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ENDS FOR MONOIDS AND SEMIGROUPS

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Abstract

We give a graph-theoretic definition for the number of ends of Cayley digraphs for finitely generated semigroups and monoids. For semigroups and monoids, left Cayley digraphs can be very different from right Cayley digraphs. In either case, the number of ends for the Cayley digraph does not depend upon which finite set of generators is used for the semigroup or monoid. For natural numbers m and n, we exhibit finitely generated monoids for which the left Cayley digraphs have m ends while the right Cayley digraphs have n ends. For direct products and for many other semidirect products of a pair of finitely generated infinite monoids, the right Cayley digraph of the semidirect product has only one end. A finitely generated subsemigroup of a free semigroup has either one end or else has infinitely many ends.

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1. Ends for graphs and digraphs

A *digraph* is a quadruple $\Gamma = (V_\Gamma, E_\Gamma, \iota_\Gamma, \tau_\Gamma)$ where $V = V_\Gamma$ is a set of vertices, $E = E_\Gamma$ is a set of edges and $\iota_\Gamma, \tau_\Gamma : E \to V$ are functions designating initial and terminal vertices for each edge. A *graph* is a quintuple $\Gamma = (V_\Gamma, E_\Gamma, \iota_\Gamma, \tau_\Gamma, \mathrm{inv}_\Gamma)$ where inv_Γ is a function $E \to E$ and we require axiomatically, for each $e \in E$, that $e \neq \mathrm{inv}_\Gamma(e)$, that $\mathrm{inv}_\Gamma(\mathrm{inv}_\Gamma(e)) = e$, that $\iota_\Gamma(\mathrm{inv}_\Gamma(e)) = \tau_\Gamma(e)$ and that $\tau_\Gamma(\mathrm{inv}_\Gamma(e)) = \iota_\Gamma(e)$. We omit the subscripts on the functions ι_Γ and τ_Γ whenever context makes these unnecessary and we routinely write e^{-1} for $\mathrm{inv}_\Gamma(e)$.

When we imagine some geometric realization of a graph, we regard e and e^{-1} as occupying the same arc of points, but traversing these arcs in opposite directions. In a geometric realization for a digraph, each edge has an associated direction for traversal. We allow loops and multiple edges in graphs and digraphs.

A graph $(V_\Upsilon, E_\Upsilon, \iota_\Upsilon, \tau_\Upsilon, \mathrm{inv}_\Upsilon)$ is a *subgraph* of $(V_\Gamma, E_\Gamma, \iota_\Gamma, \tau_\Gamma, \mathrm{inv}_\Gamma)$ if V_Υ and E_Υ are subsets of V_Γ and E_Γ, respectively, and the functions $\iota_\Upsilon, \tau_\Upsilon$ and inv_Υ are the respective restrictions of the functions $\iota_\Gamma, \tau_\Gamma$ and inv_Γ to E_Υ. Subdigraphs of digraphs are defined analogously, but we will routinely use the word subgraph for both subgraphs and subdigraphs.

If Γ is a digraph (V, E, ι, τ) or a graph $(V, E, \iota, \tau, {}^{-1})$ and V_0 is a subset of V, then the *full* subgraph of Γ on V_0 is the digraph $\Gamma_0 = (V_0, E_0, \iota, \tau)$ or the graph $\Gamma_0 = (V_0, E_0, \iota, \tau, {}^{-1})$ where E_0 consists of all of the edges in E having both vertices in V_0. If $\mathfrak{F}$ is a subset of V, we write $\Gamma - \mathfrak{F}$ for the full subgraph of Γ on $V - \mathfrak{F}$.

Suppose $\Gamma = (V, E, \iota, \tau)$ is a digraph and that E^{-1} is a set in one-to-one correspondence with E, but disjoint from E. Then there is a graph $\overleftrightarrow{\Gamma} = (V, E \cup E^{-1}, \iota, \tau, {}^{-1})$ where we extend ι, τ and ${}^{-1}$ to $E \cup E^{-1}$ by $\iota(e^{-1}) = \tau(e)$, $\tau(e^{-1}) = \iota(e)$ and $(e^{-1})^{-1} = e$. If $\xi : \Gamma_1 \to \Gamma_2$ is a morphism of digraphs, we obtain a graph morphism $\overleftrightarrow{\xi} : \overleftrightarrow{\Gamma}_1 \to \overleftrightarrow{\Gamma}_2$ by setting $\overleftrightarrow{\xi}(e^{-1}) = (\xi(e))^{-1}$. It is then easily seen that $\leftrightarrow$ is a functor from the category of digraphs to the category of graphs.

Conversely, we obtain a digraph (V, E, ι, τ) from a graph $(V, E, \iota, \tau, {}^{-1})$ by simply discarding the function ${}^{-1} : E \to E$. This extends to a functor U from the category of graphs to the category of digraphs. The functor $\leftrightarrow$ is a left adjoint to this functor U.

Suppose that $\Gamma = (V, E, \iota, \tau)$ is a digraph. A *positive walk* ω of length n in Γ is a sequence $\omega = (e_1, \ldots, e_n)$ of edges of Γ with $\tau(e_i) = \iota(e_{i+1})$ for $1 \le i < n$. Similarly, if $\Gamma = (V, E, \iota, \tau, {}^{-1})$ is a graph, then a *walk* ω of length n in Γ is a sequence $\omega = (e_1, \ldots, e_n)$ of edges of Γ with $\tau(e_i) = \iota(e_{i+1})$ for $1 \le i < n$. If $\xi : \Gamma_1 \to \Gamma_2$ is a digraph morphism and $\omega = (e_1, \ldots, e_n)$ is a positive walk in Γ_1, then $\xi(\omega) = (\omega(e_1), \ldots, \omega(e_n))$ is a positive walk in Γ_2. Likewise, if $\xi : \Gamma_1 \to \Gamma_2$ is a graph morphism and ω is a walk in Γ_1, then the sequence $\xi(\omega)$ is a walk in Γ_2. It is more convenient to write just $e_1 e_2 \cdots e_n$ rather than $(e_1, e_2, \ldots, e_n)$ for a walk or positive walk ω. We define a *walk* in the digraph $\Gamma = (V, E, \iota, \tau)$ to be a walk in $\overleftrightarrow{\Gamma}$: that is, we allow edges to be traversed in either direction. The vertices of ω are the vertices $\iota(e_i)$ and $\tau(e_i)$ such that e_i is an edge of ω. The initial vertex of ω is $\iota(e_1)$, the initial vertex of e_1, and the terminal vertex of ω is $\tau(e_n)$, the terminal vertex of e_n. We allow at each vertex v of a digraph Γ or a graph Γ an empty walk of length 0 having v as both its initial and terminal vertex. A walk $\omega = e_1 e_2 \cdots e_n$ in a graph or in a digraph is a *trail* if whenever $e_i = e$ for some i, then e_j is neither e nor e^{-1} for $j \ne i$. That is, identifying e with e^{-1}, all of the edges on ω are distinct. A walk in a graph or in a digraph is a *path* if all of its vertices are distinct. An *interior* vertex on a path ω is any vertex on ω other than its initial and terminal vertices. We are largely concerned with paths rather than walks or trails. The reader should be aware that these words are not consistently defined in the literature and that the distinction between walks and paths is sometimes important in this work. In a digraph, a positive walk is a *positive trail* if all of its edges are distinct and is a *positive path* if all of its vertices are distinct.

A graph Γ is *connected* if there is a path in Γ from any vertex v_1 to any vertex v_2. We define a digraph Γ to be *connected* if $\overleftrightarrow{\Gamma}$ is connected. A *component* of a graph or of a digraph Γ is a maximal connected subgraph of Γ.

We write $|X|$ for the cardinality of a set X. If v is a vertex in a graph or in a digraph, then indegree(v), outdegree(v) and degree(v) are defined respectively

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by indegree$(v) = |\{e \mid \tau(e) = v\}|$, outdegree$(v) = |\{e \mid \iota(e) = v\}|$ and degree$(v) =$ indegree(v) + outdegree(v). A graph is locally finite if each vertex has finite degree. It is usual practice to define the number of ends of a graph only for locally finite graphs. For finitely generated semigroups and monoids we define the number of ends for Cayley graphs which may have vertices with infinite indegree.

We state several possible definitions for the number of ends of a graph and additional possible definitions for the number of ends of a digraph. For all of these definitions, Γ is a graph or digraph, $\mathfrak{F}$ a finite set of vertices of Γ and, for various subscripts x, $\mathfrak{C}_x = \mathfrak{C}_x(\Gamma - \mathfrak{F})$ is some set whose elements are infinite components, C, of $\Gamma - \mathfrak{F}$. For each subscript x, we define a number, $\mathrm{e}_x(\Gamma)$, of ends of Γ by

$$\mathrm{e}_x(\Gamma) = \max_{\mathfrak{F} \subseteq V, \mathfrak{F} \text{ finite}} |\mathfrak{C}_x(\Gamma - \mathfrak{F})|.$$

When $\mathfrak{C}_x$ and e_x are defined for graphs rather than for digraphs, we extend the definition for e_x to digraphs by $\mathrm{e}_x(\Gamma) = \mathrm{e}_x(\overleftrightarrow{\Gamma})$ for any digraph Γ.

If v_1, v_2 are vertices in the graph Γ, then the *distance* $d_\Gamma(v_1, v_2)$ between v_1 and v_2 in Γ is the length of the shortest path in Γ from v_1 to v_2. This distance is not defined if v_1 and v_2 are in different components of Γ. A path π having initial vertex v_1 and terminal vertex v_2 is a *geodesic* in Γ if the length of π is $d_\Gamma(v_1, v_2)$. In a digraph Γ, a positive path π having initial vertex v_1 and terminal vertex v_2 is a *digeodesic* in Γ if π is a positive path of minimal length in Γ from v_1 to v_2. There is a digeodesic in Γ from v_1 to v_2 if and only if there is a positive path in Γ from v_1 to v_2.

Suppose that Φ is a subgraph of a graph Γ or a digraph Γ. Then it is easy to see that a path in Φ which is a geodesic or digeodesic in Γ is also a geodesic or digeodesic in Φ. However, a geodesic or digeodesic in Φ need not be a geodesic or digeodesic in Γ.

A graph Γ *has unbounded paths (geodesics)* if for every natural number n there is a path (geodesic) of length n in Γ. A digraph Γ *has unbounded positive paths (digeodesics)* if for every natural number n there is a positive path (digeodesic) of length n in Γ. A vertex v in a graph Γ *initiates unbounded paths (geodesics)* if for every natural number n there is a path (geodesic) of length n in Γ with initial vertex v. A vertex v in a digraph Γ *initiates unbounded positive paths (digeodesics)* if for every natural number n there is a positive path (digeodesic) of length n in Γ with initial vertex v. A vertex v in a digraph Γ *terminates unbounded positive paths (digeodesics)* if for every natural number n there is a positive path (digeodesic) of length n in Γ with terminal vertex v.

To implement the definitions displayed above for $\mathrm{e}_x(\Gamma)$, we need to define various sets $\mathfrak{C}_x(\Gamma - \mathfrak{F})$ where Γ is a graph or digraph and $\mathfrak{F}$ is a finite set of vertices of Γ. For the sake of brevity and generality, we state these definitions for $\mathfrak{C}_x(\Gamma)$ in terms of Γ rather than $\Gamma - \mathfrak{F}$. For a graph Γ, we define

$$\mathfrak{C}_\infty(\Gamma) = \{C \mid C \text{ is a component of } \Gamma \text{ having infinitely many vertices}\},$$
$$\mathfrak{C}_p(\Gamma) = \{C \mid C \text{ is a component of } \Gamma \text{ having unbounded paths}\},$$

$$\mathfrak{C}_g(\Gamma) = \{C \mid C \text{ is a component of } \Gamma \text{ having unbounded geodesics}\},$$
$$\mathfrak{C}_*(\Gamma) = \{C \mid C \text{ contains a vertex which initiates unbounded paths}\},$$
$$\mathfrak{C}_\dagger(\Gamma) = \{C \mid C \text{ contains a vertex which initiates unbounded geodesics}\}.$$

Similarly, for a digraph Γ, we define

$$\mathfrak{C}_{+p}(\Gamma) = \{C \mid C \text{ is a component of } \Gamma \text{ having unbounded positive paths}\},$$
$$\mathfrak{C}_\delta(\Gamma) = \{C \mid C \text{ is a component of } \Gamma \text{ having unbounded digeodesics}\},$$
$$\mathfrak{C}_{\overrightarrow{*}}(\Gamma) = \{C \mid C \text{ contains a vertex which initiates unbounded positive paths}\},$$
$$\mathfrak{C}_{\overleftarrow{*}}(\Gamma) = \{C \mid C \text{ contains a vertex which terminates unbounded positive paths}\},$$
$$\mathfrak{C}_{\overrightarrow{\delta}}(\Gamma) = \{C \mid C \text{ contains a vertex which initiates unbounded digeodesics}\},$$
$$\mathfrak{C}_{\overleftarrow{\delta}}(\Gamma) = \{C \mid C \text{ contains a vertex which terminates unbounded digeodesics}\}.$$

LEMMA 1. *Let Γ be a connected graph.*

(1) *If Γ has unbounded paths, then every vertex in Γ initiates unbounded paths.*
(2) *If Γ has unbounded geodesics, then every vertex in Γ initiates unbounded geodesics.*

PROOF. Let $\hat{v}$ be an arbitrary vertex in Γ. Write $|\pi|$ for the length of a walk π.

(1) For each natural number n, we want to show the existence of a path π_n in Γ with $\iota(\pi_n) = \hat{v}$ and $|\pi_n| = n$. Let χ_n be a path in Γ with $|\chi_n| = 2n$. Since Γ is connected, the distance $d_\Gamma(u, v)$ is defined and finite for any two vertices $u, v \in \Gamma$. Choose v_n to be a vertex v on χ_n for which $d_\Gamma(\hat{v}, v)$ is minimized as v ranges over the vertices on χ_n. Let γ_n be a path with length $d_\Gamma(\hat{v}, v_n)$ from $\hat{v}$ to v_n. Write χ_n as $(\alpha_n)^{-1}\beta_n$ where $\iota(\alpha_n) = \iota(\beta_n) = v_n$. Since $|\chi_n| = 2n$, either $|\alpha_n| \geq n$ or $|\beta_n| \geq n$. Assume, without loss of generality, that $|\alpha_n| \geq n$, so $|\gamma_n\alpha_n| \geq n$. Observe that $\gamma_n\alpha_n$ must be a path by the minimality of $d_\Gamma(\hat{v}, v_n)$. Let π_n be the initial subpath of $\gamma_n\alpha_n$ having length n.

