J. Austral. Math. Soc. (Series A) **55** (1993), 403–410

ON THE CHROMATIC UNIQUENESS OF CERTAIN TREES OF POLYGONS

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(Received 18 January; revised 27 September 1991)

Communicated by Louis Caccetta

Abstract

We establish a characterization of certain trees of polygons similar to that of n-gon-trees given by Chao and Li.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 05 C 75; secondary 05 C 15.

1. Introduction

The graphs which we consider here are finite, undirected, simple, and loopless. For a graph G, let $P(G; \lambda)$ denote its chromatic polynomial. Two graphs X and Y are said to be *chromatically equivalent* if $P(X; \lambda) = P(Y; \lambda)$. If H_1 and H_2 are graphs, we shall say that a graph H is *of type* (H_1, H_2) if it can be formed from the disjoint union $H_1 \cup H_2$ by identifying an edge of H_1 with an edge of H_2.

Let m and $n_1, n_2, \ldots, n_m$ be integers satisfying $m \geq 1$ and $n_m > \cdots > n_2 > n_1 \geq 3$. Let $\mathcal{G}$ be the class of graphs defined recursively by the rules: the n_i-cycle C_{n_i} is in $\mathcal{G}$ for each i ($1 \leq i \leq m$), and if H_1 and H_2 belong to $\mathcal{G}$ then so does any graph of type (H_1, H_2). The graphs in $\mathcal{G}$ are called $(n_1, n_2, \ldots, n_m)$*-gon-trees*. They are evidently 2-connected planar graphs. If $m = 1$ and $n_1 = n$ then the graphs in $\mathcal{G}$ are called n*-gon-trees*.

Let H be a graph of type (H_1, H_2). Then every chordless cycle (induced cycle) in H is a chordless cycle in H_1 or a chordless cycle in H_2. Also, the chromatic polynomial $P(H;\lambda)$ of H satisfies

$$P(H;\lambda) = \frac{P(H_1;\lambda)P(H_2;\lambda)}{\lambda(\lambda-1)}. \tag{1}$$

It is well known that the chromatic polynomial of an n-cycle is given by $P(C_n;\lambda) = \lambda(\lambda-1)Q(C_n;\lambda)$ where

$$Q(C_n;\lambda) = (-1)^n \sum_{i=0}^{n-2} (1-\lambda)^i. \tag{2}$$

We thus see that if G is an $(n_1, n_2, \dots, n_m)$-gon-tree then every chordless cycle in G has one of the lengths $n_1, \dots, n_m$. An easy inductive argument using (1) and (2) also gives the following theorem.

THEOREM 1. *If G is a $(n_1, n_2, \dots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \le i \le m)$, then*

$$P(G;\lambda) = \lambda(\lambda-1)\prod_{i=1}^{m}\Big(Q(C_{n_i};\lambda)\Big)^{k_i}, \tag{3}$$

where $Q(C_{n_i};\lambda)$ is defined by (2).

For n-gon-trees $(m = 1)$, Chao and Li [1] proved that the converse of Theorem 1 also holds. The purpose of this paper is to prove the converse of Theorem 1 for a wider class of $(n_1, n_2, \dots, n_m)$-gon-trees.

In the remainder of this section we shall state some known results that will be useful in proving our characterization theorems.

THEOREM A. (Whitney [7]). *Let G be a graph of order p and size q. Then*

$$P(G;\lambda) = \sum_{k=1}^{p}\Big(\sum_{r=0}^{q}(-1)^r N(k,r)\Big)\lambda^k,$$

where $N(k,r)$ denotes the number of spanning subgraphs of G having exactly k components and r edges.

THEOREM B. (Chao and Zhao [3]). *Let G be a connected graph of order at least 3, and with $P(G;\lambda) = (\lambda - 1)T(G;\lambda)$. Then*

(a) $T(G;1) = 0$, *if and only if G has at least one cut-vertex;*

(b) $|T(G;1)| = 1$, *if and only if G is a 2-connected graph and has no subgraph homeomorphic to the complete graph K_4 with 4 vertices;*

(c) $|T(G;1)| \geq 2$, *if and only if G is a 2-connected graph and has at least one subgraph homeomorphic to K_4.*

We also need the next result which gives explicit expressions for the first four coefficients of the chromatic polynomial of a graph.

