_i then f is a projection.*

Proof Statements (1), (2) and (3) are proved in [14] (Lemma 3.2) in the case where f is a graph homomorphism, but the proof is easily modified to accommodate the weaker hypothesis. So now we proceed to show (4). Let $I_s{}^*$ denote the subgraph of I_s consisting of those functions f that satisfy $f(x_1,\ldots,x_s)\in\{x_1,\ldots,x_s\}$ for all x_i.

Claim 0

(i) *If g is adjacent to a projection in $E_s(G)$ then g is equal to that projection.*
(ii) *If $f\in I_s$ and g is adjacent to f then $g\in I_s$.*

Proof of Claim 0 (i) Fix $(x_1,\ldots,x_s)$ and let $(y_1,\ldots,y_s)$ be a neighbour. Then $g(x_1,\ldots,x_s)$ is adjacent to $\pi_i(y_1,\ldots,y_s)=y_i$. Since this holds for any neighbour y_i of x_i and G is ramified we conclude that $g(x_1,\ldots,x_s)=x_i$. (ii) Restrict both f and g to the diagonal. You obtain elements f' and g' of $E_1(G)$ which are adjacent. But f' is the identity so by (i) with $s=1$ we get that g' is the identity and hence $g\in I_s$.

Claim 1 *If f and g are adjacent and $f\in I_s{}^*$ then $g\in I_s{}^*$.*

Proof of Claim 1 If $g(x_1,\ldots,x_s)$ is not equal to any of $x_1,x_2,\ldots,x_s$ then because G is ramified there exist neighbours u_i of x_i such that $g(x_1,\ldots,x_s)$ is not adjacent to u_i for all i. But $g(x_1,\ldots,x_s)$ is adjacent to $f(u_1,\ldots,u_s)$ which is equal to some u_i, a contradiction.

Claim 2 *Let $f\in I_s{}^*$ which has a neighbour g. For every $X_i\in\mathcal{P}$, $\hat{f}(X_1,\ldots,X_s)$ contains some X_i.*

Proof of Claim 2 Suppose that $\hat{f}(X_1,\ldots,X_s)$ contains none of $X_2,\ldots,X_s$. For each $j\geq 2$ choose $x_j\in X_j$ which is not in $\hat{f}(X_1,\ldots,X_s)$. But for any $a\in X_1$ we have that $\hat{f}(X_1,\ldots,X_s)$ contains $g(a,x_2,\ldots,x_s)$ which must be equal to a by Claim 1. Hence $\hat{f}(X_1,\ldots,X_s)$ contains X_1.

Claim 3 *Let P be any connected ramified poset, let $s\geq 3$ and let F be an s-ary isotone operation on P that satisfies the identity $F(x,x,x_3,\ldots,x_s)\approx x_s$. Then F is the s-th projection.*

Proof of Claim 3 Define an isotone map Φ from P^{s-1} to P^P by $\Phi(a_1,\ldots,a_{s-1})(t)=F(a_1,\ldots,a_{s-1},t)$. Notice that $\Phi(a_1,\ldots,a_{s-1})$ is the identity map whenever $a_1=a_2$. But P^{s-1} is connected and hence so is its image under Φ. Since P is ramified the identity is alone in its component; hence Φ is a constant map and we are done.

Claim 4 *Let P be any connected ramified poset, let $s\geq 2$ and let F be an s-ary isotone operation on P that satisfies the following condition $(*)$: for every $x_i\in P$ there exists some i such that $F(x_1,\ldots,x_s)$ is comparable to x_i. Then F is a projection.*

Proof of Claim 4 If $s = 2$ then this is a result of Hazan [8]. Now suppose that the result holds for s and let F be $(s+1)$-ary with property $(*)$. For $1 \leq i \leq s$ define operations

$$g_i(x_1, \ldots, x_s) = F(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_s).$$

Clearly each g_i is isotone and idempotent and satisfies (up to a permutation of variables) property $(*)$; hence each g_i is a projection. By Claim 3 either F is a projection and we are done, or else we must have that $g_i = \pi_i$ for all i. But then $s = 3$ otherwise we get, for a and b distinct in P that

$$a = g_1(a, b, \ldots, b) = F(a, a, b, \ldots, b) = g_s(a, a, b, \ldots, b) = b.$$

Now consider the operation $h(x, y) = F(x, y, x)$. Applying once again Claim 3 we get that either F is a projection, or h is the first projection. In this case we find that F satisfies the identities

$$F(x, x, y) \approx F(x, y, x) \approx F(y, x, x) \approx x$$

i.e. F is a so-called *majority operation* on P. But it is known that [16] if a poset admits a majority operation then it must be dismantlable (and hence cannot be ramified). It follows that F is a projection.