(2) For each natural number n, we want to show the existence of a geodesic π_n in Γ with $\iota(\pi_n) = \hat{v}$ and $|\pi_n| = n$. Let χ_n be a geodesic in Γ with $|\chi_n| = 2n$. Since Γ is connected, we can find geodesics α_n, β_n in Γ with $\iota(\alpha_n) = \iota(\beta_n) = \hat{v}$, $\tau(\alpha_n) = \iota(\chi_n)$ and $\tau(\beta_n) = \tau(\chi_n)$. Then $(\alpha_n)^{-1}\beta_n$ is a walk in Γ from $\iota(\chi_n)$ to $\tau(\chi_n)$, hence $|(\alpha_n)^{-1}\beta_n| \geq 2n$. But then either $|\alpha_n| \geq n$ or $|\beta_n| \geq n$. Without loss of generality, assume $|\alpha_n| \geq n$ and let π_n be the initial subpath of α_n having length n. □

COROLLARY 2. *Let Γ be any graph.*

(1) $\mathfrak{C}_p(\Gamma) = \mathfrak{C}_*(\Gamma)$ *and* $\mathfrak{C}_g(\Gamma) = \mathfrak{C}_\dagger(\Gamma)$.
(2) $\mathrm{e}_p(\Gamma) = \mathrm{e}_*(\Gamma)$ *and* $\mathrm{e}_g(\Gamma) = \mathrm{e}_\dagger(\Gamma)$.

PROOF. (1) It is clear that $\mathfrak{C}_*(\Gamma) \subseteq \mathfrak{C}_p(\Gamma)$ and $\mathfrak{C}_\dagger(\Gamma) \subseteq \mathfrak{C}_g(\Gamma)$. The reverse inclusions follow by applying Lemma 1 to connected components of Γ.

(2) By the first part, $\mathfrak{C}_p(\Gamma - \mathfrak{F}) = \mathfrak{C}_*(\Gamma - \mathfrak{F})$ and $\mathfrak{C}_g(\Gamma - \mathfrak{F}) = \mathfrak{C}_\dagger(\Gamma - \mathfrak{F})$ for any finite subset $\mathfrak{F}$ of vertices of Γ. □

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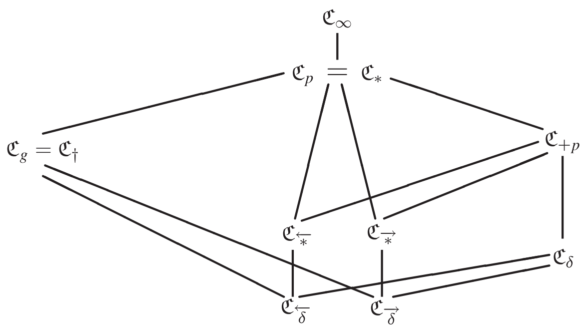

FIGURE 1. Some subset inclusions for $\mathfrak{C}_\infty$.

In the next six examples, the subscripts ∞, p, g, $*$ and $\dagger$ are graph subscripts while the subscripts $+p$, δ, $\overrightarrow{*}$, $\overleftarrow{*}$, $\overrightarrow{\delta}$ and $\overleftarrow{\delta}$ are digraph subscripts. The subscripts $\overrightarrow{*}$ and $\overrightarrow{\delta}$ are initial subscripts while $\overleftarrow{*}$ and $\overleftarrow{\delta}$ are terminal subscripts. For Cayley digraphs of semigroups and monoids, we generally ignore the two terminal subscripts. The following examples show that, except for the equalities in the corollary above, the definitions in our list are all distinct. We do not claim to have included all possible definitions.

Let Γ be a digraph and $\mathfrak{F}$ a finite set of vertices in Γ. Write just $\mathfrak{C}_x$ for $\mathfrak{C}_x(\Gamma - \mathfrak{F})$. Every $\mathfrak{C}_x$ is a subset of $\mathfrak{C}_\infty$ and we have illustrated some fairly obvious subset inclusions in Figure 1.

Since, for example,

$$\mathfrak{C}_{\overrightarrow{\delta}}(\Gamma - \mathfrak{F}) \subseteq \mathfrak{C}_{\overrightarrow{*}}(\Gamma - \mathfrak{F}) \subseteq \mathfrak{C}_*(\Gamma - \mathfrak{F}) \subseteq \mathfrak{C}_\infty(\Gamma - \mathfrak{F}),$$

we have $\mathrm{e}_{\overrightarrow{\delta}}(\Gamma) \le \mathrm{e}_{\overrightarrow{*}}(\Gamma) \le \mathrm{e}_*(\Gamma) \le \mathrm{e}_\infty(\Gamma)$, with similar inequalities following from other inclusions. An important consequence is that, for nonterminal subscripts x, all the numbers $\mathrm{e}_x(\Gamma)$ have the same value if we have $\mathfrak{C}_{\overrightarrow{\delta}}(\Gamma - \mathfrak{F}) = \mathfrak{C}_\infty(\Gamma - \mathfrak{F})$ for every finite set $\mathfrak{F}$ of vertices in Γ. Similarly, for noninitial subscripts x, all the numbers $\mathrm{e}_x(\Gamma)$ have the same value if we have $\mathfrak{C}_{\overleftarrow{\delta}}(\Gamma - \mathfrak{F}) = \mathfrak{C}_\infty(\Gamma - \mathfrak{F})$ for every finite set $\mathfrak{F}$ of vertices in Γ.

When a digraph Γ has no multiple edges, it can be notationally very convenient to write $e = (v_i, v_j)$ for the edge e which has $\iota(e) = v_i$ and $\tau(e) = v_j$. We follow this convention in the following six examples. If Γ is a digraph without multiple edges, then the dual digraph Γ^{op} has the same vertices as Γ and (v_j, v_i) is an edge in Γ^{op} if and only if (v_i, v_j) is an edge in Γ. We write $\mathbb{N}$ for the natural numbers, $\mathbb{Z}$ for the integers and $\mathbb{Z}_n$ for the integers modulo n.

EXAMPLE 1. For $z = r, a, s$, let Γ_z be the digraph (V, E_z, ι, τ) where

$$\begin{aligned}
V &= \{v_i \mid i \in \mathbb{Z}\},\\
E_r &= \{(v_i, v_{i+1}) \mid i \in \mathbb{Z}\},\\
E_a &= \{(v_{2j}, v_{2j+1}), (v_{2j}, v_{2j-1}) \mid j \in \mathbb{Z}\},\\
E_s &= \{(v_i, v_{i+1}), (v_{2j}, v_{2j+1}), (v_{2k}, v_{2k-1}) \mid i \le -1,\ j \ge 0,\ k \ge 1\}.
\end{aligned}$$

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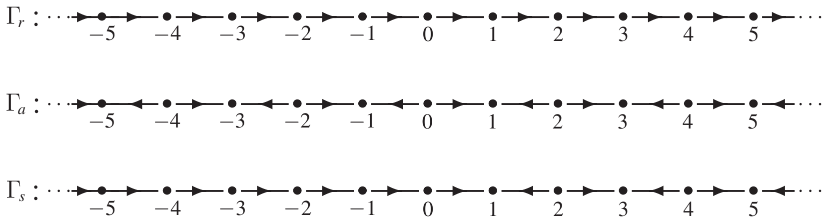

FIGURE 2. Γ_r, Γ_a and Γ_s.

The subscripts r, a, s are for right, alternating and split (see Figure 2). We observe first that $\overleftrightarrow{\Gamma}_r = \overleftrightarrow{\Gamma}_a = \overleftrightarrow{\Gamma}_s$, so that $\mathrm{e}_x(\Gamma_r) = \mathrm{e}_x(\Gamma_a) = \mathrm{e}_x(\Gamma_s)$ for any graph subscript x and then that all of these have value 2. For Γ_r, we observe that $\mathrm{e}_{+p}(\Gamma_r) = \mathrm{e}_\delta(\Gamma_r) = 2$, while $\mathrm{e}_{\overrightarrow{*}}(\Gamma_r) = \mathrm{e}_{\overrightarrow{\delta}}(\Gamma_r) = \mathrm{e}_{\overleftarrow{*}}(\Gamma_r) = \mathrm{e}_{\overleftarrow{\delta}}(\Gamma_r) = 1$. Since no positive path in Γ_a has length greater than 1, $\mathrm{e}_x(\Gamma_a) = 0$ for every digraph subscript x. Similarly, $\mathrm{e}_{+p}(\Gamma_s) = \mathrm{e}_\delta(\Gamma_s) = \mathrm{e}_{\overleftarrow{*}}(\Gamma_s) = \mathrm{e}_{\overleftarrow{\delta}}(\Gamma_s) = 1$, while $\mathrm{e}_{\overrightarrow{*}}(\Gamma_s) = \mathrm{e}_{\overrightarrow{\delta}}(\Gamma_s) = 0$.

EXAMPLE 2. Let Γ be the digraph with vertex set $V_\Gamma = \{v_{i,j} \mid i, j \in \mathbb{N}\}$ and edge set

$$E_\Gamma = \{(v_{2i,1}, v_{2i-1,1}), (v_{2i,1}, v_{2i+1,1}), (v_{i,2j}, v_{i,2j-1}), (v_{i,2j}, v_{i,2j+1}) \mid i, j \in \mathbb{N}\}.$$

Then $\mathrm{e}_x(\Gamma) = \infty$ for graph subscripts x while $\mathrm{e}_x(\Gamma) = 0$ for digraph subscripts x.

EXAMPLE 3. Let $n \geq 1$ be any natural number. For distinct symbols h and $s_{i,j}$, let $V = \{h\} \cup \{s_{i,j} \mid 1 \leq i \leq n, j \geq 1\}$ be the set of vertices for a digraph W_n. Define the set E of edges for W_n by $E = \{(h, s_{i,1}), (s_{i,j}, s_{i,j+1}), (s_{i,j}, h) \mid 1 \leq i \leq n, j \geq 1\}$. It is not difficult to see that $\mathrm{e}_x(W_n) = n$ except when x is a terminal subscript, $\overleftarrow{*}$ or $\overleftarrow{\delta}$. For those two cases, $\mathrm{e}_{\overleftarrow{*}}(W_n) = 1$ while $\mathrm{e}_{\overleftarrow{\delta}}(W_n) = 0$. Similarly, $\mathrm{e}_x(W_n^{\mathrm{op}}) = n$ for noninitial subscripts x while $\mathrm{e}_{\overrightarrow{*}}(W_n^{\mathrm{op}}) = 1$ and $\mathrm{e}_{\overrightarrow{\delta}}(W_n^{\mathrm{op}}) = 0$.

EXAMPLE 4. Let $n > 1$ be a natural number. Let $V = \{c\} \cup \{v_{i,k} \mid i \in \mathbb{Z}_n, k \in \mathbb{N}\}$ be the set of vertices for a digraph Γ_n. Define the set E of edges for Γ_n by $E = \{(v_{i,k}, v_{i+1,k}), (v_{0,k}, c) \mid i \in \mathbb{Z}_n, k \in \mathbb{N}\}$. Since we can choose finite subsets $\mathfrak{F}$ of V which exclude c, we see that $\mathrm{e}_\infty(\Gamma_n) = 1$. For any other subscript x, $\mathrm{e}_x(\Gamma_n) = 0$, since any trail in $\overleftrightarrow{\Gamma}_n$ with length greater than $2n + 2$ must pass through c at least twice and thus cannot be a path.

EXAMPLE 5. Let $V = \{c\} \cup \{v_{j,k} \mid k \in \mathbb{N}, k \geq 3, j \in \mathbb{Z}_k\}$ be the set of vertices for a digraph Θ. Define the set E of edges for Θ by $E = \{(v_{j,k}, v_{j+1,k}), (v_{j,k}, c) \mid k \in \mathbb{N}, k \geq 3, j \in \mathbb{Z}_k\}$. Here $\mathrm{e}_\infty(\Theta) = \mathrm{e}_p(\Theta) = \mathrm{e}_*(\Theta) = \mathrm{e}_{+p}(\Theta) = \mathrm{e}_\delta(\Theta) = \mathrm{e}_{\overleftarrow{*}}(\Theta) = 1$. For the other five subscripts x, $\mathrm{e}_x(\Theta) = 0$.

EXAMPLE 6. Let X be any infinite set. Let the set of all finite subsets of X be the set, $V = V_\Gamma$, of vertices for a digraph Γ. Define the set E of edges for Γ by $E = \{(A, B) \mid A, B \in V, B \subseteq A\}$. Then any positive path of length n with initial vertex $A_0 \in V$ corresponds to a chain of finite subsets $A_n \subsetneqq A_{n-1} \subsetneqq \cdots \subsetneqq A_2 \subsetneqq A_1 \subsetneqq A_0$. Observe that there is a positive path in Γ from A to B if and only if B is a subset of A, that the longest positive path having A as an initial vertex has length $|A|$ and that digeodesics in Γ have length 1.

Let $\mathfrak{F} = \{A_\psi\}_{\psi \in \Psi}$ be any finite subset of V and define $A_{\mathfrak{F}}$ by $A_{\mathfrak{F}} = \bigcup_{\psi \in \Psi} A_\psi$. Then $A_{\mathfrak{F}}$ is also finite. Choose some element $x = x_{\mathfrak{F}} \in X - A_{\mathfrak{F}}$. Given any two vertices $B_1, B_2 \in \Gamma - \mathfrak{F}$, let $B = B_1 \cup B_2 \cup \{x\}$. Then $B \in \Gamma - \mathfrak{F}$ and (B, B_1), (B, B_2) are both edges in $\Gamma - \mathfrak{F}$, so $\Gamma - \mathfrak{F}$ is connected and geodesics in $\overleftrightarrow{\Gamma - \mathfrak{F}}$ have length at most 2.

It is then clear that $\mathrm{e}_\infty(\Gamma) = \mathrm{e}_p(\Gamma) = \mathrm{e}_*(\Gamma) = \mathrm{e}_{+p}(\Gamma) = \mathrm{e}_{\overleftarrow{*}}(\Gamma) = 1$ while $\mathrm{e}_x(\Gamma) = 0$ for the other six subscripts x.

A set $X \subseteq M$ is a set of monoid generators for the monoid M if every nonidentity element of M can be written as a product of elements of X. A set $Y \subseteq M$ is a set of semigroup generators for the monoid M if every element of M can be written as a product of elements of Y.

For any semigroup S, we define S^1 to be the monoid $S \cup \{1\}$ where 1 is a new idempotent, not in S, and $1 \cdot s = s = s \cdot 1$ for every $s \in S$. For a semigroup homomorphism $f : S \to T$, we have a monoid homomorphism $f^1 : S^1 \to T^1$, extending f, if we define $f^1(1) = 1$. Then $(\ )^1$ is a functor from the category of semigroups to the category of monoids and is the left adjoint to the forgetful functor from the category of monoids to the category of semigroups. If $\langle X : R \rangle$ is a semigroup presentation for S, then we can regard $\langle X : R \rangle$ as a monoid presentation for S^1, by allowing the empty word on X.

