A NOTE ON DIVERGENT SERIES

G. M. PETERSEN

1. Methods of summation of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise, §1 and §2 are not connected.

In this section, we shall study some properties of the method (B^h) where we say the series $\sum u_\nu$ is summable (B^h) when

$$B_n^h = \sum_{\nu=0}^{n} u_\nu \cos \frac{\pi}{2}\left(\frac{\nu}{n+h}\right) \rightarrow s, \qquad n \rightarrow \infty.$$

The method (B^h) has been studied in special cases airsing from different values of h by Rogosinski [**11**; **12**], Bernstein [**2**], and more recently by Karamata [**3**; **4**].

Two methods (A) and (B) are equivalent, $(A) \equiv (B)$, when all series summable (A) are summable (B) to the same sum and inversely; on the other hand, the method (B) is more powerful than the method (A), $(A) \subset (B)$, when all series summable (A) are summable (B) to the same sum.

In the paper of Karamata [**3**] a theorem states that $(B^h) \equiv (C_1)$ if $0 < h < 1$, $|h - \frac{1}{2}| > .19$ where (C_1) denotes the Cesàro method. Lorentz [**6**] pointed out that his proof contains gaps, but can be made valid if $.69 < h < 1$. If $h = \frac{1}{2}$, then (B^h) is more powerful than (C_1) [**4**]. Here we shall prove Karamata's theorem for $\frac{1}{2} < h$; our proof will be simpler than that given in [**3**].

The partial sums B_n^h of the (B^h) method may be expressed, after easy calculations, in terms of σ_ν the partial sums of the (C_1) method. The transformation from σ_ν to B_n^h is regular and hence any (C_1)-summable series is summable (B^h) for all h, i.e., $(C_1) \subset (B^h)$.

Our main theorem is

THEOREM 1.1 $\qquad (B^h) \equiv (C_1)$ *for* $h > \frac{1}{2}$.

In our proof we shall need a theorem of Agnew [**1**], which was rediscovered by Rado [**10**]. In the formulation of Rado, if the method (T):

$$t_m = \sum_{0}^{\infty} c_{m\nu} s_\nu,$$

is regular and if $c_{m\nu} = 0$, $\nu > m$,

$$\sum_{\nu=0}^{m-1} |c_{m\nu}| < \theta |c_{mm}|, \qquad \theta < 1$$

for almost all m, then (T) is equivalent to convergence.

Received April 24, 1951. This paper is part of a thesis prepared under the supervision of Dr. G. Lorentz and submitted for a Ph.D. degree at the University of Toronto.

We shall introduce the method (K^c) where

$$K_n^c = \frac{1}{n+1}\sum_{\nu=0}^{n} [(1-c)s_{\nu-1} + cs_\nu] \to s, \qquad n \to \infty.$$

If we express the partial sums of (K^c) in terms of σ_ν, the partial sums of (C_1),

$$K_n^c = (1-c)\frac{n}{n+1}\sigma_{n-1} + c\sigma_{n1}$$

it follows at once from the theorem of Agnew that $(K^c) \equiv (C_1)$ if $c > \frac{1}{2}$. We shall now prove that $(B^h) \equiv (K^h)$ for $h > \frac{1}{2}$ and the proof of Theorem 1.1 will then follow.

THEOREM 1.2 $(B^h) \equiv (K^h)$ *if* $h > \frac{1}{2}$.

Proof. We have

$$B_n^h = \sum_{\nu=0}^{n} u_\nu \cos\frac{\pi}{2}\frac{\nu}{n+h},$$

$$K_n^c = \frac{1}{n+1}\sum_{\nu=0}^{n} [(1-c)s_{\nu-1} + cs_\nu].$$

Solving for s_ν, we have

$$cs_\nu = (\nu+1)K_\nu^c - \frac{1}{c}\nu K_{\nu-1}^c + \frac{1-c}{c^2}(\nu-1)K_{\nu-2}^c + \ldots + (-1)^\nu\left(\frac{1-c}{c}\right)^{\nu-1}\frac{1}{c}K_0^c$$