Now we can prove (4): let $f \in I_s{}^*$ have a neighbour. By Claim 2 $\hat{f}$ satisfies condition $(*)$ of Claim 4, and hence is a projection. Then by (2) we conclude that f is itself a projection. ■

Let G be a ramified, connected, non-bipartite graph and let $P = \mathcal{P}_G$. If Φ is an s-ary idempotent operation on P, then we may find an operation Ψ on P with the following properties: (i) $\Psi \leq \Phi$, (ii) Ψ is idempotent and (iii) for every $g_i \in G$ there exists $a \in G$ such that

$$\Psi(\{g_1\}, \{g_2\}, \ldots, \{g_s\}) = \{a\}.$$

Indeed, let Ψ be any minimal map below Φ. Since P is ramified, any map comparable to Φ must itself be idempotent since $x = \Phi(x, \ldots, x) \geq \Psi(x, \ldots, x)$ will hold for all $x \in P$. So it suffices to prove that, if Q is any poset, any minimal element Ψ of P^Q maps minimal elements of Q to minimal elements of P. Suppose that Ψ is minimal but that $\Psi(M_1, \ldots, M_s)$ is not, where $M_1, \ldots, M_s$ are minimal in Q. Let $U < \Psi(M_1, \ldots, M_s)$ be minimal in P and define a new map

$$\Psi'(X_1, \ldots, X_s) = \begin{cases} U & \text{if } X_i = M_i \text{ for all } i, \\ \Psi(X_1, \ldots, X_s) & \text{otherwise.} \end{cases}$$

It is obvious that $\Psi' < \Psi$ and easy to verify that it is order-preserving. Hence our claim is proved.

We now proceed to prove Theorem 2.1:

Proof of Theorem 2.1 (1) $\Rightarrow$ (2): Let $f \in I_2(G)$. If f has no neighbour we are done. Otherwise, let g be a neighbour of f. Since G is strongly projective if follows that f is a projection, and by Claim 0 we get that $f = g$. Hence f is an isolated loop and we are done.

(2) $\Rightarrow$ (3): To prove that $P = \mathcal{P}_G$ is projective, it suffices to prove that the only idempotent operations of arity 2 on P are projections [10]. Since G is ramified, connected and non-bipartite it follows that P is ramified and connected. Let Φ be a binary idempotent operation on P. Since projections are isolated in P^{P^s}, we may suppose that Φ satisfies (iii) by the above discussion. So we may define an element $f \in I_2(G)$ by $f(x, y) = z$ where $\Phi(\{x\}, \{y\}) = \{z\}$, for all $x, y \in G$.

Claim *The element f has a neighbour in I_2.*

Proof of Claim By Claim 0 it suffices to show that f has a neighbour. It is clear that to show this, it suffices to find, for every pair (a, b), some $c \in G$ such that $f(N_{(a,b)}) \subseteq N_c$ (indeed, just define the neighbour g by $g(a, b) = c$). Now since Φ is isotone, for every pair (a, b) we get that

$$f(N_{(a,b)}) \subseteq \Phi(N_a, N_b)$$

and this set is a non-empty intersection of neighbourhoods, so is contained in some N_c.

Since I_2 consists of isolated vertices, we must conclude that f is a loop, *i. e.* a graph homomorphism. But G is projective, so f is a projection, say $f = \pi_1$. This means that for any X and Y in $\mathcal{P}$ we have that

$$\Phi(X, Y) \geq \Phi(\{x\}, \{y\}) = \{x\}$$

for all $x \in X$ and all $y \in Y$, *i.e.* Φ satisfies $(*)$ of Claim 4. Thus Φ is a projection and we are done.

