INDUCED REPRESENTATIONS AND ALTERNATING GROUPS

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Introduction. This paper is based on part of the thesis of one of the authors **(5)**, submitted at the University of Toronto in 1963. In the first part of the paper a result on induced representations **(2, 4, 9)** is generalized slightly and a number of corollaries are derived. In the rest of the paper a special case of this result is applied to put the representation theory of the alternating group on a par with that of the symmetric group. A knowledge of the representation theory of S_n **(7)** on the part of the reader is assumed.

1. Let G be a group of finite order $[G:1]$ and Λ be an algebraically closed field of characteristic not dividing the order of G. We denote by $\otimes$ the Kronecker product of two matrices. If T and S are any two irreducible representations of G, consider the group $[T \otimes S]G$ generated by all elements of the form $[Tu, Sv] = Tu \otimes Sv$, where u and v are elements of G and Tu is the image of u in T of G. The group $[T \otimes S]G$ is not a representation of G but of $G \times G$. We call $[T \otimes S]G$ the *outer tensor product* of T and S of G. The diagonal subgroup $(T \otimes S)G$ of $[T \otimes S]G$, which consists of elements of the form $(Tu, Su) = (T \otimes S)u = Tu \otimes Su$, is a representation of G and is called the *inner tensor product* of T and S of G. Such a representation is, in general, reducible over Λ, since we assume Λ to be a splitting field for G and for all its subgroups.

Next if p and q are any two fixed elements of G, consider the group $[T_p \otimes S_q]G$ generated by all elements of the form $[T_p\, u, S_q\, v] = T(pup^{-1}) \otimes S(qvq^{-1})$. The subgroup of this group consisting of all elements of the form $(T_p\, u, S_q\, u) = T(pup^{-1}) \otimes S(quq^{-1})$ may be denoted by $(T_p \otimes S_q)G$. Then, since the characters are equal, we have the following lemma.

1.1. LEMMA. $[T \otimes S]G \cong [T_p \otimes S_q]G$ *and* $(T \otimes S)G \cong (T_p \otimes S_q)G$.

The representation T of G restricted to a subgroup H will be denoted by $\{T\}_H$ and, conversely, if S is a representation of H, then the representation of G induced by S will be denoted by $\{S\}^G$.

Let T be an irreducible representation of G over a field of characteristic 0. If I is the identity representation of the subgroup H, then **(9)**

1.2 $$(\{I\}^G \otimes T)G \cong \{\{T\}_H\}^G.$$

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A more general result is obtained by replacing the identity representation I by any representation S of H in the form **(2, 4)**

1.3 $$(\{S\}^G \otimes T)G \cong \{(S \otimes \{T\}_H)H\}^G.$$

Robinson, Taulbee, and Littlewood made use of 1.2 and 1.3 to decompose the inner tensor product representation of two irreducible representations of the symmetric group. In the following we generalize 1.3 and obtain a number of corollaries.

1.4. THEOREM. *Let F, H, and $K = F \cap H$ be subgroups of a finite group G such that $[G:F] = [H:K]$ and H contains a set of coset representatives of F in G. If T and S are any representations of G and F respectively over the field Λ, then*

$$\{(\{S\}^G \otimes T)G\}_H \cong \{\{(S \otimes \{T\}_F)F\}_K\}^H.$$

Proof. Let $[G:F] = n$ and $g_1(=1), g_2, \ldots, g_n$ be a set of coset representatives of F in G, which are also elements of H. Without loss of generality we may assume that $g_2, g_3, \ldots, g_n$ do not belong to K. Then we have the unique decomposition

$$G = \bigcup_{r=1}^{n} Fg_r$$

into disjoint cosets of F in G, so that

$$H \cap G = H \cap \left\{\bigcup_{r=1}^{n} Fg_r\right\} = \bigcup_{r=1}^{n} (H \cap Fg_r) = \bigcup_{r=1}^{n} (H \cap F)g_r.$$

Since $H \cap G = H$ and $H \cap F = K$, we obtain the unique decomposition

$$H = \bigcup_{r=1}^{n} Kg_r$$

of H into disjoint right cosets of K in H. Moreover, if we now define a mapping ϕ: $Fg_r \to Kg_r$, then ϕ is one-to-one and ϕ^{-1}: $Kg_r \to Fg_r$ exists, so that there is a one-to-one relation between the cosets Fg_r of F in G and the cosets Kg_r of K in H. This correspondence ensures that the permutation (matrix) representation of G induced by the identity representation of F, restricted to H, is identical (not just isomorphic) with the permutation (matrix) representation of H induced by the identity representation of K.