or

$$s_\nu = \left(1-\frac{1}{c}\right)^\nu \sum_{\mu=0}^{\nu}{}' \frac{c^{\mu-1}}{(1-c)^{\mu+1}}(-1)^\mu(\mu+1)K_\mu^c,$$

where the prime means that the term with $\mu = \nu$ has the additional factor $(1-c)$. Substituting in B_n^h, we obtain, with $\theta = \pi/2(n+h)$ and $a = 1 - 1/c$,

$$B_n^h = \sum_{\mu=0}^{n} \cos\frac{\pi}{2}\frac{\mu}{n+h}(s_\mu - s_{\mu-1}) = \sum_{\mu=0}^{n-1} s_\mu\{\cos\mu\theta - \cos(\mu+1)\theta\} + s_n\cos n\theta$$

$$= -\frac{1}{c^2}\sum_{\mu=0}^{n-1}\{\cos\mu\theta - \cos(\mu+1)\theta\}a^\mu\sum_{\nu=0}^{\mu}{}' a^{-(\nu+1)}(\nu+1)K_\nu^c$$

$$- \frac{1}{c^2}\cos n\theta\, a^n\sum_{\nu=0}^{n}{}' a^{-(\nu+1)}(\nu+1)K_\nu^c,$$

and changing the order of summation in the first sum,

$$B_n^h = -\frac{1}{c^2}\sum_{\nu=0}^{n-1} a^{-(\nu+1)}(\nu+1)K_\nu^c\left[\sum_{\mu=\nu}^{n-1} a^\mu\{\cos\mu\theta - \cos(\mu+1)\theta\} + a^n\cos n\theta\right]$$

$$+ \frac{1}{c}(n+1)K_n^c\cos n\theta.$$

https://doi.org/10.4153/CJM-1952-040-7 Published online by Cambridge University Press

Here, the expression in square brackets is

$$(1 - c)a^{\nu}\{\cos \nu\theta - \cos (\nu + 1)\theta\} + a^{\nu+1}\{\cos (\nu + 1)\theta - \cos (\nu + 2)\theta\}$$
$$+ \ldots + a^{n-1}\{\cos (n - 1)\theta - \cos n\theta\} + a^{n} \cos n\theta$$
$$= - ca^{\nu}\{\cos \nu\theta - \cos (\nu + 1)\theta\} + a^{\nu} \cos \nu\theta - \frac{1}{c} a^{\nu} \cos (\nu + 1)\theta$$
$$+ \ldots - \frac{1}{c} a^{n-1} \cos n\theta.$$

Using the formula

$$a^{\nu+1} \cos (\nu + 1)\theta + \ldots + a^{n} \cos n\theta$$
$$= \Re(a^{\nu+1}e^{i(\nu+1)\theta} + \ldots + a^{n}e^{in\theta}) = \Re \frac{a^{\nu+1}e^{i(\nu+1)\theta} - a^{n+i}e^{i(n+1)\theta}}{1 - ae^{i\theta}}$$
$$= \frac{a^{\nu+1} \cos (\nu + 1)\theta - a^{\nu+2} \cos \nu\theta - a^{n+1} \cos (n + 1)\theta + a^{n+2} \cos n\theta}{1 - 2a \cos \theta + a^{2}},$$

we obtain, for the above expression,

$$\{- ca^{\nu+1} \cos \nu\theta + ca^{\nu} \cos (\nu + 1)\theta\}$$
$$- \frac{a^{\nu}}{c} \frac{\cos (\nu + 1)\theta - a \cos \nu\theta}{1 - 2a \cos \theta + a^{2}} - \frac{a^{n}}{c} \frac{a \cos n\theta - \cos (n + 1)\theta}{1 - 2a \cos \theta + a^{2}};$$

so we have

$$B_{n}^{h} = - \frac{1}{c^{2}} \sum_{\nu=0}^{n-1} a^{-(\nu+1)} (\nu + 1)K_{\nu}^{c}\left\{ - \frac{a^{n}}{c} \frac{a \cos n\theta - \cos (n + 1)\theta}{1 - 2a \cos \theta + a^{2}}\right.$$
$$\left. + \frac{4ca^{\nu+1} \sin^{2}\tfrac{1}{2}\theta[\cos (\nu + 1)\theta - a \cos \nu\theta]}{1 - 2a \cos \theta + a^{2}}\right\} + \frac{n + 1}{c} K_{n}^{c} \cos n\theta.$$

We shall now estimate the sum of the absolute values of the coefficients of K_{ν}^{c} and show that the sum of the first $n - 1$ of them is less than that of K_{n}^{c}. Under these conditions we apply the theorem of Agnew.