(3) $\Rightarrow$ (1): Let $f \in I_s(G)$ have a neighbour. We construct $\hat{f}$ which by Lemma 2.2(2) is an idempotent operation on P. Since P is projective $\hat{f}$ is a projection and by Lemma 2.2(3) this implies that f is a projection. Since G is ramified it follows that f is adjacent only to itself, so the elements of $I_s(G)$ are all isolated vertices. ■

It would be interesting to see examples of projective graphs which are not strongly projective, if any. The following result is proved in [11], and shows that all known families of projective graphs contain only strongly projective graphs:

Theorem 2.3 *If G is of one of the following types then it is strongly projective:*

1. *Complete graphs on more than 2 vertices.*
2. *Odd cycles.*
3. *Square-free, connected, ramified, non-bipartite graphs.*
4. *Connected, ramified bipartite graphs with a universal vertex added.*
5. *Directly indecomposable primitive graphs.*
6. *Truncated simplices (see [13]).*
7. *Non-bipartite, distance-transitive graphs of diameter at least 3.* ■

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3 The Exponential Graphs $E_s(K)$

The associated poset P of the complete graph K_n on n vertices, $n \geq 3$, is a truncated boolean lattice on n atoms, *i.e.* the collection of all non-empty proper subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. Truncated boolean lattices were shown to be projective by E. Corominas [1], and it follows that the complete graphs are strongly projective. The same can be said of odd cycles: the associated poset is called a crown, and these are known to be projective [1]. Hence odd cycles are strongly projective. We shall now describe in more detail the graphs $E_s(K)$ where K is either an odd cycle or a (non-bipartite) complete graph. We start with the case of complete graphs:

Lemma 3.1 *Let f and g be adjacent in $E_s(K_n)$ where K_n is the complete graph on n vertices, $n \geq 3$. If f is onto, then there exist permutations $\sigma_1, \ldots, \sigma_s$ such that*

$$f\big(\sigma_1(x), \ldots, \sigma_s(x)\big) = x$$

for all $x \in K_n$.

Proof For convenience let $K = K_n$. Since f is onto there exist $\{x^i_j\}$, $1 \leq i \leq n$, $1 \leq j \leq s$ such that $f(x^i_1, x^i_2, \ldots, x^i_s) = i$ for all $1 \leq i \leq n$. For $1 \leq j \leq s$ let $X_j = \{x^1_j, x^2_j, \ldots, x^n_j\}$. We claim that if $X_j = \{1, 2, \ldots, n\}$ for all j then we are done. Indeed, if this is the case just define $\sigma_j(i) = x^i_j$ for all i and j. So we choose the elements $\{x^i_j\}$ in such a way that the number r of sets X_j equal to $K = \{1, 2, \ldots, n\}$ is as large as possible, and such that the union of the remaining X_j's has maximum cardinality. Notice that $r \geq 1$ for otherwise we could find elements $u_1, \ldots, u_s$ such that $u_j \notin X_j$ for all j, and thus $g(u_1, \ldots, u_n)$ would be adjacent to $f(x^i_1, x^i_2, \ldots, x^i_s) = i$ for all $1 \leq i \leq n$, a contradiction. By rearranging the variables of f we may assume that $X_1, \ldots, X_r$ are equal to K. Suppose for a contradiction that $r < s$.

We define two functions F and G from $E_1(K)^s$ to $E_1(K)$ as follows: if $h_j \in E_1(K)$, $j = 1, \ldots, s$ put

$$F(h_1, \ldots, h_s)(i) = f\big(h_1(i), \ldots, h_s(i)\big)$$

and

$$G(h_1, \ldots, h_s)(i) = g\big(h_1(i), \ldots, h_s(i)\big)$$

for all i. It is easy to see that F and G are adjacent (*i.e.* if h_j is adjacent to h'_j for all j then $F(h_1, \ldots, h_s)$ is adjacent to $G(h'_1, \ldots, h'_s)$ in $E_1(K)$.) It is clear that a map is in the component of the constants of $E_1(K)$ if and only if it is not onto, and otherwise it is an isolated loop.

