Appendix D

Dirac equation and matrices

D.1 Definition and notations

If ψ is a generic notation of a fermion field, it can be expressed in terms of the usual annihilation and creation operators as:

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E} \sum_{\lambda} [u(\vec{p}, \lambda) a(\vec{p}, \lambda) e^{-ipx} v(\vec{p}, \lambda) b^{\dagger}(\vec{p}, \lambda) e^{ipx}] \tag{D.1}$$

where the integration is over the mass hyperboloid with $p^2 = m^2$ and $p^0 > 0$. λ is the two possible fermion helicities. The annihilation and creation operators satisfy the commutation relations:

$$\begin{aligned} [a(p), a^{\dagger}(p')] = [b(p), b^{\dagger}(p')] = (2\pi)^3 2E \delta^3(p' - p)\,, \\ [a(p), a(p')] = 0 = [b(p), b(p')]\,. \end{aligned} \tag{D.2}$$

The fermion spinors $u(p)$ (particle) and $v(p)$ (anti-particle) of mass m obey the Dirac equation:

$$\begin{aligned} (\hat{p} - m) u(p) = 0 = \bar{u}(p)(\hat{p} - m)\,, \\ (\hat{p} + m) v(p) = 0 = \bar{v}(p)(\hat{p} - m)\,, \end{aligned} \tag{D.3}$$

and normalized as:

$$\bar{u}(\vec{p}, \lambda) u(\vec{p}, \lambda) = 2m = -\bar{v}(\vec{p}, \lambda) v(\vec{p}, \lambda) \tag{D.4}$$

with:

$$\begin{aligned} \bar{u} &= u^{\dagger} \gamma^0 \\ \bar{v} &= v^{\dagger} \gamma^0\,, \\ \hat{p} &= \gamma_{\mu} p^{\mu} = \gamma_0 p^0 - \gamma \cdot \mathbf{p}\,, \end{aligned} \tag{D.5}$$

where γ_{μ} are the Dirac matrices. In four dimensions, these matrices can be defined as:

$$\gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ -\sigma_{\mu} & 0 \end{pmatrix} \quad \text{for } \mu = 1, 2, 3 \quad \gamma_0 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{D.6}$$

in terms of the Pauli matrices σ:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{D.7}$$

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They obey the properties:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \,, \qquad \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu] \,, \tag{D.8}$$

and:

$$(\gamma_5)^2 = \mathbf{1} \,, \quad \text{and} \quad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5 \,, \tag{D.9}$$

with the definition:

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{D.10}$$

or:

$$\gamma_5 = \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \,. \tag{D.11}$$

The Dirac matrices are (anti)hermitians:

$$\gamma_\mu = -\gamma_\mu^+ \,, \qquad \mu = 1, 2, 3 \,, \qquad \gamma_0^+ = \gamma_0 \quad \text{and} \quad \gamma_5^+ = \gamma_5 \,. \tag{D.12}$$

D.2 CPT transformations

The action of the operators:
C $\equiv$ charge conjugation, P $\equiv$ parity transformation, T $\equiv$ time reversal ,
on the fermion field $\psi(t, \vec{r})$ are:

$$\begin{aligned}
\mathrm{C}\ \psi(t,\vec{r}) &= \gamma_2\psi^\dagger(t,\vec{r}) \\
\mathrm{T}\ \psi(t,\vec{r}) &= -i\gamma_1\gamma_3\psi^\dagger(-t,\vec{r}) \\
\mathrm{PT}\ \psi(t,\vec{r}) &= \gamma_0\gamma_1\gamma_3\psi^\dagger(-t,-\vec{r}) \\
\mathrm{CPT}\ \psi(t,\vec{r}) &= \gamma_2\gamma_0\gamma_1\gamma_3\psi(-t,-\vec{r}) \\
&= i\gamma_5\psi(-t,-\vec{r}) \,,
\end{aligned} \tag{D.13}$$

where:

$$\psi^\dagger = \bar{\psi}\gamma_0 \,. \tag{D.14}$$

D.3 Polarizations

In the evaluation of unpolarized cross-section, one has to sum over polarizations of, for example, fermions:

$$\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = \hat{p} + m \,, \qquad \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) = \hat{p} - m \,, \tag{D.15}$$

while for polarized cross-section, one inserts the projection matrices:

$$\begin{aligned}
u\left(p, \lambda = \pm\frac{1}{2}\right)\bar{u}\left(p, \lambda = \pm\frac{1}{2}\right) &= \frac{1}{2}(\hat{p} + m)\left(\frac{1 \pm \gamma_5\hat{s}}{2}\right) \,, \\
v\left(p, \lambda = \pm\frac{1}{2}\right)\bar{v}\left(p, \lambda = \pm\frac{1}{2}\right) &= \frac{1}{2}(\hat{p} - m)\left(\frac{1 \pm \gamma_5\hat{s}}{2}\right) \,,
\end{aligned} \tag{D.16}$$

where: s is the polarization four-vector of the (anti-)particle with energy-momnetum p:

$$s \cdot p = 0 \quad \text{and} \quad s^2 = -1 \,. \tag{D.17}$$

For a photon or massless vector boson, the polarization is transverse:

$$\epsilon^\mu = (0, \vec{\epsilon}) \quad \text{with} \quad \vec{p} \cdot \vec{\epsilon} = 0 \,. \tag{D.18}$$

For unpolarized cross-section involving (massless) photons, one has to sum over polarizations:

$$\sum_{polar.} \epsilon^*_\mu \epsilon^\mu = -g_{\mu\nu} \,. \tag{D.19}$$

D.4 Fierz identities

In some calculations, it is useful to arrange products of fermion bilinears using Fierz identities. Denoting by ψ_i the field of a fermion i, one has in four dimensions:

$$(\bar{\psi}_1 \psi_4)(\bar{\psi}_3 \psi_2) = \frac{1}{4} \sum_\mu (\bar{\psi}_1 \gamma_\mu \psi_2)(\bar{\psi}_3 \gamma^\mu \psi_4) \,. \tag{D.20}$$

Similar relation can be obtained by the substitution:

$$\psi_4 \to \gamma_\nu \psi_4 \,, \qquad \psi_2 \to \gamma_\rho \psi_2 \,, \tag{D.21}$$

and by using the decomposition:

$$\gamma_\mu \gamma_\nu = \frac{1}{4} \sum_\sigma (Tr\ \gamma_\mu \gamma_\nu \gamma_\sigma) \gamma^\sigma \,. \tag{D.22}$$

A typical Fierz rearrangement is the one of weak four-fermion operator:

$$(\bar{\psi}_{1L} \gamma^\mu \psi_{2L})(\bar{\psi}_{3L} \gamma_\mu \psi_{4L}) = -(\bar{\psi}_{1L} \gamma^\mu \psi_{4L})(\bar{\psi}_{3L} \gamma_\mu \psi_{2L}) \tag{D.23}$$

where:

$$\psi_{iL} \equiv \frac{1}{2}(1 - \gamma_5) \psi_i \,. \tag{D.24}$$

Additional relations can be obtained by using:

$$(\sigma_\mu)_{\alpha\beta} (\sigma^\mu)_{\gamma\delta} = 2 \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} \,. \tag{D.25}$$

D.5 Dirac algebra in n-dimensions

The (anti)-commutation properties of the Dirac matrices in four dimensions given in Eq. (D.8) are maintained, but the algebra becomes:[1]

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= n\mathbf{1} = g_{\mu\nu} g^{\mu\nu} \,, \\
\gamma_\mu \gamma_\alpha \gamma^\mu &= (2 - n) \gamma_\alpha \,, \\
\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu &= 4 g_{\alpha\beta} \mathbf{1} + (n - 4) \gamma_\alpha \gamma_\beta \,, \\
\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma^\mu &= -2 \gamma_\gamma \gamma_\beta \gamma_\alpha - (n - 4) \gamma_\alpha \gamma_\beta \gamma_\gamma \,.
\end{aligned} \tag{D.26}$$

[1] See also the discussions in Section 8.2 for different aspects of dimensional regularization.

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The traces in n dimensions can be chosen to be the same as in four dimensions. The usual properties are:

$$Tr\ \mathbf{1} = 4 \tag{D.27}$$

and:

$$\begin{aligned}
&Tr\ (\gamma^{\mu_1}\dots\gamma^{\mu_m}) = 0 \quad \text{for } m \text{ odd}\ ,\\
&Tr\ (\gamma^{\mu_1}\dots\gamma^{\mu_m}) = -Tr\ (\gamma^{\mu_m}\gamma^{\mu_1}\dots\gamma^{\mu_{m-1}})\\
&\qquad\qquad + 2\sum_{i=1}^{m-1}(-1)^{i+1}Tr\ (\gamma^{\mu_1}\dots\gamma^{\mu_{i-1}}\gamma^{\mu_{i+1}}\gamma^{\mu_{m-1}})\, g_{\mu_i\mu_m}\ .
\end{aligned} \tag{D.28}$$