Denote by $C^G(g)$ the conjugate class of G to which the element g belongs and by $[C^G(g) \cap F:C^G(g)]$ the ratio of the number of elements of $C^G(g) \cap F$ to the number of elements of $C^G(g)$. Then the permutation character of g in $\{I\}^G$ is

1.5 $$[G:F][C^G(g) \cap F:C^G(g)].$$

If we now restrict ourselves to elements of the subgroup H, then 1.5 takes the form

1.6 $$\begin{aligned}\{[G:F][C^G(g) \cap F:C^G(g)]\}_H &= [G \cap H:F \cap H][C^G(g) \cap F \cap H:C^H(g)]\\ &= [H:K][C^H(g) \cap K:C^H(g)],\end{aligned}$$

where g is an element of H and $[G:F] = [H:K]$. If the character of an element h of H in S of F is $\zeta^S(h)$, then from 1.6 the character on each side of the desired relation is

$$[H:K][C^H(h) \cap K:C^H(h)]\zeta^S(h)\zeta^T(h),$$

where $\zeta^S(h)$ is 0 if h lies in $(G - F) \cap H$, i.e. if h lies in $H - K$, the complement of K in H, which proves 1.4.

Next, if we set $T = I$ in 1.4, the result reduces to

1.7 $$\{\{S\}^G\}_H = \{\{S\}_K\}^H.$$

If $F \cap H = 1$, the right side gives a regular representation of H with multiplicity equal to deg (S).

Our next corollary is a counterpart of Mackey's subgroup theorem **(3)** and gives information about the representation on the right side of 1.7, which may be stated as follows:

Let N be a ΛF-module, where Λ is any field and ΛF is the group algebra of F over Λ. For each (F, H)-double coset $D = FaH$, $(a \otimes N)$ is a left $\Lambda\bar{F}$-module for the subgroup $\bar{F} = aFa^{-1} \cap H$ of F and $N(D) = \{(a \otimes N)_{\bar{F}}\}^H$ is a left ΛH-module which depends only on the double coset D. Moreover,

(1.8) $$\{\{N\}^G\}_H = \sum_D N(D)$$

is a left ΛH-module, where the sum is taken over all (F, H)-double cosets D in G.

From 1.7 it follows that $\{\{S\}_K\}^H$ also depends on the double coset decomposition of G with respect to F and H, which we may state in the form of the following corollary.

1.9. Corollary. *Let N be a ΛF-module of F and*

$$G = \bigcup_{i=1}^{t} Fa_i H,$$

where t is the number of double cosets. Then

$$\{\{N\}_K\}^H = \sum_{i=1}^{t} N(D_i),$$

where $N(D_i) = \{(a_i \otimes N)_{\bar{F}}\}^H$ with $\bar{F} = a_i Fa_i^{-1} \cap H$.

From now on we explicitly assume that Λ is a field whose characteristic does not divide the order $[G:1]$ of G. If θ and ϕ are any two class functions belonging to the algebra cf(G) of all class functions of G, an inner product may be defined thus:

(1.10) $$\langle\theta, \phi\rangle = \frac{1}{[G:1]} \sum_g \theta(g)\phi^*(g),$$

where ϕ^* is the conjugate of ϕ. If ϕ is an irreducible character of G, then $\langle\theta, \phi\rangle$ is the multiplicity of the character ϕ in θ.

1.11. COROLLARY. *If θ is a common irreducible character of F and K, and ϕ is an irreducible character of H, then*

$$\langle\{\{\theta\}^G\}_H, \phi\rangle = \langle\{\theta\}_K, \{\phi\}_K\rangle.$$

Proof. Since θ is an irreducible character of K, Frobenius' Reciprocity Formula gives

$$\langle\{\theta\}_K, \{\phi\}_K\rangle = \langle\{\{\theta\}_K\}^H, \phi\rangle.$$

But $\{\{\theta\}_K\}^H = \{\{\theta\}^G\}_H$ from 1.7, where on the right side θ is viewed as a representation of F. Hence, we have

$$\langle\{\{\theta\}^G\}_H, \phi\rangle = \langle\{\theta\}_K, \{\phi\}_K\rangle.$$

2. In this section, let Λ be the field of complex numbers and ΛS_n and ΛA_n be the group algebras of the symmetric group S_n and the alternating group A_n. Each class of conjugate elements of S_n is defined by means of a partition $(\lambda) = (\lambda_1\lambda_2\ldots\lambda_h)$; $n = \lambda_1 + \lambda_2 \ldots + \lambda_h$ and $\lambda_i \geqslant \lambda_{i+1} \geqslant 0$. We use the symbol (λ) to denote the conjugate class of S_n determined by the corresponding partition. Corresponding to each class (λ), there exists a uniquely defined irreducible representation $[\lambda]$ and a (left) irreducible ΛS_n-module defined in term of the Young diagram

$$[\lambda] \quad \begin{array}{ll} \cdot\;\cdot\;\cdot\qquad\cdot\;\cdot & (\lambda_1 \text{ nodes}) \\ \cdot\;\cdot\;\cdot\qquad\cdot & (\lambda_2 \text{ nodes}) \\ \\ \cdot\;\cdot\;\cdot & (\lambda_h \text{ nodes}) \end{array}$$

where we use the same symbol $[\lambda]$ to denote the corresponding Young diagram. The Young diagram $[\lambda]$ can also be described by a Frobenius symbol