For a digraph Γ and a semigroup S, let ϕ be a function from S to $E = E_\Gamma$. Then the pair (Γ, ϕ) is a *diagram* over the semigroup S, ϕ is the *label*, or labelling function for the diagram and, for any edge $e \in E$, $\phi(e) \in S$ is the *label* of e. We extend the label by concatenation to a label on positive paths in Γ.

If S is any semigroup with a finite set X of generators and $s \in S$, we write $L_X(s)$ for the smallest positive integer n such that $s = x_{i_1} x_{i_2} \cdots x_{i_n}$ with $x_{i_j} \in X$ for $1 \le j \le n$. Below, we will see that $L_X(s)$ is the length of a digeodesic from 1 to s in either the right Cayley digraph $\Gamma_r(S^1, X)$ or the left Cayley digraph ${}_\ell\Gamma(X, S^1)$. When M is a monoid and X is a set of monoid generators for M, the identity 1 is the product of 0 elements from X and we define $L_X(1) = 0$. For any nontrivial $m \in M$, we write $L_X(m)$ for the smallest positive integer n such that $m = x_{i_1} x_{i_2} \cdots x_{i_n}$ with $x_{i_j} \in X$ for $1 \le j \le n$. When X is a set of monoid generators for the monoid M and $m \in M$, we will see that $L_X(m)$ is the length of a digeodesic from 1 to m in either the right Cayley digraph $\Gamma_r(M, X)$ or the left Cayley digraph ${}_\ell\Gamma(X, M)$.

2. Cayley digraphs for semigroups and monoids

Suppose that $X \subseteq T$ is a set of semigroup generators for the semigroup T or that $X \subseteq T$ is a set of monoid generators for the monoid T. The *right Cayley digraph* for

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T with respect to X is the digraph $\Gamma_r(T, X) = (V, E, \iota, \tau)$ where $V = T$,

$$E = T \times X = \{(t, x) \mid t \in T, x \in X\}, \quad \iota((t, x)) = t, \quad \tau((t, x)) = tx.$$

Dually, the *left Cayley digraph* for T with respect to X is the digraph ${}_\ell\Gamma(X, T) = (V, E, \iota, \tau)$ where $V = T$,

$$E = X \times T = \{(x, t) \mid x \in X, t \in T\}, \quad \iota((x, t)) = t, \quad \tau((x, t)) = xt.$$

The monoid Cayley digraphs are connected, but the semigroup Cayley digraphs need not be. For example, if n is any natural number and F_n is the free semigroup on the set $X_n = \{x_1, \ldots, x_n\}$ of generators, then both $\Gamma_r(F_n, X_n)$ and ${}_\ell\Gamma(X_n, F_n)$ have n components. For a given semigroup or monoid, these left and right Cayley digraphs can be quite dissimilar and will not in general have the same number of ends. Given a Cayley digraph Γ, the graph $\overleftrightarrow{\Gamma}$ is the Cayley graph. Any reference in this work to a positive path in a Cayley graph will always mean a positive path in the Cayley digraph. To define the corresponding right and left *Cayley diagrams*, we define a label ϕ by $\phi(t, x) = x$ or $\phi(x, t) = x$, respectively. We can regard this label as having values in the semigroup or monoid T or as having values in the free semigroup or free monoid on the set X of generators for T.

LEMMA 3. *Suppose that X is a finite set of monoid generators for the monoid T. Let Γ be the right (left) Cayley digraph, $\Gamma_r(T, X)$ (${}_\ell\Gamma(X, T)$). If $\mathfrak{F}$ is any finite set of vertices of Γ and C is an infinite component of $\Gamma - \mathfrak{F}$, then there is a vertex $\hat{v}$ in C which initiates unbounded digeodesics.*

PROOF. The proof is the same for the right and left Cayley digraphs. We want to find a vertex $\hat{v}$ in C and, for each natural number n, a digeodesic $\pi_n \in C$ with $\iota(\pi_n) = \hat{v}$ and $|\pi_n| = n$.

Since C is infinite and there are only finitely many products of elements of X having any fixed length, for each natural number N, we can find an element v_N of C such that $L_X(v_N) \geq N$. Select a digeodesic γ_N in Γ from 1 to v_N.

Suppose first that, for infinitely many values of N, the digeodesic γ_N is contained in C. Then $1 \in C$. We let $\hat{v} = 1$ and, for each natural number n, we select a digeodesic $\gamma_N \in C$ with $N \geq n$. Since $|\gamma_N| = L_X(v_N) \geq N$, the digeodesic γ_N has length at least n and we let π_n be the initial subpath of γ_N of length n.

Suppose then that γ_N is in C for only finitely many values of N. Then, for some sufficiently large N_0, the digeodesic γ_N contains some vertex $a_N \in \mathfrak{F}$ whenever $N \geq N_0$. Let $\delta\mathfrak{F} = \{v \in \Gamma - \mathfrak{F} \mid \text{there is an edge } e \text{ of } \Gamma \text{ with } \iota(e) \in \mathfrak{F} \text{ and } \tau(e) = v\}$. Since $\mathfrak{F}$ and X are finite, $\delta\mathfrak{F}$ is finite. For $N \geq N_0$, we have $a_N \in \mathfrak{F}$ and $v_N \in C \subsetneqq \Gamma - \mathfrak{F}$, so γ_N contains at least one vertex which is in $\delta\mathfrak{F}$. For $N \geq N_0$, define u_N to be the unique vertex u on γ_N such that $u \in \delta\mathfrak{F}$, but no vertex following u on γ_N is in $\delta\mathfrak{F}$. Since $\delta\mathfrak{F}$ is finite, there is at least one vertex $\hat{v}$ in $\delta\mathfrak{F}$ such that $\hat{v} = u_N$ for infinitely many different values of N. This is the vertex $\hat{v}$ required for the conclusion of the lemma. For any natural number n, choose N so that $u_N = \hat{v}$ and $N \geq L_X(\hat{v}) + n$. Then the terminal subpath of γ_N from $\hat{v}$ to v_N has length at least n. Let π_n be the subpath of γ_N having length n and initial vertex $\hat{v}$. Since π_n is a subpath of a digeodesic, π_n is a digeodesic. $\square$

COROLLARY 4. *Suppose that X is a finite set of monoid generators for the monoid T. Let Γ be the right Cayley digraph, $\Gamma_r(T, X)$. Then $\mathrm{e}_x(\Gamma) = \mathrm{e}_\infty(\Gamma)$ if x is any of $p, g, *, \dagger, +p, \overrightarrow{*}$ or $\overrightarrow{\delta}$. Similarly, $\mathrm{e}_x(\Gamma) = \mathrm{e}_\infty(\Gamma)$ if Γ is the left Cayley digraph $_\ell\Gamma(X, T)$ and x is any of these subscripts.* $\square$

REMARK. The previous two results and arguments also hold for finitely generated semigroups.

EXAMPLE 7. If X is a finite alphabet with $|X| > 1$, F is the free monoid generated by X and Γ is the corresponding right Cayley digraph, then $\mathrm{e}_\infty(\Gamma) = \infty$, but $\mathrm{e}_{\overleftarrow{*}}(\Gamma) = \mathrm{e}_{\overleftarrow{\delta}}(\Gamma) = 0$.

LEMMA 5.

(a) *Suppose that M is a monoid with a finite set X of monoid generators, that Γ is the right (left) Cayley digraph for M with respect to X and that $\mathfrak{F}$ is any finite subset of M. Then $\Gamma - \mathfrak{F}$ has at most $1 + |X|\,|\mathfrak{F}|$ components.*

(b) *Suppose that S is a semigroup with a finite set X of generators, that Γ is the right (left) Cayley digraph for S with respect to X and that $\mathfrak{F}$ is any finite subset of S. Then $\Gamma - \mathfrak{F}$ has at most $(1 + |\mathfrak{F}|)|X|$ components.*

PROOF. The proof is the same for the right and left Cayley digraphs.

(a) One of the components of $\Gamma - \mathfrak{F}$ might contain the identity element of M. We regard this potential component as accounting for the '1' in $1 + |X|\,|\mathfrak{F}|$. Suppose that C is a component of $\Gamma - \mathfrak{F}$ which does not contain the identity element. Choose an element $m \in C$ for which $L_X(m)$ is minimal and, for the sake of notation, assume that $L_X(m) = k$. Choose a positive path of length k from the identity element to m in Γ and let v be the vertex occurring just before m on this path. Observe that we must have $v \in \mathfrak{F}$: otherwise $v \in \Gamma - \mathfrak{F}$ and the edge from v to m is in $\Gamma - \mathfrak{F}$, hence in C, but this contradicts our choice of $m \in C$ with $L_X(m)$ minimal. Then the edge from v to m is one of the $|X|$ edges having v as its initial vertex and we can have at most $|X|\,|\mathfrak{F}|$ such edges from vertices of $\mathfrak{F}$ to components of $\Gamma - \mathfrak{F}$ which do not contain the identity of M.

(b) Suppose that S is a semigroup, that $\Gamma = \Gamma(S, X)$ and that $\mathfrak{F}$ is a finite subset of S. By part (a), $\Gamma(S^1, X) - \mathfrak{F}$ has at most $1 + |X|\,|\mathfrak{F}|$ components. One of these components contains the identity element of S^1. With this identity element removed in Γ, the remaining elements of this component partition into at most $|X|$ components of $\Gamma - \mathfrak{F}$. Hence $\Gamma - \mathfrak{F}$ has at most $|X| + |X||\mathfrak{F}|$ components. $\square$

It is not difficult to see that the bounds given in Lemma 5 are attained when we let $\mathfrak{F}$ be the set of all words of some fixed length on the free generators of a free monoid or a free semigroup.

The following three corollaries of Lemma 5 are useful, expected and easy. We state and prove these for finitely generated monoids, but we remark that essentially the same proofs hold for finitely generated semigroups.

COROLLARY 6. *Suppose that X is a finite set of monoid generators for the monoid T. Let Γ be the right (left) Cayley digraph, $\Gamma_r(T, X)$ (${}_\ell\Gamma(X, T)$). If T is infinite, then* $\mathrm{e}_\infty(\Gamma) \geq 1$.

PROOF. For any finite $\mathfrak{F}$, T is the disjoint union of $\mathfrak{F}$ and the components of $\Gamma - \mathfrak{F}$. By Lemma 5, there are only finitely many components, so at least one of these must be infinite. □

COROLLARY 7. *Suppose that X is a finite set of monoid generators for the monoid T. Let Γ be the right (left) Cayley digraph, $\Gamma_r(T, X)$ (${}_\ell\Gamma(X, T)$). If $\mathfrak{F}$ and $\hat{\hat{\mathfrak{F}}}$ are finite subsets of T with $\mathfrak{F} \subseteqq \hat{\hat{\mathfrak{F}}}$, then $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| \leq |\mathfrak{C}_\infty(\Gamma - \hat{\hat{\mathfrak{F}}})|$.*

PROOF. Observe that each component of $\Gamma - \hat{\hat{\mathfrak{F}}}$ must be contained in some component of $\Gamma - \mathfrak{F}$. Since $\Gamma - \hat{\hat{\mathfrak{F}}}$ has only finitely many components, an infinite component, C, of $\Gamma - \mathfrak{F}$ can contain only finitely many components of $\Gamma - \hat{\hat{\mathfrak{F}}}$ and finitely many elements of $\hat{\hat{\mathfrak{F}}}$, so at least one of the components of $\Gamma - \hat{\hat{\mathfrak{F}}}$ contained in C must also be infinite. □

EXAMPLE 8. The conclusion of Corollary 7 need not hold for arbitrary digraphs Γ. Let Γ_n be the digraph of Example 4, $\mathfrak{F}$ any finite set of vertices of Γ_n which does not include the vertex c and $\hat{\hat{\mathfrak{F}}} = \mathfrak{F} \cup \{c\}$. Then $|\mathfrak{C}_\infty(\Gamma_n - \mathfrak{F})| = 1$ but $|\mathfrak{C}_\infty(\Gamma_n - \hat{\hat{\mathfrak{F}}})| = 0$.

COROLLARY 8. *Suppose that X is a finite set of monoid generators for the monoid T. Let Γ be the right (left) Cayley digraph, $\Gamma_r(T, X)$ (${}_\ell\Gamma(X, T)$). For every natural number n, define $\mathfrak{F}_n$ to be $\{t \in T \mid L_X(t) \leq n\}$. Then $\mathfrak{F}_n$ is finite and* $\mathrm{e}_\infty(\Gamma) = \lim_{n\to\infty} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)|$.

PROOF. It is clear that $|\mathfrak{F}_n| \leq \sum_{j=0}^{n} |X|^j$ and that

$$\max_{\mathfrak{F}_n} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)| \leq \mathrm{e}_\infty(\Gamma) = \max_{\mathfrak{F} \subseteq V_\Gamma,\, \mathfrak{F}\text{ finite}} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F})|.$$

If $\mathrm{e}_\infty(\Gamma)$ is finite, then, for some finite $\mathfrak{F} \subseteq T$, $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| = \mathrm{e}_\infty(\Gamma)$. Let $m = \max_{t\in\mathfrak{F}}\{L_X(t)\}$. Then $\mathfrak{F} \subseteq \mathfrak{F}_n$ for $n \geq m$, hence $\mathrm{e}_\infty(\Gamma) = |\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| \leq |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)|$ by Corollary 7. It follows that $\lim_{n\to\infty} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)| = \mathrm{e}_\infty(\Gamma)$.

If $\mathrm{e}_\infty(\Gamma) = \infty$, then for every natural number k we can find a finite $\mathfrak{F} \subseteq T$ with $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| \geq k$. Given such an $\mathfrak{F}$, let $n = \max_{t\in\mathfrak{F}}\{L_X(t)\}$. Then $\mathfrak{F} \subseteq \mathfrak{F}_n$ so $k \leq |\mathfrak{C}_\infty(\Gamma - \mathfrak{F})| \leq |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)|$ and $\lim_{n\to\infty} |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_n)| = \infty$. □

LEMMA 9. *If X and Y are finite sets of semigroup generators for the semigroup S, then $\mathrm{e}_\infty(\Gamma_r(S, X)) = \mathrm{e}_\infty(\Gamma_r(S, Y))$ and $\mathrm{e}_\infty({}_\ell\Gamma(X, S)) = \mathrm{e}_\infty({}_\ell\Gamma(Y, S))$.*

If X and Y are finite sets of monoid generators for the monoid M, then $\mathrm{e}_\infty(\Gamma_r(M, X)) = \mathrm{e}_\infty(\Gamma_r(M, Y))$ and $\mathrm{e}_\infty({}_\ell\Gamma(X, M)) = \mathrm{e}_\infty({}_\ell\Gamma(Y, M))$.