Now define elements h_j of $E_1(K)$ by $h_j(i) = x^i_j$ for all i, j. Then $h_1, \ldots, h_r$ are isolated loops and $h_{r+1}, \ldots, h_s$ are in the component of the constants. Since $F(h_1, \ldots, h_s)$ is the identity (which is isolated), and F is adjacent to G, and the component of the constants of $E_1(K)$ is non-bipartite, it follows that $F(h_1, \ldots, h_r, k_{r+1}, \ldots, k_s)$ is the identity function for any constant maps $k_{r+1}, \ldots, k_s$. In other words, we have that

$$f(x^i_1, \ldots, x^i_r, w_{r+1}, \ldots, w_s) = i$$

for all $i \in K$ and any $w_j \in K$. Define y_j^i to be equal to x_j^i when $1 \le j \le r$ and equal to i otherwise. Then $f(y_1^i, y_2^i, \dots, y_s^i) = i$ for all $1 \le i \le n$, and $Y_i = \{y_j^1, y_j^2, \dots, y_j^n\} = K$ for all i, contradicting our hypothesis that $r < s$. ■

Theorem 3.2 *Let $s \ge 1$ and let K_n be the complete graph on n vertices, $n \ge 3$. Let $f \in E_s(K_n)$. Then f is in the component of the constants if and only if it is not onto. If it is onto then it is an isolated vertex of $E_s(K_n)$; furthermore, it is a loop if and only if there exists an i and an automorphism σ of K_n such that $f(x_1, \dots, x_s) = \sigma(x_i)$ for all $x_1, \dots, x_s$.*

Proof Let $K = K_n$. Let $f \in E_s(K)$. If f is not onto then it is adjacent to a constant function. Now assume that f is onto and that it has a neighbour g. By the last lemma there exist permutations σ_j such that $f\big(\sigma_1(x), \dots, \sigma_s(x)\big) = x$ for all $x \in K$. Define new elements f' and g' of $E_s(K)$ by $f'(x_1, \dots, x_s) = f\big(\sigma_1(x_1), \dots, \sigma_s(x_s)\big)$ and $g'(x_1, \dots, x_s) = g\big(\sigma_1(x_1), \dots, \sigma_s(x_s)\big)$ for all x_j. It is easy to see that f' and g' are adjacent, and that f' is idempotent. Since K is strongly projective it follows that $f' = g'$ is a projection, say the i-th projection. So we have that

$$f\big(\sigma_1(x_1), \dots, \sigma_s(x_s)\big) = x_i = g\big(\sigma_1(x_1), \dots, \sigma_s(x_s)\big)$$

for all x_j. It follows easily that

$$f(x_1, \dots, x_s) = \sigma(x_i) = g(x_1, \dots, x_s)$$

where $\sigma = \sigma_i^{-1}$. ■

We note in passing the following interesting corollary:

Corollary 3.3 *Let P be a truncated boolean lattice on n atoms, $n \ge 3$. Let F be an order-preserving s-ary operation on P. If F is not onto, then F is in the component of the constants of P^{P^s}. Otherwise F is essentially unary.*

Proof Let F be any s-ary operation on P. Let G be a minimal element of P^{P^s} such that $G \le F$. Then G maps minimal elements of P^s to minimal elements of P, and we may define an element g of $E_s(K_n)$ by $g(a_1, \dots, a_s) = b$ where $G(\{a_1\}, \{a_2\}, \dots, \{a_s\}) = \{b\}$. As in the claim in Theorem 2.1 we see that g has a neighbour in $E_s(K_n)$. If this map g is not onto, it means that the image of G misses some minimal element of P, and it is known that the remaining poset which contains the image of G is dismantlable; it follows that G, and hence F is in the component of the constants ([12], Lemme 1 and Théorème 1). So we may assume that g is onto. By Theorem 3.2 we have that g is a projection followed by an automorphism, say $g(x_1, \dots, x_s) = \sigma(x_l)$ for some $1 \le l \le s$.