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_h \\ b_1 & b_2 & \ldots & b_h \end{bmatrix},$$

where a_i is the number of nodes to the right of the diagonal in the ith row and b_i is the number of nodes below the diagonal in the ith column. The diagram $[\lambda']$, obtained from $[\lambda]$ by interchanging rows and columns, is the conjugate of $[\lambda]$ and corresponds to the conjugate representation of $[\lambda]$ and the dual of the irreducible ΛS_n-module defined by $[\lambda]$. If $[\lambda] = [\lambda']$, then $[\lambda]$ is a self-conjugate diagram and in this case we call the conjugate class $(2\lambda_1 - 1, 2\lambda_2 - 3, 2\lambda_3 - 5, \ldots)$ a splitting class of S_n. Frobenius **(1)** proved the following theorem.

2.1. THEOREM. *If the diagram $[\lambda] \neq [\lambda']$, then the representations $[\lambda]$ and $[\lambda']$ of S_n restricted to A_n are equivalent irreducible representations of A_n, while if the diagram $[\lambda] = [\lambda']$ the representation $[\lambda]$ of S_n restricted to A_n splits into two (conjugate) irreducible representations $[\lambda]^+$ and $[\lambda]^-$ of the same degree.*

Thus, every irreducible character of a self-conjugate representation of S_n

is the sum of two characters of A_n. The class $(2\lambda_1 - 1, 2\lambda_2 - 3, \ldots) = (p_1, p_2, \ldots, p_h)$ for which the character $\chi^{\lambda}{}_{(p)}$ of S_n is

$$\chi^{\lambda}{}_{(p)} = (-1)^{\frac{1}{2}(p_1 p_2 \ldots p_h - 1)}$$

splits into two classes $(p)^+$ and $(p)^-$ in A_n. The corresponding characters of the two classes $(p)^+$ and $(p)^-$ in $[\lambda]^+$ are

$$\zeta, \zeta^* = \tfrac{1}{2}\{(-1)^{\frac{1}{2}(p_1 p_2 p_3 \ldots - 1)} \pm \sqrt{p_1 p_2 p_3 \ldots}\,(i)^{\frac{1}{2}(p_1 p_2 p_3 \ldots - 1)}\}, \tag{2.2}$$

which are interchanged between the two classes in $[\lambda]^-$.

2.3. Lemma. *The characters of the representations* $[\lambda]^+$ *and* $[\lambda]^-$ *of* A_n *are real or complex according as the number of nodes above or below the diagonal of the Young diagram* $[\lambda]$ *is even or odd respectively.*

Proof. From 2.2, the character of the representation $[\lambda]^+$ is real or complex according as $\frac{1}{2}(p_1 p_2 p_3 \ldots - 1)$ is even or odd respectively. The proof turns out to be trivial if we express the Young diagram in terms of the Frobenius symbol

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_h \\ a_1 & a_2 & \ldots & a_h \end{bmatrix}.$$

Then $\zeta^{\lambda^+}{}_{(p)^+}$ is real or complex according as

$$\tfrac{1}{2}[(2a_1 + 1)(2a_2 + 1) \ldots (2a_h + 1) - 1]$$

is even or odd, i.e. according as

$$a_1 + a_2 + a_3 + \ldots + a_h$$

is even or odd. But this is precisely the number of nodes above or below the diagonal of the diagram so that the lemma is proved.

From now on we shall use the same symbols to denote the irreducible representations of A_n and we shall not distinguish between $[\lambda]$ and $[\lambda']$ as representations of A_n. Consider the $n!$ Young tableaux obtained by placing the symbols $1, 2, 3, \ldots, n$ in $[\lambda]$ in all possible ways. In f^{λ} of these tableaux the symbols appear in their natural order in both row and column, and these tableaux are called *standard*. Making the following construction, we obtain the actual matrices of the irreducible representation $[\lambda]$ of S_n **(7)**.

2.4. Theorem (Young). *To construct the matrix representing* $(r, r+1)$ *in* $[\lambda]$ *of* S_n*, arrange the* f^{λ} *standard tableaux* $\ldots t^{\lambda}{}_u \ldots t^{\lambda}{}_v \ldots$ *in dictionary order and set*

(i) 1 (*and* -1) *in the leading diagonal where* t^{λ} *has* r *and* $r+1$ *in the same row* (*column*),

(ii) *a quadratic matrix*

$$\begin{pmatrix} -\rho & \sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{pmatrix}$$

at the intersection of rows and columns corresponding to $t^\lambda{}_u$ *and* $t^\lambda{}_v$ *where* $u < v$ *and* $t^\lambda{}_v$ *is obtainable from* $t^\lambda{}_u$ *by interchanging* r *and* $r+1$. *If* r *appears in the* (i,j) *position and* $r+1$ *in the* (k,l) *position of* $t^\lambda{}_u$ *with* $i < k, l < j$, *then* $\rho^{-1} = (j-i) - (l-k)$,

(iii) *zeros elsewhere.*

If $[\lambda] \neq [\lambda']$ the matrices representing $(12)(r, r+1)$ can be constructed according to 2.4 and these will generate the corresponding representation of A_n. If $[\lambda] = [\lambda']$, the representation will split into $[\lambda]^+$ and $[\lambda]^-$, in which case we have the following construction **(5, 10)**.