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PROOF. We state the argument in terms of the right Cayley digraphs for semigroups: the argument for the left Cayley digraphs is dual and the monoid argument is essentially the same as the semigroup argument. Observe first that, by symmetry, it suffices to prove that $\mathrm{e}_\infty(\Gamma_r(S, X)) = \mathrm{e}_\infty(\Gamma_r(S, X \cup Y))$, and second that we can reduce to the case of proving that $\mathrm{e}_\infty(\Gamma_r(S, X)) = \mathrm{e}_\infty(\Gamma_r(S, X \cup \{y\}))$ where $y \in Y$ by using induction on $|X \cup Y| - |X|$.

Suppose then that S is a semigroup, that X is a finite set of generators for S and that $y \in S - X$. For brevity, write Γ for the right Cayley digraph $\Gamma_r(S, X)$, and Γ' for the right Cayley digraph $\Gamma_r(S, X \cup \{y\})$. Then $S = V_\Gamma = V_{\Gamma'}$ and $E_\Gamma \subset E_{\Gamma'}$, so we may regard Γ as a proper subgraph of Γ'.

We want first to show that $\mathrm{e}_\infty(\Gamma') \le \mathrm{e}_\infty(\Gamma)$. Let $\mathfrak{F}$ be any finite subset of S. It suffices to show that each of the finitely many infinite components of $\Gamma' - \mathfrak{F}$ must contain one or more of the finitely many infinite components of $\Gamma - \mathfrak{F}$. Since all of the edges in $\Gamma - \mathfrak{F}$ are also edges in $\Gamma' - \mathfrak{F}$, every component of $\Gamma - \mathfrak{F}$ is entirely contained in some one component of $\Gamma' - \mathfrak{F}$. Since $\mathfrak{F}$ is finite and $\Gamma - \mathfrak{F}$ has only finitely many finite components, the union of the infinite components of $\Gamma - \mathfrak{F}$ contains all but finitely many of the vertices of Γ and hence the union of the components of $\Gamma' - \mathfrak{F}$ which contain infinite components of $\Gamma - \mathfrak{F}$ contains all but finitely many of the vertices in $V_{\Gamma'} = V_\Gamma$. Hence, every infinite component of Γ' must be one of those which contains some infinite component of $\Gamma - \mathfrak{F}$.

Suppose next that $\mathrm{e}_\infty(\Gamma)$ is finite. Then there is a finite subset $\mathfrak{F}_1$ of S such that $\Gamma - \mathfrak{F}_1$ has exactly $\mathrm{e}_\infty(\Gamma)$ infinite components. Let $\mathfrak{F}_+$ be the set of vertices in Γ which can be reached from some vertex of $\mathfrak{F}_1$ by a positive path in Γ having length at most $L_X(y)$. Observe that $\mathfrak{F}_+$ is a finite set and let $\mathfrak{F}_2 = \mathfrak{F}_1 \cup \mathfrak{F}_+$. Then $\Gamma - \mathfrak{F}_2$ also has $\mathrm{e}_\infty(\Gamma)$ infinite components. Each infinite component, C, of $\Gamma - \mathfrak{F}_2$ is contained in some infinite component, D, of $\Gamma' - \mathfrak{F}_2$. To show that $\Gamma' - \mathfrak{F}_2$ has $\mathrm{e}_\infty(\Gamma)$ infinite components it suffices to show that no two distinct infinite components of $\Gamma - \mathfrak{F}_2$ are contained in the same infinite component of $\Gamma' - \mathfrak{F}_2$. To this purpose, observe also that each infinite component, C, of $\Gamma - \mathfrak{F}_2$ is contained in an infinite component, $\hat{C}$, of $\Gamma - \mathfrak{F}_1$. Since $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, it follows from Corollary 7 that $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_1)| = |\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_2)|$ and hence that if C_1, C_2 are distinct infinite components of $\Gamma - \mathfrak{F}_2$, then $\hat{C}_1, \hat{C}_2$ are distinct infinite components of $\Gamma - \mathfrak{F}_1$. For the sake of obtaining a contradiction, suppose that two distinct infinite components, C_1 and C_2, of $\Gamma - \mathfrak{F}_2$ are contained in the same infinite component, D, of $\Gamma' - \mathfrak{F}_2$. By choosing C_1 and C_2 to minimize the length of a path in $\Gamma' - \mathfrak{F}_2$ connecting them, we may assume that there is an edge f in $\Gamma' - \mathfrak{F}_2$, labelled by y, having its initial vertex in C_1 and its terminal vertex in C_2. Write v_1 for the initial vertex of f and v_2 for the terminal vertex of f. Then v_1 is also in $\hat{C}_1$ and v_2 is also in $\hat{C}_2$. Since the edge f of Γ' has v_1 and v_2 for its endpoints, there is a path π_f in Γ having length $L_X(y)$ from v_1 to v_2. If π_f connects $\hat{C}_1$ and $\hat{C}_2$ in $\Gamma - \mathfrak{F}_1$, then $\hat{C}_1 = \hat{C}_2$ and $C_1 = C_2$, so there must be some vertex on π_f which occurs in $\mathfrak{F}_1$. But then $v_2 \in \mathfrak{F}_+ \subset \mathfrak{F}_2$, contradicting our assumption that v_2 is in a component of $\Gamma' - \mathfrak{F}_2$.

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In the case where $\mathrm{e}(\Gamma)$ is infinite, we need to show that for every natural number N, there is a finite subset $\mathfrak{F}$ of S such that $\Gamma' - \mathfrak{F}$ has at least N infinite components. Let $\mathfrak{F}_1$ be a finite subset of S such that $\Gamma - \mathfrak{F}_1$ has at least N distinct infinite components. Use $\mathfrak{F}_1$ and $L_X(y)$ to construct $\mathfrak{F}_2$ just as in the preceding paragraph. Then $\Gamma - \mathfrak{F}_2$ also has at least N infinite components and, moreover, we can select N of these which are contained in N different components of $\Gamma - \mathfrak{F}_1$. It suffices to show that no two of these can be contained within the same component of $\Gamma' - \mathfrak{F}_2$. For that, we may repeat the argument above using some edge f and corresponding path π_f. □

For a finitely generated semigroup S, we define $\mathcal{E}_r(S)$ and $\mathcal{E}_\ell(S)$ by $\mathcal{E}_r(S) = \mathrm{e}_\infty(\Gamma_r(S, X))$ and $\mathcal{E}_\ell(S) = \mathrm{e}_\infty({}_\ell\Gamma(S, X))$ for any finite set X of semigroup generators for S. It is clear that we can state analogous definitions for $\mathcal{E}_r(M)$ and $\mathcal{E}_\ell(M)$ when M is a finitely generated monoid.

We observe that when M is a finitely generated monoid, the values for $\mathcal{E}_r(M)$ and $\mathcal{E}_\ell(M)$ do not change if we consider M as a semigroup rather than as a monoid. By Lemma 9, we may assume that our set X of monoid generators for M includes the identity element of M. Then X is also a set of semigroup generators for M. When we regard M as a semigroup, we obtain the same right (left) Cayley digraph for M with respect to X that we obtain when we regard M as a monoid.

If G is a finitely generated group with a finite set X of group generators, then $X \cup X^{-1}$ is a finite set of monoid generators for G. It is usual to consider a Cayley graph rather than a Cayley digraph for a group. Typically, this is the right Cayley graph, but it is isomorphic to the left Cayley graph. All vertices in the Cayley graph $\Gamma(G, X)$ have degree $2|X|$, so $\Gamma(G, X)$ is locally finite. There are numerous equivalent definitions for the number of ends of a finitely generated group (see [2, 3, 10–12]). One of these (see [10]) is to define the number of ends of G to be $\lim_{n\to\infty} |\mathfrak{C}_\infty(\Gamma(G, X) - \mathfrak{F}_n)|$ where $\mathfrak{F}_n = \{g \in G \mid L_{X\cup X^{-1}}(g) \le n\}$. By Corollary 8, when a group G is considered as a monoid, then its number of ends (considered as a group) is equal to both of the monoid values $\mathcal{E}_r(G)$ and $\mathcal{E}_\ell(G)$.

A function ψ from a semigroup S_1 to a semigroup S_2 is an *anti-homomorphism* (see [1, Volume 1, p. 9]) if $\psi(ab) = \psi(b)\psi(a)$ for all $a, b \in S_1$. It is easy to see that the composition of two anti-homomorphisms is a homomorphism. An anti-homomorphism is an *anti-automorphism* if $S_1 = S_2$ and ψ is a bijection.

For any semigroup $(S, \cdot)$ with associative multiplication $\cdot$ the dual semigroup $S^{\mathrm{op}} = (S, *)$ has the same set of elements as S and has associative multiplication $*$ defined by $s_1 * s_2 = s_2 \cdot s_1$. Then the identity function on the set S is an anti-automorphism between the semigroup S and the semigroup S^{op}. Using this and composition, we may regard any anti-homomorphism from S_1 to S_2 as either a homomorphism from S_1^{op} to S_2 or else as a homomorphism from S_1 to S_2^{op}. In particular, anti-automorphisms of S correspond to isomorphisms between S and S^{op}. Any set of generators for S is also a set of generators for S^{op} and whenever X is a finite set of generators for S, then $\Gamma_r(S, X) = {}_\ell\Gamma(X, S^{\mathrm{op}})$, so $\mathcal{E}_r(S) = \mathcal{E}_\ell(S^{\mathrm{op}})$ and $\mathcal{E}_\ell(S) = \mathcal{E}_r(S^{\mathrm{op}})$. It is worth observing that ${}^{\mathrm{op}}$ is a functor from the category of semigroups to itself. If S_1 and S_2 are

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semigroups and $f : S_1 \to S_2$ is a homomorphism, then we may define $f^{\mathrm{op}} : S_1^{\mathrm{op}} \to S_2^{\mathrm{op}}$ by $f^{\mathrm{op}}(s) = f(s)$ for $s \in S_1$. It is easy to verify that f^{op} is a homomorphism. Generally, it is notationally convenient to suppress the distinction between f and f^{op}.

PROPOSITION 10. *If the semigroup S is isomorphic to S^{op}, then $\mathcal{E}_r(S) = \mathcal{E}_\ell(S)$.*

A semigroup S is an *inverse semigroup* if for every element $s \in S$ there is a unique element, denoted s^{-1}, in S such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. A monoid is an *inverse monoid* if it is an inverse semigroup when regarded as a semigroup. It is easy to see that ss^{-1} and $s^{-1}s$ are idempotent elements in any inverse semigroup, and it is very well known (see [1] or [8]) that idempotents in an inverse semigroup commute with each other. From these, it follows easily that for $s, t \in S$, $(st)^{-1} = t^{-1}s^{-1}$ and hence that $()^{-1}$ is an anti-automorphism.

COROLLARY 11. *If T is a finitely generated inverse semigroup or a finitely generated inverse monoid, then $\mathcal{E}_r(T) = \mathcal{E}_\ell(T)$.*

PROOF. As noted, the inversion operation is an anti-automorphism. □

3. Some constructions and examples

We give three equivalent descriptions for the construction of a useful monoid $\hat{M}$ followed by some comments about the corresponding semigroup construction.

Suppose that M and T are monoids and that $M = S^1$ for some semigroup S, or equivalently that the identity element of M is the only element of M that is a left unit or a right unit. Define a multiplication $*$ on the set $T \times M$ by

$$(t_1, m_1) * (t_2, m_2) = \begin{cases} (t_1t_2, m_2) & \text{if } m_1 = 1, \\ (t_1, m_1m_2) & \text{otherwise.} \end{cases}$$

We need the assumption that $M = S^1$ to prove that $*$ is associative. It is easy to see that $(1, 1)$ is an identity element, so $(T \times M, *)$ is a monoid. We use $\hat{M}$ as a succinct notation for this monoid. If T is also a monoid in which the identity element is the only left or right unit, then this assumption also holds for $\hat{M}$.

For the sake of notation, let $\langle A : R_1 \rangle$ be a monoid presentation for T and $\langle X : R_2 \rangle$ a monoid presentation for M. Then the monoid $\hat{M}$ has monoid presentation

$$\langle A \cup X \mid R_1 \cup R_2 \cup \{(xa, x) \mid a \in A, x \in X\}\rangle.$$

We want to identify the ordered pair (t, m) with the word tm where t is a word on A and m is a word on X. It is easy to show that every element of the presentation can be written in the form tm. It is more tedious to show that $tm = t'm'$ in $\hat{M}$ implies that $t = t'$ in T and that $m = m'$ in M: the assumption that $M = S^1$ for some semigroup S is necessary for this uniqueness result.

Later in this paper we provide a more elaborate discussion of semidirect products of monoids. Define $\theta_T \in \mathrm{End}(T)$ to be the monoid endomorphism of T which

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takes every element of T to the identity element 1_T and write ι_T for the identity automorphism of T. Define a monoid homomorphism $\Phi_0 : M \to \mathrm{End}(T)$ to be the monoid homomorphism which takes 1_M to ι_T and takes every other element of M to θ_T. Here, we need the assumption that $M = S^1$ to ensure that Φ_0 is a homomorphism. Then the multiplication $*$ above and the monoid presentation above are the multiplication and a presentation for the monoid semidirect product $T \rtimes_{\Phi_0} M$. This approach to describing $\hat{M}$ has the notational advantage that we can conveniently describe different $\hat{M}$s for different choices of M and T. We utilize this notation in some examples.

If S and T are semigroups with semigroup presentations $\langle X : R_2 \rangle$ and $\langle A : R_1 \rangle$, respectively, then we can define a semigroup $\hat{S}$ having semigroup presentation

$$\langle A \cup X \mid R_1 \cup R_2 \cup \{(xa, x) \mid a \in A, x \in X\} \rangle$$

which is clearly analogous to the monoid presentation for $\hat{M}$ given above. We could also describe this semigroup $\hat{S}$ by defining the obvious multiplication on the set $(T \times S) \cup T \cup S$ or by recognizing $\hat{S}$ as the subsemigroup of nonidentity elements of the monoid semidirect product $T^1 \rtimes_{\Phi_0} S^1$. We will not need the semigroup version of the following lemma.

LAYER LEMMA. *Let T be a finite monoid and M a finitely generated monoid. Assume that $M = S^1$ for some semigroup S. Then $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = |T|\mathcal{E}_r(M)$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = \mathcal{E}_\ell(M)$.*

PROOF. Let X be a finite set of monoid generators for M and let A be a finite set of monoid generators for T. We write x or x_j for elements of X and a or a_k for elements of A. Write $T = \{1_T = t_1, t_2, \ldots, t_n\}$ for some fixed ordering of the $|T| = n$ elements of T. We use m, m_1, m_2 as notation for arbitrary elements of M and use t, t_i, t_{i_1}, t_{i_2} as notation for arbitrary elements of T. We use $\hat{M}$ as an abbreviation for $T \rtimes_{\Phi_0} M$. We write just tm, t and m for elements (t, m), $(t, 1_M)$ and $(1_T, m)$ of $\hat{M}$ and we similarly identify X with $\{1_T\} \times X$ and A with $A \times \{1_M\}$.