It is an easy exercise to verify that since G is minimal we have that

$$G(X_1, \dots, X_s) = \bigvee_{\substack{Y_i \le X_i \\ Y_i \text{ minimal}}} G(Y_1, \dots, Y_s)$$

and thus

$$G(X_1, \dots, X_s) = \bigcup_{a \in X_l} \{\sigma(a)\}.$$

Define an automorphism S of P by $S(X) = \{\sigma(x) : x \in X\}$; then clearly $G(X_1, \dots, X_s) = S(X_l)$, and $S^{-1}G(X_1, \dots, X_s) = X_l$ is the l-th projection. Since $S^{-1}F \geq S^{-1}G$ it follows that $S^{-1}F$ must be the same projection and thus $F = G$ and we are done. ■

We shall now prove that the graphs $E_s(C_{2k+1})$ have essentially the same structure, where C_{2k+1} is the cycle on $2k+1$ vertices, $k \geq 1$. However we shall achieve this using the poset correspondence described earlier. A poset is called *Słupecki* if it satisfies the following condition: if F is an s-ary operation on P, either it is not surjective or it depends on only one variable, *i.e.* there exists some i and a unary operation σ on P such that $F(x_1, \dots, x_s) = \sigma(x_i)$ for all $x_1, \dots, x_s$. Notice that this is a stronger condition than projectivity.

Lemma 3.4 *Let P be a crown. Then*

1. *P is Słupecki,*
2. *if* $F\colon P^s \to P$ *is not in the component of the constants of* P^{P^s} *then it is onto.*

Proof (1) Crowns are shown to be Słupecki in [3] (for a more general result, including the case of truncated boolean lattices, see [2]).

(2) Let F be an s-ary operation on P which is not onto. Then the image of F is contained in a fence (a crown with one element removed), which is a dismantlable poset. By Lemme 1 of [12] it follows that F must be in the component of the constants. ■

Lemma 3.5 *Let G and H be ramified, connected non-bipartite graphs and let* $P = \mathcal{P}_G$ *and let* $Q = \mathcal{P}_H$*. Let* $f \in H^G$ *have a neighbour.*

1. *If* $\hat{f}$ *is in the component of the constants in* Q^P *then* f *is in the component of the constants in* H^G.
2. *If* $G = H^s$ *and* $\hat{f}$ *depends only on one variable then* f *depends on only one variable.*

Proof We shall divide the proof into several claims. Recall from earlier that, if $F\colon P \to Q$ is minimal in Q^P then it will map minimal elements of P to minimal elements of Q. The dual statement holds if F is maximal. So if F is minimal, define a map $F_0\colon P \to Q$ by $F_0(p) = q$ if $F(\{p\}) = \{q\}$. If F is maximal then define a map $F^0\colon P \to Q$ by $F_0(p) = q$ if $F(N_p) = N_q$.

Claim 0 *Let* $f \in Q^P$ *have a neighbour. If* $\hat{f} \leq F$ *and F is maximal then* $f = F^0$.

Proof of Claim 0 By definition of $\hat{f}$ we have for all $x \in P$ that

$$N_{f(x)} = \hat{f}(N_x) \subseteq F(N_x) = N_{F^0(x)}$$

and hence $f(x) = F^0(x)$.

Claim 1 *Let* $f \in Q^P$ *have a neighbour. If* $\hat{f} \geq F$ *and F is minimal then f is adjacent to* F_0.

Proof of Claim 1 Let a and b be adjacent vertices in G. Put $c = F_0(a)$, *i.e.* $F(\{a\}) = \{c\}$. We must show that c and $f(b)$ are adjacent in H. We have

$$\{c\} = F(\{a\}) \subseteq \hat{f}(\{a\}) = \bigcap_{x \in N_a} N_{f(x)}$$

so that $c \in N_{f(b)}$, *i.e.* c is adjacent to $f(b)$.

Claim 2 *If $F \leq G$ in Q^P, where F is minimal and G is maximal, then F_0 is adjacent to G^0.*

Proof of Claim 2 Let a and b be adjacent vertices in G. Put $c = F_0(a)$, *i.e.* $F(\{a\}) = \{c\}$ and let $d = G^0(b)$ *i.e.* $G(N_b) = N_d$. We must show that c and d are adjacent. Indeed, we have that $\{a\} \subseteq N_b$ so

$$\{c\} = F(\{a\}) \subseteq F(N_b) \subseteq G(N_b) = N_d$$

and we are done.