THEOREM C. *Let G be a graph of order p and size q. If $P(G;\lambda) = \sum_{i=0}^{p} a_i \lambda^{p-i}$ is the chromatic polynomial of G, then*

(a) $a_0 = 1,\ a_1 = -q;$

(b) $a_2 = \binom{q}{2} - N_{Q_3};$

(c) $a_3 = -\binom{q}{3} + (q-2)N_{Q_3} + N_{Q_4} - 2N_{K_4};$

where N_{K_i} and N_{Q_i} denote the number of complete graphs K_i and chordless cycles C_i respectively in G, and $\binom{q}{i}$ is the binomial coefficient.

We end this section with the following result which is a corollary of a more general result established in [5].

LEMMA D. *Let G be a bipartite graph which has no subgraph $K(2,3)$. If a graph H is chromatically equivalent with G, then $N_{Q_6}(H) = N_{Q_6}(G)$.*

2. Tree of Polygons

In this section we shall prove the converse of Theorem 1 for a wider class of $(n_1, n_2, \ldots, n_m)$-gon-trees. We first establish the following key result by using Whitney's theorem and a technique introduced by Farrell [4].

THEOREM 2. *Let G be a graph of order p and size q with girth $k \geq 3$. If $P(G;\lambda) = \sum_{i=0}^{p} a_i \lambda^{p-i}$ is the chromatic polynomial of G, then*

(a) $a_i = (-1)^i \binom{q}{i}$, *for* $i = 0, 1, \ldots, k-2;$

(b) $a_{k+i-1} = (-1)^{k+i-1}\left\{\binom{q}{k+i-1} - \sum_{j=0}^{i} \binom{q-k-j+1}{i-j} N_{Q_{k+j}}\right\}$, *for* $i = 0, 1, \ldots, \lfloor (k-3)/2 \rfloor$.

PROOF. (a) Let i be any integer such that $0 \leq i \leq k-2$. By Theorem A, the coefficient of λ^{p-i} is

$$a_i = \sum_{r=0}^{q} (-1)^r N(p-i, r),$$

where $N(p-i, r)$ is the number of spanning subgraphs of G with $p-i$ components and r edges. But in a spanning subgraph with $p-i$ components, no component can have more than $i+1$ vertices. Since $i \leq k-2$ and G has girth k, all such spanning subgraphs are forests with exactly i edges, and every set of i edges gives such a forest. So the contribution of these graphs to a_i is therefore

$$(-1)^i \binom{q}{i}.$$

Thus we have the required result.

(b) Let i be any integer satisfying $0 \leq i \leq \lfloor (k-3)/2 \rfloor$. By Theorem A,

$$a_{k+i-1} = \sum_{r=0}^{q} (-1)^r N(p-k-i+1, r),$$

where $N(p-k-i+1, r)$ is the number of spanning subgraphs of G with $p-k-i+1$ components and r edges. No component of such a spanning subgraph can have more than $k+i$ vertices. Since the girth of G is k and $i < \lfloor (k-1)/2 \rfloor$, these spanning subgraphs of G are forests with exactly $k+i-1$ edges or unicyclic graphs of girth $\geq k$ with $k+i$ edges. Thus we can categorize them as follows:

(i) $S_j = \{$ Unicyclic graphs with one chordless $(k+j)$-cycle plus $i-j$ edges $\}$, for $j = 0, 1, \ldots, i$.

(ii) $S_{i+1} = \{$ Forests with $k+i-1$ edges $\}$.

We shall now calculate the contributions of all the graphs in S_j $(0 \leq j \leq i+1)$ to the coefficient a_{k+i-1}.

Let j be any integer such that $0 \leq j \leq i$. All the graphs in S_j have one chordless $(k+j)$-cycle plus $i-j$ edges. Since G has girth k and $i < \lfloor (k-1)/2 \rfloor$, there is no other such graph which is not in S_j. Therefore the contribution of these graphs to a_{k+i-1} is

$$\xi_j = (-1)^{k+i} \binom{q-k-j}{i-j} N_{Q_{k+j}}.$$

All the graphs in S_{i+1} contain $k+i-1$ edges. Since G has girth k and $i < \lfloor (k-1)/2 \rfloor$, the only other graphs with $k+i-1$ edges which are not in S_{i+1}

are unicyclic graphs with one chordless $(k+j)$-cycle plus $i-j-1$ edges, for $j=0,1,\ldots,i-1$. Hence the contribution of all the graphs in S_{i+1} to a_{k+i-1} is

$$\xi_{i+1} = (-1)^{k+i-1}\left\{\binom{q}{k+i-1} - \sum_{j=0}^{i-1}\binom{q-k-j}{i-j-1}N_{Q_{k+j}}\right\}.$$

By adding all the contributions ξ_j $(0 \le j \le i+1)$ to a_{k+i-1}, and using the binomial identity $\binom{r}{s}+\binom{r}{s-1}=\binom{r+1}{s}$, we get the required result.