2.5. THEOREM. *To construct the matrix representing* $(12)(r, r+1)$ *in* $[\lambda]^+$ $([\lambda]^-)$ *of* A_n *arrange the* f^λ *standard tableaux in dictionary order and assign the first* $\frac{1}{2}f^\lambda$ *tableaux to* $[\lambda]^+$ *and the remainder to* $[\lambda]^-$. *Construct the matrix representing* (12) *according to* 2.4 *and the matrix representing* $(r, r+1)$ *in the same manner, as long as the two associated tableaux in* 2.4 (ii) *both belong to the same set* $[\lambda]^+$ $([\lambda]^-)$.

If $t^\lambda{}_u$ *lies in* $[\lambda]^+$ *and* $t^\lambda{}_v$ *in* $[\lambda]^-$, *we must have* $r = n-2$ *or* $n-1$. *Denote by* $t^\lambda{}_{u_1}$ *that tableau in which the part of the tableau in* $n-2$ *letters is conjugate to the same part in* $t^\lambda{}_u$, *so that* $t^\lambda{}_{u_1}$ *also lies in* $[\lambda]^+$. *Set the quadratic matrix*

$$\begin{pmatrix} -\rho & \epsilon\sqrt{1-\rho^2} \\ \epsilon^{-1}\sqrt{1-\rho^2} & \rho \end{pmatrix}$$

at the intersection of the rows and columns corresponding to $t^\lambda{}_u$ *and* $t^\lambda{}_{u_1}$, *where* ρ *has the same meaning as in* 2.4 (ii) *and* $\epsilon = (i)^{\frac{1}{2}(p_1 p_2 \ldots -1)}$ *with* $p_i = 2\lambda_i - (2i-1)$. *Putting zeros elsewhere leads to the matrix representing* $(12)(r, r+1)$ *in* $[\lambda]^+$.

A similar construction applied to $t^\lambda{}_v$ *yields* $t^\lambda{}_{v_1}$ *and with* ϵ *and* ϵ^{-1} *interchanged leads to the matrix representing* $(12)(r, r+1)$ *in* $[\lambda]^-$.

Proof. The standard tableaux of $[\lambda]$ arranged in dictionary order are such that the second half of the f^λ tableaux are the conjugates of the first half, but in *reverse order*.

Consider the matrices representing A_n constructed according to 2.4. If the order of the tableaux in the second half of the set is reversed, the two representations obtained by restricting $[\lambda]$ to A_{n-2} (or A_{n-1} as the case may be) on the symbols $1, 2, \ldots, n-2$ (or $n-1$) will be equivalent by 2.1, so that a further transformation will make them identical if necessary. It follows from Schur's lemma that there exists a commuting matrix which should enable us to reduce the representation $[\lambda]$ of A_n, so transformed, into $[\lambda]^+$ and $[\lambda]^-$ leaving the two identical representations of A_{n-2} (or A_{n-1}) unchanged.

Combining these three transformations leads to a matrix

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$$P = \left[\begin{array}{cccccccccccc}
a_1 & & & & & & & & & & & b_1 \\
 & a_2 & & & & & & & & & b_2 & \\
 & & \cdot & & & & & & & \cdot & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & & & \cdot & & & \cdot & & & & \\
 & & & & & a_{\frac{1}{2}f^\lambda} & b_{\frac{1}{2}f^\lambda} & & & & & \\
 & & & & & c_{\frac{1}{2}f^\lambda} & -d_{\frac{1}{2}f^\lambda} & & & & & \\
 & & & & \cdot & & & \cdot & & & & \\
 & & & \cdot & & & & & \cdot & & & \\
 & & \cdot & & & & & & & \cdot & & \\
 & c_2 & & & & & & & & & -d_2 & \\
c_1 & & & & & & & & & & & -d_1
\end{array}\right]$$

where

$$\sigma_j a_j = \sigma_j d_j = 1/\sqrt{2}, \; \bar{b}_j = c_j = (i)^{\frac{1}{2}(p_1 p_2 p_3 \cdots - 1)}/\sqrt{2},$$

and $a_j d_j + b_j c_j = -1$ with $\sigma_j = \pm 1$ according as the jth tableau is obtained from the first by an even or an odd number of interchanges of symbols. The effect of transformation by P is as desired in the theorem.