Therefore, one can deduce:

$$\begin{aligned}
&Tr\ \gamma_\mu\gamma_\nu = 4g_{\mu\nu}\ ,\\
&Tr\ \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})\ ,\\
&Tr\ \gamma_{\lambda\mu\nu\rho\sigma\tau} = g_{\lambda\mu}T_{\nu\rho\sigma\tau} - g_{\nu\lambda}T_{\mu\rho\sigma\tau} + g_{\lambda\rho}T_{\mu\nu\sigma\tau} - g_{\lambda\sigma}T_{\mu\nu\rho\tau} + g_{\lambda\tau}T_{\mu\nu\rho\sigma}\ ,
\end{aligned} \tag{D.29}$$

with:

$$\gamma_{\lambda\mu\nu\rho\sigma\tau} \equiv \gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau\ , \qquad T_{\mu\nu\rho\sigma} \equiv Tr\ \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\ . \tag{D.30}$$

The definition of γ_5 is more delicate in n-dimensions. There are many definitions in the literature (see e.g. [116] and the review in [2]). These definitions are good if the corresponding Green's functions satisfy constraints imposed by the Ward identities, and do not induce *unphysical* anomalous term [116,119], which cannot be absorbed in the Lagrangian counterterms. The most convenient and unambiguous definition is the one encountered in four dimensions, which is either the one in Eq. (D.10) or the one in Eq. (D.11), although the one in Eq. (D.10) does not exist for $n < 4$. In both cases, the most important properties are:

$$(\gamma_5)^2 = \mathbf{1}, \qquad \gamma_5^\dagger = \gamma_5, \quad \text{and} \quad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5\ . \tag{D.31}$$

and:

$$[\gamma_5, \sigma_{\mu\nu}] = 0 \quad \text{and} \quad \gamma_5\sigma_{\mu,\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}\ . \tag{D.32}$$

The traces involving γ_5 are:

$$\begin{aligned}
&Tr\ \gamma_5 = 0\ ,\\
&Tr\ \gamma_5\gamma_\mu\gamma_\nu = 0\ ,\\
&Tr\ \gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4i\epsilon_{\mu\nu\rho\sigma}\ ,\\
&Tr\ \gamma_5\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau = 4i[g_{\mu\lambda}\epsilon_{\nu\rho\sigma\tau} - g_{\lambda\nu}\epsilon_{\mu\rho\sigma\tau} + g_{\mu\nu}\epsilon_{\lambda\rho\sigma\tau} + g_{\sigma\tau}\epsilon_{\lambda\mu\nu\rho}\\
&\qquad\qquad - g_{\rho\tau}\epsilon_{\lambda\mu\nu\sigma} + g_{\rho\sigma}\epsilon_{\lambda\mu\nu\tau}]\\
&Tr\ \gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_m} \ = \ \text{for } m \text{ odd}\ .
\end{aligned} \tag{D.33}$$

Finally, in order to complete the presentation of the Dirac algebra in n dimensions, it is also useful to remind the hermiticity:

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger\ , \qquad \gamma^0\gamma_5\gamma^0 = -\gamma_5^\dagger = -\gamma_5\ , \tag{D.34}$$

and the parity properties:

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \qquad C\gamma_5 C^{-1} = \gamma_5^T \ ,$$
$$C\sigma_{\mu\nu} C^{-1} = -\sigma_{\mu\nu}^T \qquad C(\gamma_5\gamma_\mu)C^{-1} = (\gamma_5\gamma_\mu)^T \ , \tag{D.35}$$

where C is the charge conjuguate operator normalized as:

$$C^2 = -1 \ . \tag{D.36}$$

D.6 The totally anti-symmetric tensor

The totally anti-symmetric tensor has the same definition as in four dimensions:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 0, & \text{if two indices are equal} \\ -1, & \text{if } \mu\nu\rho\sigma = 0123 \\ +1, & \text{if } \mu\nu\rho\sigma = 1230 \ , \end{cases} \tag{D.37}$$

while one can choose its properties as:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\nu\alpha\beta} = -(n-3)(n-2)(n-1)g_\mu^\rho \ ,$$
$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = -(n-3)(n-2)\left(g_\mu^\rho g_\nu^\sigma - g_\nu^\rho g_\mu^\sigma\right) \ ,$$
$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\tau\beta} = -(n-3)\begin{vmatrix} g_\mu^\rho & g_\mu^\sigma & g_\mu^\tau \\ g_\nu^\rho & g_\nu^\sigma & g_\nu^\tau \\ g_\alpha^\rho & g_\alpha^\sigma & g_\alpha^\tau \end{vmatrix} \tag{D.38}$$

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