COROLLARY 3. *If two graphs G and H are chromatically-equivalent, then they have the same girth. Furthermore, if the girth of G is $k \ge 3$, then H and G have the same number of chordless i-cycles for $3 \le i \le k + \lfloor (k-3)/2 \rfloor$.*

Our characterization theorems depend on the following lemma.

LEMMA 4. *Let G be a connected graph of order p, size q, and with chromatic polynomial given by* (3). *Then*

(a) $p = \sum_{i=1}^{m}(n_i-2)k_i\ +2$ *and* $q=\sum_{i=1}^{m}(n_i-1)k_i\ +1$;

(b) *G is a 2-connected planar graph with $\sum_{i=1}^{m} k_i$ interior regions.*

PROOF. (a) It is easy to see that $\left(Q(C_{n_i};\lambda)\right)^{k_i}$ has degree $(n_i-2)k_i$ and that the coefficient of $\lambda^{(n_i-2)k_i-1}$ is $-(n_i-1)k_i$. Thus

$$P(G;\lambda) = \lambda(\lambda-1)\times\prod_{i=1}^{m}\left(Q(C_{n_i};\lambda)\right)^{k_i}$$

has degree $h=\sum_{i=1}^{m}(n_i-2)k_i+2$ and its coefficient of λ^{h-1} is $-\left(\sum_{i=1}^{m}(n_i-1)k_i\ +1\right)$. Hence we conclude that (a) holds.

(b) Let $P(G;\lambda)$ be written as $P(G;\lambda)=(\lambda-1)T(G;\lambda)$, that is,

$$T(G;\lambda) = \lambda\prod_{i=1}^{m}\left(Q(C_{n_i};\lambda)\right)^{k_i}.$$

Then $|T(G;1)| = |1\prod_{i=1}^{m}\left(Q(C_{n_i};1)\right)^{k_i}| = 1$, since $|Q(C_{n_i};1)|=1$ for $1 \le i \le m$. By Theorem B(b), G is a 2-connected graph and has no subgraph homeomorphic to K_4, and hence G is a planar graph (see [1, Lemma 6]).

The well-known theorem of Euler states that the p vertices, q edges, and f regions of a planar graph satisfying $p+f=q+2$. Thus for the graph G, we have $\sum_{i=1}^{m}(n_i-2)k_i\ +2+f=\sum_{i=1}^{m}(n_i-1)k_i\ +1+2$, and $f=\sum_{i=1}^{m}k_i\ +1$, that is, G has $\sum_{i=1}^{m}k_i$ interior regions.

https://doi.org/10.1017/S1446788700034121 Published online by Cambridge University Press

We are now ready to state and prove our main result.

THEOREM 5. *Let $m \geq 2$ and let n_i be integers satisfying $3 \leq n_1 < n_2 < \cdots < n_m \leq n_1 + \lfloor (n_1 - 3)/2 \rfloor$. Then a graph G is a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m)$ if and only if*

$$P(G;\lambda) = \lambda(\lambda - 1)\prod_{i=1}^{m}\left(Q(C_{n_i};\lambda)\right)^{k_i},$$

where $Q(C_{n_i};\lambda)$ is defined as in (2).

PROOF. The necessity follows from Theorem 1. To prove the sufficiency we proceed as follows. We first claim that G has k_i chordless n_i-cycles for $1 \leq i \leq m$. Since G is chromatically-equivalent to a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m)$, the claim follows from the corollary to Theorem 2.