Example. As an illustration, consider the construction of the matrices of $[31^2]^+$ of the alternating group A_5:

$$[31^2]^+: \quad \begin{array}{ccc} 1\,2\,3 & 1\,2\,4 & 1\,3\,4 \\ 4 & 3 & 2 \\ 5 & 5 & 5 \end{array}$$

The group A_5 can be generated by (12)(23), (12)(34), and (12)(45). We construct the matrices representing (12)(23) or (12)(34) according to 2.4, but for the matrix representing (12)(45) we must use the construction of 2.5, obtaining

$$(12)(23): \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -\frac{1}{2} & \sqrt{3}/2 \\ \cdot & -\sqrt{3}/2 & -\frac{1}{2} \end{pmatrix}, \qquad (12)(34): \begin{pmatrix} -1/3 & \sqrt{8}/3 & \cdot \\ \sqrt{8}/3 & 1/3 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

and

$$(12)(45): \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1/4 & \epsilon\sqrt{15}/4 \\ \cdot & \epsilon^{-1}\sqrt{15}/4 & 1/4 \end{pmatrix}$$

where $\epsilon = (i)^{\frac{1}{2}(5-1)} = -1$.

3. Next let us determine the minimal left ideals of ΛA_n over Λ. For this purpose, it is necessary to consider the cases $[\lambda] \neq [\lambda']$ and $[\lambda] = [\lambda']$ separately. First let $[\lambda] \neq [\lambda']$.

Let $t^\lambda{}_i$ be any standard tableau belonging to $[\lambda]$ of A_n. Denote by P_i the product of the symmetric groups of the symbols of the rows and by Q_i the product of the symmetric groups of the symbols of the columns of the tableau $t^\lambda{}_i$. Then clearly

$$P_i \cap Q_i = 1 \quad \text{and} \quad P_i Q_i \neq Q_i P_i$$

in general. Define

$$2e^\lambda = \sum_i \sum_{p \epsilon P_i, q \epsilon Q_i} (\epsilon_q + \epsilon_p) pq,$$

where $\epsilon_p = \pm 1$ according as p is an even or an odd permutation. It is well known that the expressions

$$\sum_{p,q} \epsilon_p pq \quad \text{and} \quad \sum_{q,p} \epsilon_q pq, \qquad \Big(\sum_{p,q} \epsilon_p pq\Big)\Big(\sum_{p,q} \epsilon_q pq\Big) = 0,$$

are primitive characteristic units, i.e. non-zero idempotents of the group algebra ΛS_n corresponding to $[\lambda]$ and $[\lambda']$, which do not belong to the centre of the algebra. Hence, e^λ is a central idempotent, but for a constant. Thus we may associate the minimal ideal $\Lambda A_n e^\lambda$ with $[\lambda]$ of A_n. It can be shown that these ideals are the desired irreducible ΛA_n-modules.

Secondly let $[\lambda] = [\lambda']$. Then define

$$2e^{\lambda+} = \sum_i \sum_{p \epsilon P_i, q \epsilon Q_i} f(pq)(\epsilon_q + \epsilon_p) pq,$$

where $f(pq) = 2\zeta$ and $2\zeta^*$ for elements of the split classes and $f(pq) = 1$ otherwise and ζ, ζ^* are defined by 2.2. From 2.1, 2.2, and the above argument we conclude that $e^{\lambda+}$ is also a central idempotent and the roles of ζ and ζ^* are interchanged for the conjugate representation.

4. We next show how the irreducible components in an induced permutation representation of A_n can be determined. For this purpose denote by $[\lambda_1].[\lambda_2]. \ldots [\lambda_h]$ the permutation representation of S_n induced by the I-representation of the subgroup $F = S_{\lambda_1} \times S_{\lambda_2} \ldots \times S_{\lambda_h}$ whose reduction into its irreducible components is given by **(7)**

$$[\lambda_1].[\lambda_2]. \ldots . [\lambda_h] = \prod_{i,j} (1 - R_{ij})^{-1} [\lambda_1 \lambda_2 \ldots \lambda_h]. \tag{4.1}$$

R_{ij} for $i < j$ indicates the process of raising a node from the jth row to the ith row of $[\lambda]$ to yield a new diagram and $\prod R_{ij}$ indicates the successive raising of the nodes subject to the restrictions that the result is to be disregarded (i) if any row contains more symbols than a previous row or (ii) if two symbols from the same row appear in the same column.

If K is the subgroup of all even permutations of F, then by 1.7, we have

$$\{\{(I)F\}^{S_n}\}_{A_n} = \{\{(I)F\}_K\}^{A_n},$$

which may also be written as

$$\{[\lambda_1].[\lambda_2]. \ldots [\lambda_h]\}_{A_n} = \{(I)K\}^{A_n},$$

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where $(I)F$ denotes the identity representation of F. If we use the same symbol $[\lambda_1].[\lambda_2]. \ldots [\lambda_h]$ $(\lambda_1 \geqslant 2)$ to denote the permutation representation of A_n induced by $(I)K$, then its reduction is given by 4.1, where along with the above restrictions on the raising operators, we have the further restrictions that (iii) $[\lambda]$ and $[\lambda']$ are equivalent if $[\lambda] \neq [\lambda']$ and (iv) $[\lambda] = [\lambda]^+ + [\lambda]^-$ if $[\lambda] = [\lambda']$.