Now we prove (1) of the Lemma: suppose that there is a path from $\hat{f}$ to a constant map. Clearly we may assume that every map in this path (except $\hat{f}$) is either minimal or maximal. By considering F^0 or F_0 as the case may be, for every map in this path, we build a path in H^G from f to a constant map, by the claims above. Hence f is in the component of the constants.

To prove (2): suppose without loss of generality that $\hat{f}$ depends only on its first variable, so that

$$\hat{f}(X_1, \ldots, X_s) = \sigma(X_1)$$

for some map σ. Then we have by definition of $\hat{f}$ that

$$\sigma(N_{a_1}) = \hat{f}(N_{a_1}, \ldots, N_{a_s}) = N_{f(a_1,\ldots,a_s)}$$

for all a_i in G. It follows that f cannot depend on any variable but the first, since for any b_i in G we get

$$N_{f(a_1,a_2,\ldots,a_s)} = \sigma(N_{a_1}) = N_{f(a_1,b_2,\ldots,b_s)}$$

and since H is ramified this proves that $f(a_1, a_2, \ldots, a_s) = f(a_1, b_2, \ldots, b_s)$. ■

We can now describe the structure of the graph $E_s(C_{2k+1})$ for $k \geq 1$ and $s \geq 1$.

Theorem 3.6 *Let $s \geq 1$ and let C_{2k+1} be the cycle on $2k+1$ vertices, $k \geq 1$. Let $f \in E_s(C_{2k+1})$. Then f is in the component of the constants if and only if it is not onto. If it is onto then it is an isolated vertex of $E_s(C_{2k+1})$; furthermore, it is a loop if and only if there exist an i and an automorphism σ of C_{2k+1} such that $f(x_1, \ldots, x_s) = \sigma(x_i)$ for all $x_1, \ldots, x_s$.*

Proof For convenience let $K = C_{2k+1}$. Recall that since K is an odd cycle then the associated poset P is a crown. Let f and g be adjacent in $E_s(K)$. If $\hat{f}$ is not onto, then by Lemma 3.4 it is in the component of the constants and by Lemma 3.5 f and g are in the component of the constants of $E_s(K)$. So we may now assume that $\hat{f}$ is onto, and it is easy to see that this implies that f in onto. By Lemma 3.4 the poset P is Słupecki so it follows that $\hat{f}$ depends on one variable only. Hence by the last lemma f is unary and onto, *i.e.* there exist an index i and a permutation σ such that $f(x_1, \ldots, x_s) = \sigma(x_i)$. By the same argument we have that $g(x_1, \ldots, x_s) = \tau(x_j)$ for some permutation τ. It follows easily that $i = j$ and that σ and τ are adjacent in $E_1(K)$; since K is ramified it means that $\sigma = \tau$ and thus $f = g$ is a homomorphism. Hence if f is not in the component of the constants, then it is onto, and either it has no neighbours or it is a projection followed by an automorphism of K. ■

In [6] Greenwell and Lovász show that, if G is not n-colourable then the graph $G \times K_n$ is uniquely n-colourable. Here is an extension of this result to higher powers and an analog for odd cycles.

Corollary 3.7 *Let K be any odd cycle (or K_n, $n \geq 3$). Let G be a connected graph which does not admit a homomorphism into K. Then for every $s \geq 1$ the graph $G \times K^s$ admits exactly s homomorphisms into K (up to automorphisms of K).*

Proof Let $\Phi\colon G \times K^s \to K$ be any homomorphism. Then Φ induces a homomorphism $\phi\colon G \to K^{K^s}$ where $\phi(g)(x_1, \ldots, x_s) = \Phi(g, x_1, \ldots, x_s)$. We know that the component of the constants admits a homomorphism into K because K is a vertex-transitive projective core [15]. Since G admits no homomorphism into K it follows from the last result that the image of G under ϕ must be a loop, since it is connected. Hence it is, up to an automorphism of K, one of the s projections. ■

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Department of Mathematics
Champlain Regional College
900 Riverside St-Lambert, Quebec
J4P 3P2

Department of Mathematics
and Statistics
Concordia University
1455 de Maisonneuve West
Montréal, Quebec
H3G 1M8
e-mail: larose@mathstat.concordia.ca