By Lemma 4 and the above claim, $P(G;\lambda) = \lambda(\lambda - 1)\prod_{i=1}^{m}\left(Q(C_{n_i};\lambda)\right)^{k_i}$ implies that G is a 2-connected planar graph with $\sum_{i=1}^{m}(n_i - 2)k_i + 2$ vertices, $\sum_{i=1}^{m}(n_i - 1)k_i + 1$ edges, $k = \sum_{i=1}^{m} k_i$ interior regions, and k_i chordless n_i-cycles $(1 \leq i \leq m)$. We now proceed by induction on k to show that G is indeed a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m)$. For $k = 1$, that is, $k_i = 0$ for all i, $1 \leq i \leq m$ except $k_r = 1$ for some r, $1 \leq r \leq m$, it is clear that G is a (n_r)-gon-tree with one chordless n_r-cycle. Assume that the conclusion holds for $k - 1 = \sum_{i=1, i\neq t}^{m} k_i + (k_t - 1)$, for some t, $1 \leq t \leq m$ $(k \geq 3)$. That is, if G^* is a 2-connected planar graph with $\sum_{i=1}^{m}(n_i - 2)k_i + 2 - (n_t - 2)$ vertices, $\sum_{i=1}^{m}(n_i - 1)k_i + 1 - (n_t - 1)$ edges, $k - 1$ interior regions, and k_i chordless n_i-cycles $(1 \leq i \leq m, i \neq t)$ and $k_t - 1$ chordless n_t-cycles, then G^* is a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m, i \neq t)$ and $k_t - 1$ chordless n_t-cycles.

We now consider k. Suppose that G is not a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m)$. If G contains h $(h \geq 2)$ chordless cycles C_{r_j} $(1 \leq j \leq h)$ which share exactly one common edge e, then $G - e$ is a 2-connected planar graph with $k - 1$ interior regions, $\sum_{i=1}^{m}(n_i - 2)k_i + 2$ vertices, $\sum_{i=1}^{m}(n_i - 1)k_i$ edges, and k_i chordless n_i-cycles $(1 \leq i \leq m, i \neq r_j$ for $1 \leq j \leq h)$, $k_{r_j} - 1$ chordless n_{r_j}-cycles, and one chordless $n_{r_1+r_j-2}$-cycle for each $2 \leq j \leq h$. But $G - e$ is not a $(n_1, n_2, \ldots, n_m, n_{r_1+r_2-2}, n_{r_1+r_3-2}, \ldots, n_{r_1+r_h-2})$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m$, $i \neq r_j$ for $1 \leq j \leq h)$, $k_{r_j} - 1$ chordless n_{r_j}-cycles, and one chordless $n_{r_1+r_j-2}$-cycle for $2 \leq j \leq h$, contradicting our induction hypothesis. So any two chordless cycles of G

have either no or at least two edges in common. But then, since G is a 2-connected planar graph with $\sum_{i=1}^{m} k_i$ interior regions and k_i chordless n_i-cycles $(1 \leq i \leq m)$, it is not difficult to see that G has no more than $\sum_{i=1}^{m}(n_i - 2)k_i + 2$ edges, which is strictly less than $\sum_{i=1}^{m}(n_i - 1)k_i + 1$, again a contradiction.

3. Special Cases

We shall in this section characterize two other trees of polygons which are not established in Theorem 5. Firstly we need the following two results.

LEMMA 6. *If a graph G is chromatically-equivalent to a $(3,4)$-gon-tree with k_1 triangles and k_2 chordless 4-cycles, then G has the same number of triangles and chordless 4-cycles.*

PROOF. This follows from Theorem C.

LEMMA 7. *If a graph G is chromatically-equivalent to a $(4,6)$-gon-tree with k_1 chordless 4-cycles and k_2 chordless 6-cycles, then G has the same number of chordless 4 and 6-cycles.*

PROOF. This follows from the corollary to Theorem 2 and Lemma D.

We are now ready to give the characterizations of (3,4)-gon-trees and (4,6)-gon-trees which can easily be proved using Lemmas 6 and 7, and similar arguments to those used in establishing Theorem 5.

THEOREM 8. *A graph G is a $(3,4)$-gon-tree with k_1 triangles and k_2 chordless 4-cycles, where $k_1 + k_2 \geq 1$, if and only if*

$$P(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^{k_1}(\lambda^2-3\lambda+3)^{k_2}.$$

THEOREM 9. *A graph G is a $(4,6)$-gon-tree with k_1 chordless 4-cycles and k_2 chordless 6-cycles, where $k_1 + k_2 \geq 1$, if and only if*

$$P(G;\lambda) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{k_1}(\lambda^4-5\lambda^3+10\lambda^2-10\lambda+5)^{k_2}.$$

In concluding this paper, we note that the converse of Theorem 1 holds if the following conjecture is true.

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CONJECTURE. *If a graph G is chromatically equivalent to a $(n_1, n_2, \ldots, n_m)$-gon-tree with k_i chordless n_i-cycles $(1 \leq i \leq m)$, then G has k_i chordless n_i-cycles for $i = 1, 2, \ldots, m$.*

Acknowledgement

The author is grateful to the referee for his comments and suggestions.

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