Write Γ for the right Cayley digraph $\Gamma_r(M, X)$ and $\hat{\Gamma}$ for the right Cayley digraph $\Gamma_r(\hat{M}, A \cup X)$. Here, it will be useful to regard Γ and $\hat{\Gamma}$ as diagrams with labelled edges. For $1 \le i \le n$, we define the ith *layer* in $\hat{\Gamma}$ to be the subdiagram Γ_i of $\hat{\Gamma}$ having vertices $V_i = \{t_i m \mid m \in M\}$ and edges $E_i = \{f \mid f \in E_{\hat{\Gamma}}, \iota(f) = t_i m, \tau(f) = t_i mx, x \in X\}$. Then, for each i with $1 \le i \le n$, there is a diagram isomorphism $\phi_i : \Gamma \to \Gamma_i$, defined on vertices and edges by $\phi_i(m) = t_i m$, and $\phi_i(f)$ is the edge from $t_i m$ to $t_i mx$ with label x if f is the edge from m to mx having label x. Since $t_i m a_k = t_i m$ in $\hat{M}$ if $m \ne 1_M$, the edges of $\hat{\Gamma}$ having labels in A are loops at the given vertex except for the edges from t_i to $t_i a_k$. We build $\hat{\Gamma}$ from the n layers by constructing the finite right Cayley digraph $\Gamma_r(T, A)$, identifying the vertex t_i in this digraph with the vertex $t_i 1$ in Γ_i and adding loops labelled by each element of A at each vertex $t_i m$, $m \ne 1$ of Γ_i.

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Suppose first that $\mathcal{E}_r(M)$ is finite and choose a finite subset $\mathfrak{F} \subseteq M$ such that $\Gamma - \mathfrak{F}$ has $\mathcal{E}_r(M)$ infinite components. We may assume by Corollary 7 that $1 \in \mathfrak{F}$. For $1 \leq i \leq n$, let $\mathfrak{F}_i = \{t_i m \mid m \in \mathfrak{F}\} \subseteq V_i$ and let $\hat{\mathfrak{F}} = \bigcup_{i=1}^{n} \mathfrak{F}_i$. Observe that $\hat{\mathfrak{F}}$ is finite. Since $\Gamma_i - \mathfrak{F}_i$ has $\mathcal{E}_r(M)$ infinite components for each i and the edges of $\hat{\Gamma} - \hat{\mathfrak{F}}$ having labels in A are all loops, $\hat{\Gamma} - \hat{\mathfrak{F}}$ has $n\mathcal{E}_r(M)$ infinite components. This shows that $\hat{\Gamma}$ has at least $n\mathcal{E}_r(M)$ ends. Suppose that $\bar{\mathfrak{F}}$ is an arbitrary finite subset of $\hat{M}$. Let

$$\mathfrak{F} = \mathfrak{F}_1 = \{m \mid t_i m \in \bar{\mathfrak{F}} \text{ for some } i \text{ with } 1 \leq i \leq n\}, \quad \mathfrak{F}_i = \{t_i m \mid m \in \mathfrak{F}\},$$

for $1 \leq i \leq n$ and $\hat{\mathfrak{F}} = \bigcup_{i=1}^{n} \mathfrak{F}_i$. Observe again that $\hat{\mathfrak{F}}$ is finite. Because $\bar{\mathfrak{F}} \subseteq \hat{\mathfrak{F}}$, we know that $\hat{\Gamma} - \hat{\mathfrak{F}}$ has at least as many infinite components as $\hat{\Gamma} - \bar{\mathfrak{F}}$. It suffices to prove that $\hat{\Gamma} - \hat{\mathfrak{F}}$ has at most $n\mathcal{E}_r(M)$ infinite components. If this were not the case, then by the pigeonhole principle, we would have more than $\mathcal{E}_r(M)$ infinite components in some $\Gamma_i - \mathfrak{F}_i$. Since all of these are isomorphic to $\Gamma_1 - \mathfrak{F}_1$, we would then have the contradiction that $\Gamma - \mathfrak{F}$ has more than $\mathcal{E}_r(M)$ infinite components.

The argument when $\mathcal{E}_r(M)$ is infinite is similar. For every natural number h we can find a finite subset $\mathfrak{F}$ of M such that $1 \in \mathfrak{F}$ and $\Gamma - \mathfrak{F}$ has at least h infinite components. Then, with $\hat{\mathfrak{F}}$, as above, $\hat{\Gamma} - \hat{\mathfrak{F}}$ has at least hn infinite components.

Now write Γ for the left Cayley digraph ${}_\ell\Gamma(X, M)$ and $\hat{\Gamma}$ for the left Cayley digraph ${}_\ell\Gamma(A \cup X, \hat{M})$. Write Υ for the left Cayley digraph ${}_\ell\Gamma(A, T)$. For any $m \in M$, define the *tower* at m to be the the subdiagram Υ_m of $\hat{\Gamma}$ having vertex set $V_m = \{tm \mid t \in T\}$ and edge set $E_m = \{f \mid f \in E_{\hat{\Gamma}}, \iota(f) = tm, \tau(f) = atm, a \in A\}$. Then, for every $m \in M$, there is a diagram isomorphism $\phi_m : \Upsilon \to \Upsilon_m$ defined on vertices and edges by $\phi_m(t) = tm$, and $\phi_m(f)$ is the edge from tm to atm if f is the edge from t to at in Υ having label a. We can think of constructing $\hat{\Gamma}$ from the set of towers by identifying each vertex $1_T m$ in the tower Υ_m with the vertex m in Γ and for each $x \in X$ and $m \neq 1$ in M adjoining an edge from tm in Υ_m to xm in Γ having label x.

Corresponding to any finite subset $\mathfrak{F}$ of M, we define $\hat{\mathfrak{F}}$ to be the finite subset $\hat{\mathfrak{F}} = \{t_i m \mid 1 \leq i \leq n, m \in \mathfrak{F}\}$ of $\hat{M}$. Corresponding to any finite subset $\bar{\mathfrak{F}}$ of $\hat{M}$, let $\mathfrak{F} = \mathfrak{F}_1 = \{m \mid t_k m \in \bar{\mathfrak{F}} \text{ for some } 1 \leq k \leq n\}$. Since $\bar{\mathfrak{F}} \subseteq \hat{\mathfrak{F}}$, we obtain $\mathrm{e}_\infty(\hat{\Gamma}) = \mathrm{e}_\infty(\Gamma)$ if we show that $\Gamma - \mathfrak{F}$ and $\hat{\Gamma} - \hat{\mathfrak{F}}$ have the same number of infinite components for any finite $\mathfrak{F}$. After discussing some technical details, we construct a bijection $\widehat{\ }$ between infinite components C of $\Gamma - \mathfrak{F}$ and infinite components $\hat{C}$ of $\hat{\Gamma} - \hat{\mathfrak{F}}$.

If, for some $x \in X$, the edge $\hat{f} \in E_{\hat{\Gamma}}$ has label x, $\iota(\hat{f}) = tm$ and $\tau(\hat{f}) = xm$, define the projection of $\hat{f}$ to be the edge $f = \pi(\hat{f})$ in E_Γ having label x, $\iota(f) = m$ and $\tau(f) = xm$. If the edge $\hat{f} \in E_{\hat{\Gamma}}$ has label a for $a \in A$, $\iota(\hat{f}) = tm$ and $\tau(\hat{f}) = atm$, define the projection, $\pi(\hat{f})$, of $\hat{f}$ to be the empty path in Γ at the vertex m. We regard the inverse of an empty path at any vertex to be the same empty path. If $\omega = \hat{f}_1^{\varepsilon_1} \hat{f}_2^{\varepsilon_2} \cdots \hat{f}_k^{\varepsilon_k}$, where $\varepsilon_j = \pm 1$ and $\hat{f}_j \in E_{\hat{\Gamma}}$ for $1 \leq j \leq k$, is a walk in $\hat{\Gamma} - \hat{\mathfrak{F}}$ from $\iota(\omega)$ to $\tau(\omega)$, then by induction on k, $(\pi(\hat{f}_1))^{\varepsilon_1} (\pi(\hat{f}_2))^{\varepsilon_2} \cdots (\pi(\hat{f}_k))^{\varepsilon_k}$ is a walk in $\Gamma - \mathfrak{F}$ from $\iota((\pi(\hat{f}_1))^{\varepsilon_1})$ to $\tau((\pi(\hat{f}_k))^{\varepsilon_k})$. Denote this walk by $\pi(\omega)$. In this paragraph and the next, walks and paths (allowing negative edges) in a digraph are to be interpreted as walks in the corresponding graph.

If C is an infinite component of $\Gamma - \mathfrak{F}$, define $\hat{C}$ by $\hat{C} = \{tm \mid t \in T, m \in C\}$. Then $\hat{C}$ is an infinite set, since $\{1m \mid m \in C\} \subseteq \hat{C}$. To see that $\hat{C}$ is connected in $\hat{\Gamma} - \hat{\mathfrak{F}}$, suppose that $t_{i_1}m_1, t_{i_2}m_2 \in \hat{C}$. Then $m_1, m_2 \in C$, so there is a path from m_1 to m_2 in $\Gamma - \mathfrak{F}$ and hence a corresponding path from $1m_1$ to $1m_2$ in $\hat{\Gamma} - \hat{\mathfrak{F}}$. There are also paths, with all edges labelled by elements of A in $\hat{\Gamma} - \hat{\mathfrak{F}}$, from $1m_1$ to $t_{i_1}m_1$ and from $1m_2$ to $t_{i_2}m_2$. We compose paths and their inverses to find a path from $t_{i_1}m_1$ to $t_{i_2}m_2$. If C^* is the component of $\hat{\Gamma} - \hat{\mathfrak{F}}$ which contains $\hat{C}$, let t^*m^* be any vertex in C^*, $\hat{t}\hat{m}$ any vertex in $\hat{C}$ and ω a path in $\hat{\Gamma} - \hat{\mathfrak{F}}$ from t^*m^* to $\hat{t}\hat{m}$. Then $\pi(\omega)$ is a walk in $\Gamma - \mathfrak{F}$ from m^* to $\hat{m}$, hence $m^* \in C$, $t^*m^* \in \hat{C}$ and $C^* = \hat{C}$. That is, $\hat{C}$ is always a component of $\hat{\Gamma} - \hat{\mathfrak{F}}$. If C_1 and C_2 are any two infinite components of $\Gamma - \mathfrak{F}$ and ω is a path in $\hat{\Gamma} - \hat{\mathfrak{F}}$ from some vertex in $\hat{C}_1$ to some vertex in $\hat{C}_2$, then $\pi(\omega)$ is a walk in $\Gamma - \mathfrak{F}$ from $\iota(\pi(\omega)) \in C_1$ to $\tau(\pi(\omega)) \in C_2$, hence $C_1 = C_2$.

To see that $\widehat{\ }$ is onto, suppose that $\bar{C}$ is some infinite component of $\hat{\Gamma} - \hat{\mathfrak{F}}$. Define C by $C = \{m \mid tm \in \bar{C}$ for some $t \in T\}$. It is then routine to verify that C is an infinite set, that C is connected, that C is a component of $\Gamma - \mathfrak{F}$ and that $\hat{C} = \bar{C}$. □

It is well known that, for any finitely generated group G and finite set X of generators for G, the left and right Cayley graphs for G with generating set X are isomorphic, and that such a Cayley graph has $0, 1, 2$ or else infinitely many ends (see [2, 3, 10–12]). The next six examples illustrate that these conclusions do not hold for left and right Cayley graphs for semigroups and monoids.

EXAMPLE 9. For $n \geq 2$, define A_n to be the monoid semidirect product $T \rtimes_{\Phi_0} M$ where M is the infinite cyclic monoid having generator x and T is the monogenic monoid having monoid presentation $\langle t : t^n = t^{n-1}\rangle$. It is apparent that $M = S^1$ where S is the infinite cyclic semigroup having generator x. It is easy to see that the left and right Cayley digraphs for M have one end. By the Layer Lemma, $\mathcal{E}_r(A_n) = n$ and $\mathcal{E}_\ell(A_n) = 1$.

To greatly generalize, let S_1 be any semigroup with $\mathcal{E}_r(S_1) = \mathcal{E}_\ell(S_1) = 1$ and let S_2 be any semigroup with $n-1$ elements. With $M = S_1^1$ and $T = S_2^1$, we have $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = n$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = 1$. The monoid A_n is arguably the most elementary example of this construction. Another elementary possibility is to replace $T = \langle t : t^n = t^{n-1}\rangle$ by a cyclic group of order n having generator t. This has the disadvantage that the element $(t, 1) \in T \rtimes_{\Phi_0} M$ is then a left and right unit, preventing us from using this $T \rtimes_{\Phi_0} M$ as the monoid M^{op} in the next example.

EXAMPLE 10. For arbitrary natural numbers $m, n \geq 2$, let $M = A_n^{\mathrm{op}}$ and let $T = T_m$ now be the the monogenic monoid having monoid presentation $\langle t : t^m = t^{m-1}\rangle$. Then by the Layer Lemma, $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = m$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = n$. For later reference, denote this monoid $T \rtimes_{\Phi_0} M$ by $J_{n,m}$.

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EXAMPLE 11. Let B be the monoid having monoid presentation $\langle a, b : ba = a^2\rangle$. We want to show that the right Cayley digraph $\Gamma_r(B, \{a, b\})$ has one end while the left Cayley digraph ${}_\ell\Gamma(\{a, b\}, B)$ has infinitely many ends. Every element of B can be uniquely written in the form $a^j b^k$ where $j, k \geq 0$. For either the right or the left Cayley digraph for B, we place the vertex $a^j b^k$ at the lattice point (j, k) in the first quadrant of the real plane. It is easy to show that $a^j b^k \cdot a = a^{j+k+1}$ and it is clear that $a^j b^k \cdot b = a^j b^{k+1}$. Similarly, $a \cdot a^j b^k = a^{j+1} b^k$, while $b \cdot b^k = b^{k+1}$ when $j = 0$, but $b \cdot a^j b^k = a^{j+1} b^k$ when $j > 0$.