Here, for the coefficient of K_{n},

$$\lim_{n\to\infty} \frac{1}{c}(n + 1) \cos n\theta = \frac{\pi}{2} \cdot \frac{h}{c}.$$

We break the sum of the absolute values of the other coefficients into two parts, the second part of which is

$$D_{2} = \sum_{\nu=0}^{n-1} \left| - \frac{1}{c^{2}} \frac{1}{a^{\nu+1}} (\nu + 1) \frac{4ca^{\nu+1} \sin^{2}\tfrac{1}{2}\theta\{\cos (\nu + 1)\theta - a \cos \nu\theta\}}{1 - 2a \cos \theta + a^{2}}\right|$$
$$= \left|\frac{1}{c} \frac{4 \sin^{2}\tfrac{1}{2}\theta}{1 - 2a \cos \theta + a^{2}}\right| \sum_{\nu=0}^{n-1} (\nu + 1)| \cos (\nu + 1)\theta - a \cos \nu\theta|.$$

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Since we have

$$|\cos(\nu+1)\theta - a\cos\nu\theta| \leqslant |\cos(\nu+1)\theta - \cos\nu\theta| + \left|\frac{1}{c}\right| |\cos\nu\theta|,$$

$$\sum_{\nu=0}^{n-1} (\nu+1)|\cos(\nu+1)\theta - a\cos\nu\theta| \leqslant A(n+1) + \left|\frac{1}{c}\right|\frac{n(n+1)}{2},$$

$$\sin^2\frac{\theta}{2} = \frac{1}{n^2}\left\{\frac{\pi^2}{16} + o(1)\right\},$$

and $1 - 2a\cos\theta + a^2 = (1-a)^2 + o(1) = c^{-2} + o(1)$, therefore

$$D_2 \leqslant \frac{\pi^2}{8} + o(1).$$

Now we shall turn our attention to the first part of the sum

$$D_1 \leqslant \left|\frac{1}{c^3}\right| \left|\frac{a^n\cos n\theta - a^{n-1}\cos(n+1)\theta}{1 - 2a\cos\theta + a^2}\right| \sum_{\nu=0}^{n-1} (\nu+1)|a^{-\nu}|.$$

As before $1 - 2a\cos\theta + a^2 = c^{-2} + o(1/n)$, and therefore

$$D_1 \leqslant \left|\frac{a^{n-1}}{c}\right| |a\cos n\theta - \cos(n+1)\theta|[1 + o(1/n)]\frac{-(n+1)|a|^{-n} + n|a|^{-n-1} + 1}{(1 - 1/|a|)^2}$$

$$\leqslant \left|\frac{1}{c}\right| |a\cos n\theta - \cos(n+1)\theta|\frac{n(1-|a|) + o(1)}{(1-|a|)^2}.$$

Here we have assumed that $a^n = o(1)$, that is, $|a| < 1$. We shall proceed to give an estimate of $|a\cos n\theta - \cos(n+1)\theta|$. We have

$$|a\cos n\theta - \cos(n+1)\theta| = \left|a\sin\frac{\pi}{2}\frac{h}{n+h} - \sin\frac{\pi}{2}\frac{h-1}{n+h}\right|$$

$$= \left|\left(a - \frac{h-1}{h}\right)\sin\frac{\pi}{2}\frac{h}{n+h} + \frac{h-1}{h}\sin\frac{\pi}{2}\frac{h}{n+h} - \sin\frac{\pi}{2}\frac{h-1}{n+h}\right|,$$

$$\frac{h-1}{h}\sin\frac{\pi}{2}\frac{h}{n+h} - \sin\frac{\pi}{2}\frac{h-1}{n+h} = o\left(\frac{1}{n^2}\right),$$

and so

$$|a\cos n\theta - \cos(n+1)\theta| \leqslant \left|\left(a - \frac{h-1}{h}\right)\sin\frac{\pi}{2}\frac{h}{n+h} + o\left(\frac{1}{n^2}\right)\right|$$

$$\leqslant \frac{\pi}{2n}\left|1 - \frac{h}{c}\right| + o\left(\frac{1}{n}\right).$$