Example. If $[3].[2]$ is the permutation representation of A_5 induced by the identity representation of the subgroup of all even permutations of the product $S_3 \times S_2$, then

$$[3].[2] = \prod (1 - R_{ij})^{-1} [32] = (1 + R_{12} + R^2{}_{12}) [32]$$
$$= [32] + [41] + [5]$$

If $[\lambda] \neq [\lambda']$, then we can invert 4.1 in the form

4.2 $$[\lambda_1\lambda_2 \ldots \lambda_h] = \prod (1 - \mathrm{R}_{ij})[\lambda_1].[\lambda_2]. \ldots [\lambda_h],$$

thus expressing an irreducible representation in terms of permutation representations. Here no restrictions on the R_{ij} are needed. The relations 4.1 and 4.2 can be extended to all subgroups of A_n **(6)**.

Example.

$$[3, 2] = \prod (1 - R_{ij})[3].[2] = (1 - R_{12})[3].[2] = [3].[2] - [4].[1].$$

5. Let Λ be a field of characteristic p in which all the modular irreducible representations of a finite group G of order $[G:1] = g'p^a$ with $(g', p) = 1$ can be realized. Two modular irreducible representations of G over Λ are called mutually "p-conjugate" if they are obtained from each other by a different choice of the p^a-th primitive roots of unity. If a modular representation of G is its own p-conjugate, it may be called a self p-conjugate representation. In case of S_n, a self p-conjugate representation appears as a component of an ordinary self-conjugate representation. In the following Λ is a finite extension field of the Galois field $\mathrm{GF}(p)$. Let Q be the field of rational numbers and Z be the ring of integers. If $[\lambda]$ is a Q-representation of A_n, then $[\lambda]$ is Q-equivalent to a Z-representation. In this Z-representation we reduce the coefficients modulo p and denote the resulting representation by $\overline{[\lambda]}$. Now, by 2.1, if $[\lambda] \neq [\lambda']$, then $\overline{[\lambda]}$ and $\overline{[\lambda']}$ of A_n contain the same modular irreducible components. First let $p \neq 2$. Then we have the following theorem.

5.1. THEOREM. *Two distinct p-conjugate representations of S_n restricted to A_n are Q-equivalent representations of A_n over* $\mathrm{GF}(p)$, *while a self p-conjugate representation of S_n restricted to A_n splits into two p-conjugate representations of A_n over an extension of* $\mathrm{GF}(p)$.

Proof. As in the case of ordinary irreducible representations, the characters in the p-conjugate representations of S_n are equal for all even p-regular elements

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and different only for certain odd p-regular elements of S_n. From this the first part follows. A self p-conjugate representation of S_n is necessarily a component of an ordinary self-conjugate representation of S_n. Then the second part follows from 2.1.

Next let $p = 2$. In this case $1 \equiv -1 \pmod 2$, so that every GF(2)-representation of S_n is a self 2-conjugate representation. The result 5.1 needs to be modified. If $[\lambda] = [\lambda']$ is any representation, then

$$\overline{[\lambda]} = \sum_T m^T . T,$$

where m^T is the multiplicity of the GF(2)-irreducible representation T of S_n. In this equation we have to single out those components T with odd multiplicity. Then a representation T of S_n over GF(2) restricted to A_n is an irreducible representation of A_n unless T is a component of $\overline{[\lambda]} = \overline{[\lambda']}$ with odd multiplicity. In the latter case T splits into two equivalent or two 2-conjugate representations of A_n according as all $[\lambda] = [\lambda']$ in which T appears as a component are such that (λ) are 2-singular or 2-regular classes respectively.

Example. The 4-dimensional modular irreducible representation T of S_5 over GF(2), contained in $\overline{[31^2]}$, is a self 2-conjugate representation of S_5. This representation T, restricted to A_5, is irreducible over GF(2); but in the extended field GF(2^2) consisting of four elements $0, 1, w, w^*$, it reduces into two representations T_1 and T_2 of dimension 2 each. Moreover, the class (31^2) which corresponds to $[31^2]$ is 2-regular and hence T_1 and T_2 are 2-conjugate. The matrices representing (123) and (345) in T are

$$T\colon\ (123)\colon \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad (345)\colon \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