Suppose first that Γ is the right Cayley digraph $\Gamma_r(B, \{a, b\})$ and let $\mathfrak{F}$ be a finite set of vertices in B. Let $t = \max\{j + k \mid a^j b^k \in \mathfrak{F}\}$ and define $\mathfrak{F}_t$ by $\mathfrak{F}_t = \{a^j b^k \mid j + k \leq t\}$. The graph Γ has one end by Corollary 7 if $\Gamma - \mathfrak{F}_t$ is connected. Let $a^{j_1} b^{k_1}, a^{j_2} b^{k_2}$ be vertices in $\Gamma - \mathfrak{F}_t$ and assume without loss of generality that $j_1 + k_1 \leq j_2 + k_2$. Then we have a positive or empty path in $\Gamma - \mathfrak{F}_t$ from $a^{j_1+k_1+1}$ to $a^{j_2+k_2+1}$, with edges labelled by a and edges with label a from $a^{j_1} b^{k_1}$ to $a^{j_1+k_1+1}$ and from $a^{j_2} b^{k_2}$ to $a^{j_2+k_2+1}$.

Now suppose that Γ is the left Cayley digraph ${}_\ell\Gamma(\{a, b\}, B)$. For every natural number h, define the subset $\mathfrak{F}_h$ of B by $\mathfrak{F}_h = \{b^i : 0 \leq i < h\}$. We then observe that $\Gamma - \mathfrak{F}_h$ has $h + 1$ infinite components C_i where C_i has vertices $\{a^j b^i : j > 0\}$ if $0 \leq i < h$ and C_h has vertices $\{a^j b^k : j \geq 0, k \geq h\}$.

EXAMPLE 12. We have remarked earlier that a Cayley digraph for a semigroup need not be connected. Let B be the monoid from Example 11 and S the subsemigroup of nontrivial elements in B. We observe that the right Cayley digraph $\Gamma_r(S, \{a, b\})$ is connected, but the left Cayley digraph ${}_\ell\Gamma(\{a, b\}, S)$ has two components.

EXAMPLE 13. Let B be the monoid of Example 11 and observe that the identity element is the only left or right unit in B. Let $T = T_m$ again be the the monogenic monoid having monoid presentation $\langle t : t^m = t^{m-1}\rangle$. Then by the Layer Lemma, $\mathcal{E}_r(T \rtimes_{\Phi_0} B) = m$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} B) = \infty$.

An element t in a semigroup or a monoid T is a *regular* element if there is an element $x \in T$ such that $txt = t$. The semigroup or monoid T is *regular* if every element of T is regular. An element t in a semigroup or a monoid T is *completely regular* if there is an element $x \in T$ such that $txt = t$ and $xt = tx$. A semigroup or monoid T is *completely regular* if every element of T is completely regular. There are several other equivalent characterizations of completely regular semigroups. For example, a semigroup is completely regular if and only if it is a union of groups. In [9], this is part of Theorem II.1.4, which the authors there describe as the fundamental theorem for the global structure of completely regular semigroups.

EXAMPLE 14. We know by Corollary 11 that $\mathcal{E}_r(I) = \mathcal{E}_\ell(I)$ for any finitely generated inverse monoid I. In this example and the next, we show that this equality need not hold for completely regular monoids.

Let G_1 be any finitely generated group having one end. There are many of these, and $G_1 = \mathbb{Z} \times \mathbb{Z}$ is a standard example. Regard G_1 as a semigroup with a finite set X

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of semigroup generators and form the monoid $M = (G_1)^1$. Then X is a finite set of monoid generators for M and M is both a completely regular monoid and an inverse monoid. For a natural number $n > 1$, let G_2 be any completely regular semigroup with $n - 1$ elements. For example, we might take G_2 to be any group or any semilattice with $n - 1$ elements. Let $T = (G_2)^1$. Then T is a completely regular monoid with n elements. A routine argument by cases shows that $T \rtimes_{\Phi_0} M$ is a (completely) regular monoid whenever M and T are (completely) regular. By the Layer Lemma, we have that $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = n$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = 1$. To have one explicit example of this for later reference, let D_n be $T \rtimes_{\Phi_0} M$ when $G_1 = \mathbb{Z} \times \mathbb{Z}$ and G_2 is the cyclic group with $n - 1$ elements.

EXAMPLE 15. Let $M = D_n^{\mathrm{op}}$ where D_n is the completely regular monoid from Example 14. Then $\mathcal{E}_r(M) = 1$ and $\mathcal{E}_\ell(M) = n$. For a natural number $m > 1$, let G_3 be any completely regular semigroup with $m - 1$ elements and $T = (G_3)^1$. Then $T \rtimes_{\Phi_0} M$ is a completely regular monoid and we have $\mathcal{E}_r(T \rtimes_{\Phi_0} M) = m$ and $\mathcal{E}_\ell(T \rtimes_{\Phi_0} M) = n$ by the Layer Lemma.

We now embark upon a second, more common, construction that will allow us to present some interesting examples of commutative and inverse monoids.

Let Λ be an index set and $(S_\lambda, *_\lambda)$ be a semigroup for each $\lambda \in \Lambda$. Assume that $S_{\lambda_1} \cap S_{\lambda_2} = \emptyset$ if $\lambda_1 \neq \lambda_2$ and that 0 is a new element not in $\cup S_\lambda$. Define $\vee S_\lambda$ to be $\{0\} \cup \left(\bigcup_{\lambda \in \Lambda} S_\lambda\right)$ and define a multiplication $*$ on $\vee S_\lambda$ by

$$s * t = \begin{cases} s *_\lambda t & \text{if there exists } \lambda \in \Lambda \text{ such that } s \in S_\lambda \text{ and } t \in S_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $*$ is associative. Observe that $\vee S_\lambda$ is commutative if and only if every S_λ is commutative. When $|\Lambda| = 2$ and $\{S_\lambda\} = \{A, B\}$, write $A \vee B$ for $\vee S_\lambda$ and observe that $A \vee B = B \vee A$.

For a first variant description of $\vee S_\lambda$, for any λ, define S_λ^0 to be the semigroup having elements $\{0\} \cup S_\lambda$ with the multiplication $*_\lambda$ extended by setting $s *_\lambda 0 = 0 *_\lambda s = 0 *_\lambda 0 = 0$ for all $s \in S_\lambda$. Then $\vee S_\lambda$ is the 0-direct union of the semigroups S_λ^0. See Clifford and Preston [1, Volume II, p. 13], Howie, [5, p. 71] or Higgins, [4, p. 26].

For a second variant description of $\vee S_\lambda$, write S_0 for the one-element semigroup $\{0\}$ and define a multiplication on $\Lambda^0 = \Lambda \cup \{0\}$ by $0 \cdot 0 = \lambda \cdot 0 = 0 \cdot \lambda = \lambda_1 \cdot \lambda_2 = 0$ for all $\lambda, \lambda_1, \lambda_2 \in \Lambda$. Then Λ^0 is a rather trivial lower semilattice and $\vee S_\lambda$ is a Λ^0 semilattice of the semigroups S_λ. See [1, p. 25, 26], [5, p. 89] or [4, p. 37–39].

LEMMA 12. *Suppose that Λ is a finite set and that $\{S_\lambda\}_{\lambda \in \Lambda}$ is a set of pairwise disjoint, finitely generated semigroups S_λ. Then $\vee S_\lambda$ is finitely generated, $\mathcal{E}_\ell(\vee S_\lambda) = \sum_{\lambda \in \Lambda} \mathcal{E}_\ell(S_\lambda)$ and $\mathcal{E}_r(\vee S_\lambda) = \sum_{\lambda \in \Lambda} \mathcal{E}_r(S_\lambda)$.*

PROOF. We consider the case for the number of left ends. The case for the number of right ends is dual. If X_λ is a finite set of generators for S_λ, then $X = \{0\} \cup (\bigcup_{\lambda \in \Lambda} X_\lambda)$

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is a finite set of generators for $\vee S_\lambda$. Write Γ for the left Cayley digraph ${}_\ell\Gamma(X, \vee S_\lambda)$ and write Γ_λ for the left Cayley digraph ${}_\ell\Gamma(X_\lambda, S_\lambda)$. If, for each $\lambda \in \Lambda$, $\mathfrak{F}_\lambda$ is a finite subset of S_λ, define $\mathfrak{F}$ to be the finite set $\{0\} \cup (\cup \mathfrak{F}_\lambda)$. The key observation is that if $\lambda_1 \neq \lambda_2$, $x \in X_{\lambda_1}$ and $s \in S_{\lambda_2}$, then $xs = 0$ in $\vee S_\lambda$ and the edge from x to $xs = 0$ is not in $\Gamma - \mathfrak{F}$. Hence, the digraph $\Gamma - \mathfrak{F}$ is the disjoint union of the finitely many digraphs $\Gamma_\lambda - \mathfrak{F}_\lambda$. If $\Gamma_\lambda - \mathfrak{F}_\lambda$ has m_λ infinite components then $\Gamma - \mathfrak{F}$ has $\sum m_\lambda$ infinite components. If any $\mathcal{E}_\ell(S_\lambda)$ is infinite, then we can choose $\mathfrak{F}_\lambda \subseteq S_\lambda$ for which m_λ is larger than any given natural number N and hence $\mathcal{E}_\ell(\vee S_\lambda)$ must be larger than N also. If $\mathcal{E}_\ell(S_\lambda)$ is finite for every λ, then for every λ we can choose $\mathfrak{F}_\lambda \subseteq S_\lambda$ for which $m_\lambda = \mathcal{E}_\ell(S_\lambda)$ and conclude that $\mathcal{E}_\ell(\vee S_\lambda) \geq \sum_{\lambda\in\Lambda} \mathcal{E}_\ell(S_\lambda)$. To prove $\mathcal{E}_\ell(\vee S_\lambda) \leq \sum_{\lambda\in\Lambda} \mathcal{E}_\ell(S_\lambda)$, it suffices to show that $m \leq \sum_{\lambda\in\Lambda} \mathcal{E}_\ell(S_\lambda)$, whenever $\mathfrak{F}$ is a finite subset of $\vee S_\lambda$ and $\Gamma - \mathfrak{F}$ has m infinite components. By Corollary 7, we may assume that $0 \in \mathfrak{F}$. For each λ, define $\mathfrak{F}_\lambda$ to be $\mathfrak{F} \cap S_\lambda$ and define m_λ to be the number of infinite components of $\Gamma_\lambda - \mathfrak{F}_\lambda$. Then, as above, $\Gamma - \mathfrak{F}$ is the disjoint union of the digraphs $\Gamma_\lambda - \mathfrak{F}_\lambda$ and we obtain $m = \sum m_\lambda \leq \sum_{\lambda\in\Lambda} \mathcal{E}_\ell(S_\lambda)$. □

EXAMPLE 16. For an arbitrary natural number n, let Λ be an index set with $|\Lambda| = n$, and for each $\lambda \in \Lambda$, let S_λ be a finitely generated abelian group with $\mathcal{E}_\ell(S_\Lambda) = \mathcal{E}_r(S_\Lambda) = 1$. For example, take S_λ to be the free abelian group of rank $r_\lambda \geq 2$. Let $S = \vee S_\lambda$. Then S is a finitely generated, completely regular, commutative inverse semigroup with $\mathcal{E}_r(S) = \mathcal{E}_\ell(S) = n$.

If M is any monoid, $\mathrm{End}(M)$ is standard notation for the monoid of monoid endomorphisms of M. For $m \in M$ and $f \in \mathrm{End}(M)$, mathematicians sometimes find it convenient to write the argument m to the left of the function f and other times find it more convenient to write the argument on the right. Then for $f, g \in \mathrm{End}(M)$, the composition fg has, in general, two different values depending upon which notational convention is followed. We write $\mathrm{End}_\mathrm{r}(M)$ for $\mathrm{End}(M)$ when we write functions to the right of their arguments and we write $\mathrm{End}_\ell(M)$ for $\mathrm{End}(M)$ when we write functions to the left of their arguments. Then $\mathrm{End}_\ell(M)$ and $\mathrm{End}_\mathrm{r}(M)$ are duals of each other with respect to the functor ${}^{\mathrm{op}}$. We use $\mathrm{Monic}_\mathrm{r}(M)$ for the submonoid of $\mathrm{End}_\mathrm{r}(M)$ consisting of one-to-one endomorphisms and $\mathrm{Monic}_\ell(M)$ for the submonoid of $\mathrm{End}_\ell(M)$ consisting of one-to-one endomorphisms. We note that $\mathrm{Monic}_\mathrm{r}(M)$ and $\mathrm{Monic}_\ell(M)$ are also dual.

Let A and B be monoids and let $\Phi : A \to \mathrm{End}_\ell(B)$ be a monoid homomorphism. For consistency, we would ordinarily write $\Phi(a)$ for the endomorphism of B which is the image of $a \in A$ and then write $[\Phi(a)](b)$ for the value of this endomorphism at the element $b \in B$. When the monoid homomorphism Φ is understood from context, we will abbreviate $[\Phi(a)](b)$ as ${}^a b$. Since Φ is a monoid homomorphism, we have ${}^1 b = b$ and ${}^{a_1}({}^{a_2} b) = {}^{a_1 a_2} b$. Since $\Phi(a)$ is a monoid homomorphism for every $a \in A$, we have ${}^a 1 = 1$ and ${}^a(b_1 b_2) = ({}^a b_1)({}^a b_2)$. Similarly, if $\Phi : A \to \mathrm{End}_\mathrm{r}(B)$ is a monoid homomorphism, it is often convenient and unambiguous to abbreviate $(b)[(a)\Phi]$ as b^a and then observe that $b^1 = b$, $(b^{a_1})^{a_2} = b^{a_1 a_2}$, $1^a = 1$, and $(b_1 b_2)^a = (b_1^a)(b_2^a)$.

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Suppose that A and B are monoids and that $\Phi : A \to \mathrm{End}_r(B)$ is a monoid homomorphism. We define the monoid semidirect product $A \ltimes_\Phi B$ to have elements $\{(a, b) \mid a \in A, b \in B\}$ and multiplication $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1^{a_2}b_2)$. Similarly, if $\Phi : A \to \mathrm{End}_\ell(B)$ is a monoid homomorphism, we define the monoid semidirect product $B \rtimes_\Phi A$ to have elements $\{(b, a) \mid b \in B, a \in A\}$ and multiplication $(b_1, a_1)(b_2, a_2) = ((b_1)({}^{a_1}b_2), a_1a_2)$. It is easily verified that both multiplications are associative with respective identity elements $(1_A, 1_B)$ and $(1_B, 1_A)$. If $\Phi(a)$ (or $(a)\Phi$) is the identity endomorphism $b \mapsto b$ of B for every $a \in A$, then $A \ltimes_\Phi B$ is the monoid direct product $A \times B$ while $B \rtimes_\Phi A$ is the monoid direct product $B \times A$ and thus $A \ltimes_\Phi B \cong B \rtimes_\Phi A$ in this case. More generally, we have $A \ltimes_\Phi B \cong (B^{\mathrm{op}} \rtimes_\Phi A^{\mathrm{op}})^{\mathrm{op}}$. If A and B are commutative monoids, then $B^{\mathrm{op}} = B$, $A^{\mathrm{op}} = A$ and $A \ltimes_\Phi B \cong (B \rtimes_\Phi A)^{\mathrm{op}}$.