Substituting the above estimate for $|a\cos n\theta - \cos(n+1)\theta|$ in our expression for D, we obtain

$$D_1 \leqslant \frac{\pi}{2c}\left|1 - \frac{h}{c}\right|\frac{1}{1-|a|}.$$

To satisfy the theorem of Agnew, the absolute value of the coefficient of

K_n^c must be greater than the sum of the absolute values of the other coefficients. In our case, this is true if

$$\frac{\pi}{2}\left|\frac{h}{c}\right| > \frac{\pi^2}{8} + \frac{\pi}{2c}\left|1 - \frac{h}{c}\right|\frac{1}{1 - |a|}. \tag{1.1}$$

If $c = h$, this reduces to

$$\frac{\pi}{2} \geqslant \frac{\pi^2}{8},$$

so that $(B^h) \equiv (K^h)$ whenever $|a| < 1$ or $h > \frac{1}{2}$. This completes our proof.

In the general case, (1.1) does not hold for $h < \frac{1}{2}$ while $c > \frac{1}{2}$; so that (B^h), $h < \frac{1}{2}$ can not be shown equivalent to some (K^c), $c > \frac{1}{2}$ by these means. Examples can be constructed to show (B^h) is not equivalent to (C_1) for $h < 0$. The most interesting question remaining open is whether or not (B^h) is equivalent to (C_1) in the interval $0 < h < \frac{1}{2}$.

2. Some special Nörlund methods of summation. In this section we wish to consider some elementary Nörlund methods, namely, methods of the form

$$\sigma_n = \alpha_0\, s_{n-p} + \ldots + \alpha_p\, s_n, \quad \alpha_0 + \alpha_1 + \ldots + \alpha_p = 1. \tag{A}$$

It was first proposed as a problem by Pólya, [**9**] that the method defined by

$$t_n = (1 - c)\, s_{n-1} + c\, s_n \to s, \qquad n \to \infty \quad (c \neq 0),$$

is equivalent to convergence if and only if $c > \frac{1}{2}$. Kubota [**5**] proved more generally that a transformation of type (A) is equivalent to convergence if and only if all of the roots of the "associated" equation

$$\alpha_0 + \alpha_1 z + \ldots + \alpha_p z^p = 0, \tag{2.1}$$

lie inside the unit circle.

Other results concerning the method (A) have been obtained by Lorentz [**7**] and by Silverman and Szász [**13**]. We shall show that any bounded sequence summable (A) is convergent if and only if none of the roots of (2.1) lie on the unit circle. This will easily follow from Theorem 2.2 (the main theorem of this section), where we describe all (A)-summable sequences under the above hypotheses on the roots of (2.1).

We shall first prove

LEMMA 1. *If*

$$s_n = \sum_{\nu=1}^{n} \nu^k a^\nu \qquad (a \neq 1)$$

then S_n may be written in the form $P_k(n)\, a^n + c$ where $P_k(n)$ is of the form

$$c_k n^k + c_{k-1} n^{k-1} + \ldots + c_0,$$

and $c_k, c_{k-1}, \ldots, c_0, c$ are constants depending only on a.