If we now transform by

$$\begin{bmatrix} 1 & w^* & 0 & 0 \\ 0 & 0 & w & 1 \\ 1 & w & 0 & 0 \\ 0 & 0 & w^* & 1 \end{bmatrix},$$

we obtain the two 2-conjugate representations of A_5:

$$T_1 + T_2\colon\ (123)\colon \left[\begin{array}{cc:cc} w & 0 & & \\ 0 & w^* & & \\ \hdashline & & w^* & 0 \\ & & 0 & w \end{array}\right], \qquad (345)\colon \left[\begin{array}{cc:cc} 0 & 1 & & \\ 1 & 1 & & \\ \hdashline & & 0 & 1 \\ & & 1 & 1 \end{array}\right].$$

As in the case of S_n, the block structure of A_n depends on the p-core $[\tilde{\lambda}]$ of $[\lambda]$. Let us consider first those diagrams which are themselves p-cores, i.e.

from which no p-hooks are removable. In general, the maximum power of p which divides the degree f^λ is given by (7)

5.2 $$e(f^\lambda) = e(n!) - e((n-a)!) + e(f^\lambda{}_p),$$

which reduces to

5.3 $$e(f^\lambda) = e(n!)$$

if $a = n$. If $p \neq 2$, and $[\lambda]$ is not self-conjugate,

5.4 $$e(f^\lambda) = e(\tfrac{1}{2}n!)$$

so that $[\lambda]$ of A_n is modularly irreducible and constitutes a block by itself. If $[\lambda] = [\lambda']$, then the representation will split and

5.5 $$e(\tfrac{1}{2}f^\lambda) = e(\tfrac{1}{2}n!)$$

so that the same statement applies to each of $[\lambda]^+$ and $[\lambda]^-$.

In the case $p = 2$ the situation is simplified by the fact that every 2-core is self-conjugate so that 5.5 still holds and each of the 2-conjugate components $[\lambda]^+$ and $[\lambda]^-$ is modularly irreducible and constitutes a block by itself.

Example. Let $p = 3$ and $n = 5$. The 3-core $[3, 1^2]$ containing 5 nodes yields two modularly irreducible representations of degree 3. The matrices constructed according to 2.4, on transforming by an appropriate matrix and reducing module 3, take the form

$$(123):\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & & & \\ 0 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \\ \hline & & & 1 & 1 & -1 \\ & & & 0 & 1 & 0 \\ & & & 0 & 1 & 1 \end{array}\right], \quad (345):\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right].$$

The representation $\overline{[3, 1^2]}$ of A_5 is irreducible in GF(3). But if we extend the field to GF(3^2) consisting of 9 elements

$$0, 1, -1, j, -j, 1+j, -1+j, -1-j,$$

where $j^2 + 1 = 0 \pmod 3$, the representation reduces into two 3-conjugate modularly irreducible representations:

$$(123):\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & & & \\ 0 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \\ \hline & & & 1 & 1 & -1 \\ & & & 0 & 1 & 0 \\ & & & 0 & 1 & 1 \end{array}\right], \quad (345):\left[\begin{array}{ccc|ccc} -1 & 1 & -j & & & \\ -1 & -1 & 0 & & & \\ j & j & -1 & & & \\ \hline & & & -1 & 1 & j \\ & & & -1 & -1 & 0 \\ & & & -j & -j & -1 \end{array}\right].$$

The representations are irreducible in any extension field. Also since the degrees (i.e. 3) are divisible by the characteristic of the field, they form their own 3-blocks and indecomposable components of the regular representation of A_5.

We have thus disposed of the blocks determined by the p-cores of A_n containing exactly n nodes. For the blocks determined by p-cores containing less than n nodes, we have the following theorem.

5.6. THEOREM. *All ordinary irreducible representations of A_n which have p-cores $[\tilde{\lambda}]$ or $[\tilde{\lambda}']$ belong to the same p-block of A_n.*

Proof. We deduce the result from that for S_n. The representations $\overline{[\lambda]}$ and $\overline{[\lambda']}$ of A_n contain the same irreducible representations. From this remark, the result follows.

In enumerating the ordinary and modular irreducible representations of A_n in a block, we have to consider the following three types:

(i) The block determined by the non-self-conjugate p-cores $[\tilde{\lambda}]$ and $[\tilde{\lambda}']$.

(ii) The block determined by a self-conjugate p-core $[\tilde{\lambda}]$, to which no ordinary split representations of A_n belong.

(iii) The block determined by a self-conjugate p-core $[\tilde{\lambda}]$, to which split representations of A_n do belong.

Clearly no split representation can belong to a block of the first type. For if it does, it has to belong to blocks determined by p-cores $[\tilde{\lambda}]$ and $[\tilde{\lambda}']$ in the case of S_n, which implies $[\tilde{\lambda}] = [\tilde{\lambda}']$, contrary to the assumption. Moreover, half the number of representations of S_n determined by the two p-cores $[\tilde{\lambda}]$ and $[\tilde{\lambda}']$ are also representations of A_n. Hence, in a block of type (i) of A_n, the number of ordinary or modular representations of A_n is the same as that for S_n. For actual enumeration see **(7, 8)**.