THEOREM 13. *Suppose that M_i is a finitely generated infinite monoid for $i = 1, 2$. If $\Phi : M_1 \to \mathrm{Monic}(M_2)$ is a monoid homomorphism, then $\mathcal{E}_r(M_1 \ltimes_\Phi M_2) = \mathcal{E}_\ell(M_2 \rtimes_\Phi M_1) = 1$.*

PROOF. Since $M_2 \rtimes_\Phi M_1 \cong (M_1^{\mathrm{op}} \ltimes_\Phi M_2^{\mathrm{op}})^{\mathrm{op}}$, it will be sufficient to prove that $\mathcal{E}_r(M_1 \ltimes_\Phi M_2) = 1$. Write M for $M_1 \ltimes_\Phi M_2$. For $i = 1, 2$, let X_i be a finite set of monoid generators for M_i. Let $X = \{(x, 1) \mid x \in X_1\} \cup \{(1, x) \mid x \in X_2\}$. Then X is a finite set of monoid generators for M.

For the proof below, the reader might find it useful to visualize and then generalize the Cayley digraph for a direct product of two infinite cyclic monoids.

Write Γ for the right Cayley digraph $\Gamma_r(M, X)$. Since X_i is a finite set for $i = 1, 2$, for every positive integer n, there can be only finitely many elements $m_i \in M_i$ with $L_{X_i}(m_i) < n$. Let $\mathfrak{F}$ be a finite subset of $V_\Gamma = M$. Since $\mathfrak{F}$ is finite, we may fix a natural number $N = N_{\mathfrak{F}}$, depending upon $\mathfrak{F}$, such that $\mathfrak{F} \subseteq \{(m_1, m_2) \mid L_{X_i}(m_i) < N$ for $i = 1, 2\}$. As a consequence, an element $(m_1, m_2) \in M$ is not in $\mathfrak{F}$ if either $L_{X_1}(m_1) \geq N$ or $L_{X_2}(m_2) \geq N$.

Suppose that $m \in M_1$ and $q, p \in M_2$ with $L_{X_2}(p) = t > 0$. Write p in the form $x_{j_1}x_{j_2} \cdots x_{j_t}$ with $x_{j_i} \in X_2$ for $1 \leq i \leq t$. Then we have a positive path in Γ of length t from (m, q) to (m, qp) with consecutive edges labelled by $(1, x_{j_i})$ for $1 \leq i \leq t$. For further reference, we may refer to this as the *vertical* path from (m, q) to (m, qp). If we further assume that $L_{X_1}(m) \geq N_{\mathfrak{F}}$, then all of the vertices of this path occur in $\Gamma - \mathfrak{F}$ and hence in the same component of $\Gamma - \mathfrak{F}$.

Suppose that $m \in M_2$ and $q, p \in M_1$ with $L_{X_1}(p) = t > 0$. Set $p = x_{j_1}x_{j_2} \cdots x_{j_t}$ with $x_{j_i} \in X_1$ for $1 \leq i \leq t$. Then there is a positive path in Γ of length t from (q, m) to (qp, m^p) with consecutive edges labelled by $(x_{j_i}, 1)$ for $1 \leq i \leq t$. For further reference, we may refer to this as the *oblique* path from (q, m) to (qp, m^p). We need to discuss hypotheses which will guarantee that this oblique path is in $\Gamma - \mathfrak{F}$.

Let k be a fixed natural number and $x \in X_1$ a fixed generator of M_1. Then $\{b \in M_2 \mid L_{X_2}(b) < k\}$ is a finite set and hence $\{b^x \mid b \in M_2, L_{X_2}(b) < k\}$ is also finite. For this k and x, define $\bar{k}$ by

$$\bar{k} = 1 + \max\{L_{X_2}(b^x) \mid b \in M_2, L_{X_2}(b) < k\}.$$

Then if $b \in M_2$ and $L_{X_2}(b) < k$, we also obtain that $L_{X_2}(b^x) < \bar{k}$, or equivalently, if $L_{X_2}(b^x) \geq \bar{k}$ then $L_{X_2}(b) \geq k$. We may slightly modify our definition for $\bar{k}$ and assume that $\bar{k} \geq k$. Thus, for every natural number k, there is a natural number $\bar{k} \geq k$ such that whenever $b \in M_2$ and $L_{X_2}(b^x) \geq \bar{k}$ then $L_{X_2}(b) \geq k$.

Similarly, suppose that $x \in X_1$ and that ℓ is any natural number. Since $x\Phi$ is one-to-one, there is a natural number $\hat{\ell} \geq \ell$, depending upon ℓ and x, such that whenever $b \in M_2$ and $L_{X_2}(b) \geq \hat{\ell}$, then $L_{X_2}(b^x) \geq \ell$.

We use the values $\bar{k}$ to keep an oblique path within $\Gamma - \mathfrak{F}$ by our choice of the path's final endpoint. We use the values $\hat{\ell}$ to keep an oblique path within $\Gamma - \mathfrak{F}$ by our choice of the path's initial endpoint.

Consider again oblique paths from (q, m) to (qp, m^p). We regard q as varying over M_1 and m as varying over M_2, but we want to fix $p \in M_1$ with $L_{X_1}(p) = t > 0$ and write $p = x_{j_1} \cdots x_{j_t}$ with $x_{j_i} \in X_1$ for $1 \leq i \leq t$.

Let $k_0 = N = N_{\mathfrak{F}}$ and choose $\bar{k}_0 \geq k_0$ such that $L_{X_2}(b) \geq k_0$ whenever $b \in M_2$ with $L_{X_2}(b^{x_{j_1}}) \geq \bar{k}_0$. Define k_1 to be $\bar{k}_0$. By induction on i, for $1 \leq i < t$, choose $\bar{k}_i \geq k_i$ such that whenever $b \in M_2$ with $L_{X_2}(b^{x_{j_{i+1}}}) \geq \bar{k}_i$, then $L_{X_2}(b) \geq k_i$. Define k_{i+1} to be $\bar{k}_i$. Then k_t depends upon both N and p. To emphasize this, we may write $k_{N,p}$ for k_t. Suppose that for the oblique path from (q, m) to (qp, m^p), we know that $L_{X_2}(m^p) \geq k_t$. Then it is routine to show, by induction on i, for each vertex $(qx_{j_1}x_{j_2} \cdots x_{j_{t-i}}, m^{x_{j_1}x_{j_2}\cdots x_{j_{t-i}}})$ on the path, that $L_{X_2}(m^{x_{j_1}x_{j_2}\cdots x_{j_{t-i}}}) \geq k_{t-i} \geq k_0 = N$ and that $L_{X_2}(m) \geq N$, so that all of these vertices are in $\Gamma - \mathfrak{F}$. Thus, the oblique path from (q, m) to (qp, m^p) is in $\Gamma - \mathfrak{F}$ provided that $L_{X_2}(m^p) \geq k_{N,p}$.

Let $\ell_0 = N = N_{\mathfrak{F}}$ and choose $\hat{\ell}_0 \geq \ell_0$ such that $L_{X_2}(b^{x_{j_t}}) \geq \ell_0$ whenever $b \in M_2$ with $L_{X_2}(b) \geq \hat{\ell}_0$. Define ℓ_1 to be $\hat{\ell}_0$. By induction on i, for $1 \leq i < t$, choose $\hat{\ell}_i \geq \ell_i$ such that whenever $b \in M_2$ with $L_{X_2}(b) \geq \hat{\ell}_i$, then $L_{X_2}(b^{x_{j_{t-i}}}) \geq \ell_i$. Define ℓ_{i+1} to be $\hat{\ell}_i$. Then ℓ_t depends upon both N and p. To emphasize this, we may write $\ell_{N,p}$ for ℓ_t. Suppose that for the oblique path from (q, m) to (qp, m^p), we know that $L_{X_2}(m) \geq \ell_t$. Then it is routine to show, by induction on i, that for each vertex $(qx_{j_1}x_{j_2} \cdots x_{j_i}, m^{x_{j_1}x_{j_2}\cdots x_{j_i}})$ on the path, that we have $L_{X_2}(m^{x_{j_1}x_{j_2}\cdots x_{j_i}}) \geq \ell_{t-i} \geq \ell_0 = N$, so that all of these vertices are in $\Gamma - \mathfrak{F}$. Thus, the oblique path from (q, m) to (qp, m^p) is in $\Gamma - \mathfrak{F}$ provided that $L_{X_2}(m) \geq \ell_{N,p}$.

Let C be an infinite component of $\Gamma - \mathfrak{F}$. Define sets I_C and J_C by $I_C = \{c \in M_1 \mid (c, d) \in C$ for infinitely many different $d \in M_2\}$ and $J_C = \{d \in M_2 \mid (c, d) \in C$ for infinitely many different $c \in M_1\}$. Below, we will show that I_C and J_C are nonempty, that I_C is a right ideal in M_1 and J_C is a right ideal in M_2 and finally that $I_C = M_1$ and $J_C = M_2$.

Let C_g and C_h be two infinite components of $\Gamma - \mathfrak{F}$. To prove that Γ has one end, it suffices to show that $C_g = C_h$ by exhibiting a vertex of Γ which is in $C_g \cap C_h$. Suppose that we know that $J_{C_g} = M_2$. Then $1 \in J_{C_g}$, so there are infinitely many $c \in M_1$ with $(c, 1) \in C_g$. We may then choose $p \in M_1$ with $(p, 1) \in C_g$ and $L_{X_1}(p) \geq N_{\mathfrak{F}}$. Suppose also that we know that $I_{C_h} = M_1$. Then $1 \in I_{C_h}$, so there are infinitely many $d \in M_2$ with $(1, d) \in C_h$. Choose $m \in M_2$ with $(1, m) \in C_h$ and $L_{X_2}(m) \geq \ell_{N,p}$. Since $L_{X_1}(p) \geq N_{\mathfrak{F}}$, the vertical path, labelled by m^p, in Γ from

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$(p, 1)$ to (p, m^p) is in $\Gamma - \mathfrak{F}$ and hence in C_g. Since $L_{X_2}(m) \geq \ell_{N,p}$, the oblique path in Γ from $(1, m)$ to (p, m^p) is in $\Gamma - \mathfrak{F}$ and hence in C_h. Then the vertex (p, m^p) is in both C_g and C_h, so $C_g = C_h$.

We want to show that, for any infinite component C of $\Gamma - \mathfrak{F}$, the set I_C is nonempty, is a right ideal in M_1, and is all of M_1.

If it were the case that there were only finitely many elements $c \in M_1$ with $(c, d) \in C$ for some $d \in M_2$, then, since C is infinite, for at least one such element, $\hat{c}$, there must be infinitely many different $d \in M_2$ with $(\hat{c}, d) \in C$ and then $\hat{c} \in I_C$. Suppose then that there are infinitely many different elements $c \in M_1$ with $(c, d) \in C$ for some $d \in M_2$. Then there is an element $(\hat{c}, \hat{\delta}) \in C$ with $L_{X_1}(\hat{c}) \geq N_{\mathfrak{F}}$. The vertical path from $(\hat{c}, 1)$ to $(\hat{c}, \hat{\delta})$ is in C, so $(\hat{c}, 1) \in C$. But then the vertical path from $(\hat{c}, 1)$ to $(\hat{c}, d)$ is in C for every $d \in M_2$, so $\hat{c} \in I_C$.

To show that I_C is a right ideal in M_1, it suffices to show that $cx \in I_C$ whenever $c \in I_C$ and $x \in X_1$. Choose $\hat{\ell} = \ell_{N,x}$ such that $L_{X_2}(d^x) \geq N$ whenever $d \in M_2$ and $L_{X_2}(d) \geq \hat{\ell}$. Since $c \in I_C$, there are infinitely many $d \in M_2$ with $(c, d) \in C$ and hence, for infinitely many of these, $L_{X_2}(d) \geq \hat{\ell}$. For each of these, the edge with label $(x, 1)$ from (c, d) to (cx, d^x) is in C. Since $x\Phi$ is one-to-one, these values for d^x are distinct and $cx \in I_C$.

Since I_C is a right ideal in M_1, we will obtain $I_C = M_1$ if we show that $1 = 1_{M_1} \in I_C$. Suppose that $\hat{c} \in I_C$ with $L_{X_1}(\hat{c}) \geq N_{\mathfrak{F}}$. Then, as above, $(\hat{c}, 1) \in C$ and $(\hat{c}, d) \in C$ for every $d \in M_2$. There are infinitely many $d \in M_2$ with $L_{X_2}(d) \geq \ell_{N,\hat{c}}$. For each of these, $(\hat{c}, d^{\hat{c}}) \in C$ and there is the oblique path from $(1, d)$ to $(\hat{c}, d^{\hat{c}})$ in C. If I_C is infinite, there is always some $\hat{c} \in I_C$ with $L_{X_1}(\hat{c}) \geq N_{\mathfrak{F}}$. Suppose instead that I_C is finite. Then, since I_C is a subsemigroup of M_1, it must contain an idempotent e. Then there are infinitely many $d \in M_2$ with $(e, d) \in C$, hence infinitely many such d with both $L_{X_2}(d) \geq \ell_{N,e}$ and $L_{X_2}(d^e) \geq k_{N,e}$. For the latter, we use the hypothesis that $e\Phi$ is one-to-one. Since $L_{X_2}(d) \geq \ell_{N,e}$, for each such d, the oblique path from (e, d) to $(e \cdot e, d^e) = (e, d^e)$ is in C for each of these infinitely many d. Since $L_{X_2}(d^e) \geq k_{N,e}$, the oblique path from $(1, d)$ to (e, d^e) is also in C for each of these infinitely many d and we see that $1_{M_1} \in I_C$.