Let us write

$$\Delta^k f(\nu) = \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{k-i} f(\nu+i).$$

Applying Abel's formula, we have

$$\sum_{\nu=1}^{n} \nu^k a^\nu = \frac{1}{1-a} - \frac{a}{1-a}\sum_{\nu=1}^{n} a^\nu \Delta \nu^k - \frac{a^n}{1-a}(n+1)^k.$$

Repeating this process $k+1$ times,

$$\sum_{\nu=1}^{n} \nu^k a^\nu = \left\{\frac{1}{1-a} + \frac{a}{(1-a)^2}\Delta 1^k + \ldots + (-1)^k \frac{a^k}{(1-a)^k}\Delta^k (n+1)^k\right\}$$
$$+ (-1)^k \frac{a^k}{(1-a)^k}\sum_{\nu=1}^{n} a^\nu \Delta^{k+1}\nu^k - \frac{a^n}{1-a}\left\{(n+1)^k - \frac{a}{1-a}\Delta(n+1)^k + \ldots + (-1)^k \frac{a^k}{(1-a)^k}\Delta^k (n+1)^k\right\}$$

and therefore, since $\Delta^{k+1}\nu^k = 0$,

$$\sum_{\nu=1}^{n} \nu^k a^\nu = P_k(n) a^n + c$$

as required.

In preparation for Theorem 2.2 we shall first consider the special case of (A) when $p = 1$. In this case, we may write (A) in the form

$$(A_a) \qquad \sigma_n = \frac{1}{1-a}\{-a s_{n-1} + s_n\} \qquad (a \neq 1).$$

THEOREM 2.1 *Suppose* $|a| > 1$.

(i) *If* $\sigma_n \to \sigma$, *then* $s_n = ca^n + \sigma_n'$, *where* $\sigma_n' \to \sigma$ *and* c *is a certain constant.*

(ii) *If* $\sigma_n = P_\mu(n) a^n$ *where* $P_\mu(n) = c_\mu n^\mu + c_{\mu-1} n^{\mu-1} + \ldots + c_0$, *then*

$$s_n = (c_{\mu+1}' n^{\mu+1} + c_\mu' n^\mu + \ldots + c_0') a^n = P_{\mu+1}'(n) a^n$$

and conversely.

(iii) *If* $\sigma_n = P_\nu(n) b^n$; $P_\nu(n) = c^\nu n^\nu + c_{\nu-1} n^{\nu-1} + \ldots + c_0$ *and* $b \neq a$ *then*

$$s_n = (c_\nu' n^\nu + \ldots + c_0')\, b^n + ca^n$$

and conversely.

Proof. We have for (i),

$$(2.2) \qquad \frac{1}{1-a} s_n = a^n \left[\sigma_0 + \frac{\sigma_1}{a} + \ldots + \frac{\sigma_n}{a^n}\right].$$

If we define $t_n = \sigma_n / a^n$, then part (i) of our theorem means that, for $|a| > 1$, $a^n t_n \to \sigma$ implies $t_0 + t_1 + \ldots + t_n = c + \sigma_n'/a^n$, where $(1-a)\sigma_n' \to \sigma$.

The series $\sum t_\nu$ is absolutely convergent. Set

$$c = \sum_{\nu=0}^{\infty} t_\nu,$$

then

$$t_0 + t_1 + \ldots + t_n = c - (t_{n+1} + t_{n+2} + \ldots)$$
$$= c - \frac{1}{a^n}\left[\frac{1}{a} a^{n+1} t_{n+1} + \frac{1}{a^2} a^{n+2} t_{n+2} + \ldots\right].$$

Since $a^n t_n \to \sigma$,

$$\sum_{k=1}^{\infty} \frac{1}{a^k} a^{n+k} t_{n+k}$$

converges toward

$$\sigma\left(\frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n} + \ldots\right) = \sigma \frac{1}{a} \frac{1}{1 - 1/a} = \frac{\sigma}{a-1}.$$

Therefore

$$t_0 + \ldots + t_n = c + \frac{1}{a^n}\sigma'_n, \qquad \sigma'_n \to \frac{\sigma}{a-1},$$

which proves (i).

(ii) Substituting the value of σ_n in (2.2) we have

$$\frac{1}{1-a} s_n = a^n\{c_\mu(1^\mu + 2^\mu + \ldots + n^\mu) + c_{\mu-1}(1^{\mu-1} + 2^{\mu-1} + \ldots + n^{\mu-1})$$
$$+ \ldots + c_0(1 + 1 + \ldots + 1)\}.$$

Using the well-known fact that $1^\mu + 2^\mu + \ldots + n^\mu$ is a polynomial in n of degree $\mu + 1$ with constant coefficients, we obtain

$$s_n = \{c'_{\mu+1} n^{\mu+1} + c'_\mu n^\mu + \ldots + c'_0\} a^n = P'_{\mu+1}(n) a^n.$$

The converse becomes evident on substituting the expression for s_n in (A_a).