In a block of the second type, there are no split representations. Also the number of ordinary irreducible representations in a p-block of S_n of the second type is even, for if $[\lambda]$ belongs to the block, so does its conjugate $[\lambda']$. Thus the number of ordinary and modular irreducible representations of A_n in a block of the second type is equal to half the number of the representations of the block of S_n determined by the p-core $[\tilde{\lambda}]$.

In the third case the situation is more complicated. If $n = a + bp$, and the number b of removable p-hooks is even, then the block of S_n characterized by the self-conjugate p-core $[\tilde{\lambda}]$ does contain at least one self-conjugate $[\lambda]$ and this condition is both necessary and sufficient when $p = 2$. If $p \neq 2$, however, we can have a self-conjugate $[\lambda]$ when b is odd, e.g. $[2, 1]$ for $p = 3$ and $[3^3]$ for $p = 5$. Thus, the distinction between p-blocks of types (ii) and (iii) depends upon both b and p and enumeration is more difficult.

We conclude by giving the D-matrices (mod 2, 3) of A_n for

$$n = 3, 4, 5, 6, 7, 8.$$

D-*Matrices* (mod 2) *of* A_n

		deg.	1	1	1
A_3		[3]	1		
		$[21]^+$		1	
		$[21]^-$			1

	deg.	1	1	1
A_4	[4]	1		
	[31]	1	1	1
	$[2^2]^+$	0	1	0
	$[2^2]^-$	0	0	1

	deg.	1	2	2	4
A_5	[5]	1			
	[32]	1	1	1	Core
	$[31^2]^+$	1	1	0	[1]
	$[31^2]^-$	1	0	1	
	[41]				1

	deg.	1	4	4	8	8
A_6	[6]	1				
	[51]	1	1			
	[42]	1	1	1		
	$[41^2]$	2	1	1		
	$[3^2]$	1	0	1		
	$[321]^+$				1	
	$[321]^-$					1

	deg.	1	14	20	6	4	4
A_7	[7]	1					
	[52]	0	1				
	$[51^2]$	1	1			Core	
	[421]	1	1	1		[1]	
	$[3^21]$	1	0	1			
	[61]				1		
	[43]		Core		1	1	1
	$[41^3]^+$		[2, 1]		1	1	0
	$[41^3]^-$				1	0	1

	deg.	1	6	14	4	20	64
A_8	[8]	1					Core $[\emptyset]$
	[71]	1	1				
	[62]	0	1	1			
	$[61^2]$	1	1	1			
	[53]	0	1	1	2		
	$[51^3]$	1	2	1	2		
	$[4^2]$	0	1	0	2		
	[431]	2	1	1	2	2	
	$[42^2]$	2	0	1	0	2	
	$[421^2]^+$	1	1	1	1	1	
	$[421^2]^-$	1	1	1	1	1	
	$[3^22]^+$	1	0	0	0	1	
	$[3^22]^-$	1	0	0	0	1	
	[521]	Core [321]					1

D-*Matrices* (mod 3) *of* A_n

	deg.	1
A_3	[3]	1
	$[21]^+$	1
	$[21]^-$	1

	deg.	1	3
A_4	[4]	1	Core [1]
	$[2^2]^+$	1	
	$[2^2]^-$	1	
	[31]		1

	deg.	1	4	3	3
A_5	[5]	1		Core	
	[32]	1	1	$[2] = [1^2]$	
	$[31^2]^+$			1	
	$[31^2]^-$				1

	deg.	1	4	3	3	9
A_6	[6]	1				Core $[\phi]$
	[51]	1	1			
	$[41^2]$	0	1	1	1	
	$[3^2]$	1	1	0	0	
	$[321]^+$	1	1	1	0	
	$[321]^-$	1	1	0	1	
	[42]					1

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		1	13	10	10	6	15
	deg.	1	13	10	10	6	15
	[7]	1					
	[52]	1	1				
	[43]	1	1			Core	
	[421]	2	1	1	1	[1]	
A_7	$[41^3]^+$	0	0	1	0		
	$[41^3]^-$	0	0	0	1		
	[61]					1	
	$[51^2]$		Core			0	1
	$[3^21]$		$[3, 1] = [2, 1^2]$			1	1

	deg.	1	7	13	27	35	21	35	35
	[8]	1							
	[7, 1]	0	1						
	[6, 2]	0	1	1			Core $[2] = [1^2]$		
	$[4^2]$	1	0	1					
	[5, 3]	1	0	0	1				
	[5, 2, 1]	2	0	0	1	1			
	$[5, 1^3]$	0	0	0	0	1			
A_8	[4, 3, 1]	1	1	0	1	1			
	$[4, 2^2]$	1	1	1	0	1			
	$[6, 1^2]$						1		
	$[3^2, 2]^+$						1	Core $[31^2]$	
	$[3^2, 2]^-$						1		
	$[4, 2, 1^2]^+$							1	
	$[4, 2, 1^2]^-$								1

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