We want to show, for any infinite component C of $\Gamma - \mathfrak{F}$, that $1_{M_2} \in J_C$, and that J_C is a right ideal in M_2 and hence is all of M_2. We now know that $I_C = M_1$, so there are infinitely many elements $c \in I_C$ with $L_{X_1}(c) \geq N_{\mathfrak{F}}$ and for each such c an element $d_c \in M_2$ with $(c, d_c) \in C$. For each such c, the vertical path from $(c, 1)$ to (c, d_c) is in C, so we have infinitely many different c with $(c, 1) \in C$ and $1 \in J_C$. Suppose that $d \in J_C$ and that $x \in X_2$. Then, for infinitely many $c \in M_1$, $(c, d) \in C$, and there are thus infinitely many such c with $L_{X_1}(c) \geq N_{\mathfrak{F}}$. For those c with $L_{X_1}(c) \geq N_{\mathfrak{F}}$, we can be sure that the edge labelled $(1, x)$ from (c, d) to (c, dx) is in C, so $(c, dx) \in C$ for infinitely many c and $dx \in J_C$. $\square$

EXAMPLE 17. In this example, we want to show that, in the previous theorem, the hypothesis that Φ has its range in $\mathrm{Monic}(M_2)$ rather than just in $\mathrm{End}(M_2)$ is necessary. We also see that the second conclusion of the Layer Lemma need not hold if T

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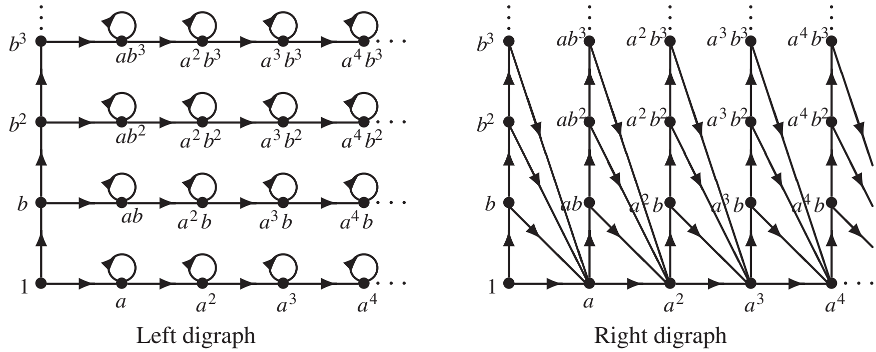

FIGURE 3. Left and right Cayley digraphs for $A \ltimes_{\Phi_0} B$.

is an infinite monoid. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be free monogenic monoids and $M = A \ltimes_{\Phi_0} B$. Here $a\Phi_0 = \theta_B$ where $b^m\theta_B = 1_B$ for every nonnegative integer m, hence θ_B is not one-to-one.

Then M has monoid presentation $\langle a, b : ba = a \rangle$ and every element of M can be represented by a unique word in the form $a^n b^m$ for nonnegative integers m, n. We illustrate the right and left Cayley digraphs for $A \ltimes_{\Phi_0} B$ in Figure 3. We remark that

$$\mathcal{E}_r(A \ltimes_{\Phi_0} B) = \mathcal{E}_\ell(A \ltimes_{\Phi_0} B) = \mathcal{E}_r(B \rtimes_{\Phi_0} A) = \mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = \infty.$$

Since $M^{\mathrm{op}} \approx B \rtimes_{\Phi_0} A$, we see that $\mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = \mathcal{E}_r(M) = \infty$. This shows that it is not necessarily the case that $\mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = 1$, as in Theorem 13, if $\Phi = \Phi_0$ nor that $\mathcal{E}_\ell(B \rtimes_{\Phi_0} A) = \mathcal{E}_\ell(A)$, as in the Layer Lemma, if B is infinite.

EXAMPLE 18. Our initial impression was that, with the hypotheses of the previous theorem, we should also be able to prove that $\mathcal{E}_\ell(M_1 \ltimes_\Phi M_2) = \mathcal{E}_r(M_2 \rtimes_\Phi M_1) = 1$. This is not valid. Here, we give an example where $\Phi : M_1 \to \mathrm{Monic}(M_2)$ is a monoid homomorphism but $\mathcal{E}_\ell(M_1 \ltimes_\Phi M_2) = \mathcal{E}_r(M_2 \rtimes_\Phi M_1) = \infty$.

Let $M_1 = \langle a \rangle$ be the free cyclic monoid with generator a. Let M_2 be the monoid having monoid presentation $\langle b, t \mid bt = t, t^2 = t \rangle$. Then one can show that every element of M_2 has a unique representative of the form $t^\varepsilon b^n$ where ε has value 0 or 1 and n is a nonnegative integer. We want to define a semidirect product, $M_1 \ltimes_\Phi M_2$, so we write functions to the right of their arguments. Define $\alpha : M_2 \to M_2$ by $b^n\alpha = b^n$ and $(tb^n)\alpha = tb^{n+1}$. Then α is a monoid homomorphism and is one-to-one. Since M_1 is free on a, we obtain a monoid homomorphism $\Phi : M_1 \to \mathrm{Monic}_{\mathrm{r}}(M_2)$ by setting $a\Phi = \alpha$. Then the semidirect product $M = M_1 \ltimes_\Phi M_2$ has elements $\{(a^m, t^\varepsilon b^n) \mid m \geq 0, n \geq 0, \varepsilon = 0, 1\}$ and is generated by $(a, 1)$, $(1, b)$ and $(1, t)$. If we identify $(a, 1)$, $(1, b)$ and $(1, t)$ with a, b and t, respectively, then M has a monoid presentation $\langle a, b, t \mid ta = atb, ba = ab, bt = t, t^2 = t \rangle$ and every element of M has a unique representative of the form $a^m t^\varepsilon b^n$ with $m \geq 0$, $n \geq 0$, $\varepsilon = 0, 1$. Write just Γ for the

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left Cayley digraph ${}_{\ell}\Gamma(\{a, b, t\}, M)$. We want to show that $\mathrm{e}_\infty(\Gamma) = \infty$. It is easily verified that

$$a \cdot (a^m t^\varepsilon b^n) = a^{m+1} t^\varepsilon b^n, \quad b \cdot (a^m b^n) = a^m b^{n+1},$$
$$b \cdot (a^m t b^n) = a^m t b^n, \quad t \cdot (a^m b^n) = a^m t b^{m+n}, \quad t \cdot (a^m t b^n) = a^m t b^n.$$

We thus account for all of the edges in Γ. To show that $\mathrm{e}_\infty(\Gamma) = \infty$, it suffices to find, for every natural number k, a finite subset $\mathfrak{F}_k \subseteq V_\Gamma$ such that $|\mathfrak{C}_\infty(\Gamma - \mathfrak{F}_k)| > k$.

Let $\mathfrak{F}_k = \{a^m t b^n \mid 0 \le m \le k, 0 \le n \le k\}$. For $0 \le i \le k$, let C_i be the full subgraph of Γ on the set $\{a^m t b^i \mid m > k\}$. We will be done when we show that each C_i is a component of $\Gamma - \mathfrak{F}_k$. (There is one more component of $\Gamma - \mathfrak{F}_k$, but we need not concern ourselves with it.) C_i is connected since we have an edge labelled by a from $a^m t b^i$ to $a^{m+1} t b^i$ for each $m > k$. For a vertex $v = a^m t b^i$ in C_i, the other edges having v as an initial vertex are loops labelled by b and t. Examining the account of the edges in Γ in the previous paragraph, we see that the only other edges in Γ having a vertex $v = a^m t b^i$ as a terminal vertex are edges with label t from $a^m b^n$ to $a^m t b^{m+n}$. But such an edge cannot have its terminal vertex in C_i if $m + n = i \le k$ and $m > k$.

COROLLARY 14. *Suppose that G_i is a finitely generated infinite group for $i = 1, 2$. If $\Phi : G_1 \to \mathrm{Aut}(G_2)$ is a group automorphism, then the group semidirect product $G_2 \rtimes_\Phi G_1$ has one end.*

PROOF. Group automorphisms are endomorphisms and are one-to-one. □

COROLLARY 15. *Suppose, for $i = 1, 2$, that M_i is an infinite monoid with a finite set of monoid generators X_i. Let $M = M_1 \times M_2$ be the monoid direct product. Then $\mathcal{E}_r(M) = \mathcal{E}_\ell(M) = 1$.*

PROOF. The direct product is a special case of Theorem 13 where Φ takes each element of M_1 to the identity automorphism of M_2. □

EXAMPLE 19. Let $m, n, \hat{m}, \hat{n}$ be arbitrary natural numbers. We construct a monoid M having a submonoid J such that $\mathcal{E}_r(M) = \hat{m}$, $\mathcal{E}_\ell(M) = \hat{n}$, $\mathcal{E}_r(J) = m$ and $\mathcal{E}_\ell(J) = n$. Let $J = J_{n,m}$ be the monoid of Example 10 and let P be the direct product of J with any infinite monoid of the form S^1 for some semigroup S. Then P contains a submonoid which is isomorphic to J, and P, like J and S^1, contains no nontrivial left or right units. By the corollary, $\mathcal{E}_r(P) = \mathcal{E}_\ell(P) = 1$. For any natural number k and generator y, let $T(y, k)$ be the the monogenic monoid having monoid presentation $\langle y : y^k = y^{k-1} \rangle$. The monoid $M = T(u, \hat{m}) \rtimes_{\Phi_0} (T(v, \hat{n}) \rtimes_{\Phi_0} P^{\mathrm{op}})^{\mathrm{op}}$ contains a submonoid isomorphic to P. With two applications of the Layer Lemma, we obtain $\mathcal{E}_r(M) = \hat{m}$ and $\mathcal{E}_\ell(M) = \hat{n}$.

4. Subsemigroups of free semigroups

Our principal results in this section are about the number of ends of subsemigroups of free semigroups. Theorem 18 says that every commutative subsemigroup of a free semigroup has one end. Theorem 20 says that every finitely generated noncommutative subsemigroup of a free semigroup has infinitely many ends. The analogous results for submonoids of free monoids follow immediately by adjoining the empty word. The next two lemmas will be used in the proof. We regard the first lemma as elementary and well known. The group versions in [6, Exercise 1.4.6] and [7, Proposition I.2.17] are easily modified to obtain the semigroup version.

LEMMA 16. *Suppose that F is the free semigroup on the alphabet A and that u, $v \in F$. If $uv = vu$, then there are natural numbers m, n and an element $w \in F$ such that $u = w^m$ and $v = w^n$.*

LEMMA 17. *If S is any subsemigroup of the additive semigroup $\mathbb{N}$ of natural numbers, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = 1$.*

PROOF. Let S be a subsemigroup of the additive semigroup $\mathbb{N}$. Since S is commutative, it is clear from Proposition 10 that $\mathcal{E}_\ell(S) = \mathcal{E}_r(S)$ when these are defined. Since we have only defined $\mathcal{E}_\ell(S)$ and $\mathcal{E}_r(S)$ for finitely generated semigroups S, we need to show that S is finitely generated.

If all of the elements of S are divisible by some natural number $d > 1$, assume that d is the largest natural number dividing all of the elements of S and define $\hat{S}$ by $\hat{S} = \{s/d \mid s \in S\}$. Then $\hat{S}$ is a subsemigroup of $\mathbb{N}$ which is isomorphic to S. We can replace S by $\hat{S}$ and assume that the greatest common divisor of the set of all elements of S is 1. It is easy to see, using elementary number theory, that S then contains all but finitely many natural numbers. If we write $n_0 - 1$ for the greatest natural number that is not in S, then we may write $S = X_0 \cup \{n \in \mathbb{N} \mid n \geq n_0\}$ for some finite set $X_0 \subseteq \mathbb{N}$. We then see that S is generated by the finite set $X = X_0 \cup \{n \in \mathbb{N} \mid n_0 \leq n < 2n_0\}$. Write Γ for $\Gamma_r(S, X)$.

Now let $\mathfrak{F}$ be any finite subset of vertices of Γ, let m be the largest element in $\mathfrak{F}$ and choose any $k \in \mathbb{N}$ which satisfies $m < kn_0$. Then the set $C = \{n \mid n \geq (k+1)n_0\}$ is an infinite subset of $\Gamma - \mathfrak{F}$ having a finite complement in $\mathbb{N}$, so we will be done when we show that C is contained in the component of $\Gamma - \mathfrak{F}$ which contains kn_0. For an arbitrary element $n \in C$, write $n = qn_0 + r$, where $0 \leq r < n_0$ and $q \geq k + 1$. Then we have a path of length $q - k$ from kn_0 to n in $\Gamma - \mathfrak{F}$ having 1 edge labelled by $n_0 + r$ and $q - k - 1$ edges labelled by n_0. □

THEOREM 18. *If S is a commutative subsemigroup of a free semigroup, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = 1$.*

PROOF. Using Lemma 16, it can be shown that any set of pairwise commuting elements in a free semigroup must consist of powers of a single word. Hence S is isomorphic to a subsemigroup of $\mathbb{N}$ and the conclusion follows from Lemma 17. □

LEMMA 19. *Let F be the free semigroup on the alphabet A and let S be a finitely generated subsemigroup of F with finite set of generators X. Let Γ be the right Cayley graph $\Gamma_r(S, X)$. If $\mathfrak{F}$ is a finite subset of S and w is a element of $S - \mathfrak{F}$, write C_w for*

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the component of $\Gamma - \mathfrak{F}$ containing w. If the length, $L_A(w)$, of w on the alphabet A is minimal among elements of $S - \mathfrak{F}$, then w is a prefix of every vertex in C_w.

PROOF. Let u be an arbitrary element of C_w. We use induction on the length of a path in C_w from w to u to show that w is a prefix of u. The base case, where $u = w$, is obvious. Suppose that $u \neq w$ and write p for the vertex before u on a path from w to u in C_w. By the induction hypothesis, w is a prefix of p and $p = wp'$ for some, possibly empty, word p'. If $u = px = wp'x$ for some generator $x \in X$, we are done. If $wp' = p = ux$, then by the minimality of $L_A(w)$ among elements of $S - \mathfrak{F}$, it must be true that $L_A(w) \leq L_A(u)$ and again w is a prefix of u. □

THEOREM 20. *If S is a finitely generated subsemigroup of a free semigroup and S is not commutative, then $\mathcal{E}_\ell(S) = \mathcal{E}_r(S) = \infty$.*

PROOF. As in Lemma 19, we write F for the free semigroup on the alphabet A, write X for some finite set of generators for the subsemigroup S of F, and Γ for the right Cayley graph $\Gamma_r(S, X)$. We have hypothesized that S is not commutative and we choose two elements $x, y \in S$ such that $xy \neq yx$. It suffices to exhibit, for every natural number n, a finite subset $\mathfrak{F}$ of S such that $\Gamma - \mathfrak{F}$ has at least $n + 1$ infinite components. Let $n \in \mathbb{N}$ be arbitrary. For integers $0 \leq i \leq n$, define the element $w_i \in S$ by $w_i = x^i y x^{n-i}$. All of the w_i have the same length, $nL_A(x) + L_A(y)$. For notational convenience, we write ℓ for this length. By the cancellative properties in the free semigroup S, all of the w_i are distinct. Let $\mathfrak{F} = \{s \in S \mid L_A(s) < \ell\}$. By construction, each w_i must have minimal length in $S - \mathfrak{F}$ and, using Lemma 19, we see that elements w_i must occur in distinct components of $\Gamma - \mathfrak{F}$. It is easily seen that these components are infinite. □

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