(iii) Again we substitute the value for σ_n in (2.2),

$$\frac{1}{1-a} s_n = a^n\left\{c_\nu\left(1^\nu \frac{b}{a} + 2^\nu \frac{b^2}{a^2} + \ldots + n^\nu \frac{b^n}{a^n}\right)\right.$$
$$\left. + c_{\nu-1}\left(1^{\nu-1}\frac{b}{a} + 2^{\nu-1}\frac{b^2}{a^2} + \ldots + n^{\nu-1}\frac{b^n}{a^n}\right) + \ldots + c_0\left(1 + \frac{b}{a} + \ldots + \frac{b^n}{a^n}\right)\right\}.$$

By Lemma 1,

$$1^\mu \frac{b}{a} + 2^\mu \frac{b^2}{a^2} + \ldots + n^\mu \frac{b^n}{a^n} = \frac{b^n}{a^n} P'_\mu(n) + c, \tag{2.3}$$

where $P'_\mu(n) = (C'_\mu n^\mu + \ldots + c'_0)$.
Using (2.3) we have

$$\frac{1}{1-a} s_n = a^n\left\{c_\nu P'_\nu(n)\frac{b^n}{a^n} + c_{\nu,1} + \ldots + c_1 P'_1(n)\frac{b^n}{a^n} + c_{11} + c_0 \frac{b^n}{a^n} + c_{01}\right\}$$

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which may be written $P_\nu^{\prime\prime}(n)b^n + ca^n$. Again the converse is evident if we substitute s_n in (A_a).

We now return to the method (A).

THEOREM 2.2 *If $a_1, a_2, \ldots, a_l$, $|a_i| \neq 1$ are all of the different roots of equation (2.1), $a_1, a_2, \ldots, a_k$ are those roots with $|a_i| > 1$ and $m_1, m_2, \ldots, m_k$ their multiplicities, then the general form of a sequence summable (A) is*

$$s_n = P_1(n)a_1^n + P_2(n)a_2^n + \ldots + P_k(n)a_k^n + s_n', \tag{2.4}$$

where

$$P_i(n) = c_{i,m_i-1}n^{m_i-1} + c_{i,m_i-2}n^{m_i-2} + \ldots + c_{i,0}$$

are polynomials in n of degree m_{i-1} with arbitrary constant coefficients and s_n' is an arbitrary convergent sequence.

Proof. (A) may be considered as an iteration of p transformations

$$\sigma_n^{j-1} = \frac{1}{1-b_j}\{-b_j\sigma_{n-1}^j + \sigma_n^j\}, \qquad j = 1, 2, \ldots, p, \ \sigma_n^0 = \sigma_n, \ \sigma_n^p = s_n.$$

The b_j are first those a_i with $|a_i| < 1$ and then the $a_1, a_2, \ldots, a_k$ all taken with their multiplicities. There will be m_1 transformations with $b_j = a_1$ and so on. The first $m = m_{k+1} + \ldots + m_l$ transformations are all equivalent to convergence by the theorem of Kubota, and therefore the convergence of σ_n will be equivalent to the convergence of σ_n^m.

Hence, in proving our theorem we may assume that all $|a_i| > 1$. For the first transformation σ_n^0 is a convergent sequence, and therefore

$$\sigma_n' = ca_1^n + \bar{\sigma}_n', \qquad \bar{\sigma}_n' \to \sigma$$

by Theorem 2.1 (i). If now we repeat this argument p times and use Theorem 2.1 (i), (ii), and (iii), we shall obtain as the final result expression (2.4) for s_n. Conversely, substituting s_n in the expression for (A), we see s_n is (A) summable. This proves the theorem.

We shall next prove a lemma that will enable us to prove a further theorem on methods of type (A).

LEMMA 2. *If $|a_i| > 1$ for $a_i \neq a_j$, $i \neq j$ $(i = 1, 2, \ldots, k)$, and*

$$P_{\mu_i}(n) = c_{i\mu_i}n^{\mu_i} + \ldots + c_{i0}, \qquad P_{\mu_i}(n) \neq 0 \textit{ for all } i,$$

then the expression

$$y_n = P_{\mu_1}(n)a_1^n + P_{\mu_2}(n)a_2^n + \ldots + P_{\mu_k}(n)a_k^n, \tag{2.5}$$

is unbounded for $n \to \infty$.

We shall show that if $y_n = O(1)$ we have a contradiction. Assume the first l of the a_i are all those having that modulus which is the maximum modulus of the a_i that is

$$|a_1| = |a_2| = \ldots = |a_l|,$$

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and

$$a_i = a_1 e^{i\alpha_i}, \quad \alpha_1 = 0, \quad \alpha_i \neq \alpha_j, \quad i \neq j \qquad (i = 1, 2, \ldots, l).$$

Then (2.5) becomes

$$y_n = a_1^n[P_{\mu_1}(n) + e^{in\alpha_2}P_{\mu_2}(n) + \ldots + e^{in\alpha_l}P_{\mu_l}(n)] + a_1^n o(1).$$

We have, since $y_n = O(1)$,

$$(2.6) \qquad P_{\mu_1}(n) + e^{in\alpha_2}P_{\mu_2}(n) + \ldots + e^{in\alpha_l}P_{\mu_l}(n) = o(1).$$

We write C'_{μ_i} for the coefficients in

$$P_{\mu_i}(n) \text{ of } n^{\mu}, \quad \mu = \max_{i=1,2,\ldots,l} \mu_i;$$

at least one of these is different from zero. We consider the l equations

$$c'_{\mu_1} + e^{i(n+j)\alpha_2}c'_{\mu_2} + \ldots + e^{i(n+j)\alpha_l}c'_{\mu_l} = \epsilon_{n+j}, \qquad j = 0, 1, 2, \ldots l-1.$$

Dividing by n^{μ} in (2.6) we have $\epsilon_{n+j} \to 0$, $n \to \infty$. The α_j are all different and different from zero. Solving these l equations

$$(2.7) \qquad c'_{\mu_j}e^{i\alpha_j n} = \begin{vmatrix} 1 & 1 & \ldots 1 & \epsilon_n & \ldots 1 \\ 1 & e^{i\alpha_2} & \ldots e^{i\alpha_{j-1}} & \epsilon_{n+1} & \ldots e^{i\alpha_l} \\ & & \ldots & & \ldots \\ 1 & e^{i\alpha_2(l-1)} & \ldots e^{i\alpha_{j-1}(l-1)} & \epsilon_{n+l-1} & \ldots e^{i\alpha_l(l-1)} \end{vmatrix} V^{-1},$$

where V is a Vandermonde determinant different from zero and independent of n.

Hence, expanding the numerator in (2.7) by the jth column, we see that $c'_{\mu_j} e^{i\alpha_j n} \to 0$ as $n \to \infty$ or $c'_{\mu_j} = 0$ for all j. This contradiction proves our lemma.

THEOREM 2.3 *Any bounded sequence summable* (A) *is convergent if and only if none of the roots of* (2.1) *lie on the unit circle.*

Proof. The sufficiency of these conditions follows from Theorem 2.2 and Lemma 2.

If we assume that the associated equation (2.1) has a root a with $|a| = 1$, then breaking (A) into an iteration of transformations as in Theorem 2.2, we can consider

$$(2.8) \qquad \sigma_n^{p-1} = \frac{1}{1-a}[-as_{n-1} + s_n],$$

last in our sequence of transformations. It is then evident that the method (2.8) and therefore (A) sums the sequence $e^{in\psi}$ where $a = e^{i\psi}$. This contradiction proves our conditions necessary.

The existence of a bounded divergent (A)-summable sequence implies [8] that sequences of his type form a non-separable subset of the space m of bounded sequences. It follows that in the case of a root $|a| = 1$ a simple enumeration of all (A)-summable sequences comparable with (2.4) is impossible.

I would express my gratitude to Dr. G. G. Lorentz for his suggestions and assistance during the preparation of this paper.

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University of Arizona