INTERVAL SEMIRINGS ON R_1 WITH ORDINARY MULTIPLICATION

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1. Introduction

The study of topological semirings, initiated by Selden [5], arises naturally from the theory of topological semigroups. It is of interest to take a known multiplication and investigate the possible additions. Selden has done this in [5] for several compact semirings.

If S is any interval of R_1 which is algebraically closed under ordinary multiplication, then there are at least two semirings on S with this (or, in fact, any) multiplication, namely those with additions $x \oplus y = x$ and $x \oplus y = y$. It is the purpose of this paper to list all the possible additions on S.

It can be easily verified that if S is an interval which is closed under ordinary multiplication, then S is of one of the following types:

I. $\{0\}$; II. $\{1\}$; III. $(0, \infty)$; IV. $[0, \infty)$; V. R_1;
VI. $(0, b)$ or $(0, b]$ where $0 < b \leqq 1$;
VII. $[0, b)$ or $[0, b]$ where $0 < b \leqq 1$;
VIII. $[a, b]$, $(a, b]$, (a, b) where $-1 \leqq a < 0 < a^2 \leqq b \leqq 1$, or $[a, b)$ where $-1 < a < 0 < a^2 < b \leqq 1$;
IX. (b, ∞) or $[b, \infty)$ where $b \geqq 1$.

The problem is a trivial one for intervals I and II.

The necessary definitions are given in §2, while in §3 we derive some results which are necessary for the rest of the paper. In §§4—6 the problem is solved for intervals III—V respectively. Intervals of types VI—VIII are dealt with in §7, and finally, in §8, the list of additions for intervals of type IX is obtained from the corresponding list for intervals of type VI.

The systems treated here can be defined in terms of functional equations, and similar problems have been considered in that field (see, for example, [1]). In particular, with the further assumption that addition satisfies the cancellation law, Aczél (Theorem 2 of [2]) has found all additions when $S \subset [0, \infty)$. Also Bohnenblust (§4 of [3]) has solved a similar problem on

$[0, \infty)$ in which, although continuity is not assumed, several other assumptions, including monotoneity, are made.

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2. Topological semirings in general

2.1 DEFINITION. By a *(topological) semiring* we mean a system $\{S, \oplus, \circ\}$ where S is a Hausdorff space and $\oplus$, $\circ$ (called *addition* and *multiplication* respectively) are continuous binary operations on S such that

(i) each is associative. That is, for all x, y, z in S,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z),$$
$$(x \circ y) \circ z = x \circ (y \circ z);$$

(ii) $\circ$ distributes across $\oplus$. That is, for all x, y, z in S,

$$x \circ (y \oplus z) = (x \circ y) \oplus (x \circ z),$$
$$(x \oplus y) \circ z = (x \circ z) \oplus (y \circ z).$$

In functional notation, this is equivalent to having two continuous functions

$$F : S \times S \to S,$$
$$G : S \times S \to S,$$

where

$$F(x, y) = x \oplus y,$$
$$G(x, y) = x \circ y,$$

and requiring that, for all x, y, z in S,

$$F[F(x, y), z] = F[x, F(y, z)],$$
$$G[G(x, y), z] = G[x, G(y, z)],$$
$$G[x, F(y, z)] = F[G(x, y), G(x, z)]$$
$$G[F(x, y), z] = F[G(x, z), G(y, z)].$$

We shall find both notations useful.

2.2 The following lemma, although obvious, will be applied several times.

LEMMA 1. *Suppose that* $\{S, \oplus, \circ\}$ *is a topological semiring and* $\{T, \cdot\}$ *is a topological semigroup* (i.e., $\cdot$ *is a continuous associative binary operation on* T). *If* h *is a homeomorphism of* S *onto* T *such that*

$$h(x \circ y) = h(x) \cdot h(y)$$

for all x, y in S, then σ, defined by

$$z \sigma w = h[h^{-1}(z) \oplus h^{-1}(w)]$$

for all z, w in T, is an addition of a semiring on $\{T, \cdot\}$.

2.3 Throughout the rest of this paper, multiplication will be ordinary multiplication. That is,

$$G(x, y) = xy.$$

Note that with this multiplication the distributive requirement is satisfied if and only if, for all x, y, z in S,

$$x(y \oplus z) = xy \oplus xz.$$

3. Preliminaries

Throughout this paper we shall use the following notations:

$$f(x) = F(1, x) = 1 \oplus x;$$
$$g(x) = F(x, 1) = x \oplus 1;$$
$$\theta = F(1, 1) = 1 \oplus 1.$$

In this section $\{S, \oplus, .\}$ will denote a semiring on an interval S of R_1 with ordinary multiplication. We establish here some apparently unrelated results which we need in the remaining sections.

3.1 Suppose that, for any x in S, we know the value of $f(x)$, and suppose further that if $x \in S$ and $x \neq 0$, then $x^{-1} \in S$. Then if $x, y \in S$ and $x \neq 0$ we see that

$$x \oplus y = x(1 \oplus yx^{-1}) = xf(yx^{-1})$$

is known. From the continuity of $\oplus$ we can then find $x \oplus y$ for all x, y in S. Similarly if we have fixed $g(x)$ for all x in S, then we can find $x \oplus y$ for all x, y. Accordingly we direct much of our attention to specifying the values of f or g.

Because $f(1) = g(1)$, there is a certain duality between f and g. This is especially true when, at various times, we assume that $f(0) = g(0)$. Several results which we state and prove for f have dual results for g. We will assume these latter results without stating them explicitly.

3.2 If $0 \in S$, then $0 \oplus 0 = 0(0 \oplus 0) = 0$.

3.3 If $x, y, z, w \in S$, then

$$(x \oplus y)(z \oplus w) = (x \oplus y)z \oplus (x \oplus y)w = xz \oplus yz \oplus xw \oplus yw.$$

3.4 If $0,1 \in S$, then, by §§ **3.3** and **3.2**,

$$[f(0)]^2 = 1 \oplus 0 \oplus 0 \oplus 0 = 1 \oplus 0 = f(0).$$

Hence $f(0)$ is either 0 or 1, and similarly for $g(0)$. Clearly $x \oplus 0 = xf(0)$ for all x, and so

$$f(0) = 0 \Rightarrow x \oplus 0 = 0 \text{ for all } x,$$
$$f(0) = 1 \Rightarrow x \oplus 0 = x \text{ for all } x.$$

3.5 For any $x \neq 0$ such that $x^{-1} \in S$, it is clear that

$$f(x) = x(x^{-1} \oplus 1) = xg(x^{-1}).$$

3.6 (i) Suppose $S = R_1$ and $g(0) = 1$. Then we see that

$$\lim_{x \to 0^-} g(x) = 1 = \lim_{x \to 0^+} g(x).$$

If we put $xf(x^{-1})$ for $g(x)$ and y for x^{-1}, we see that

$$\lim_{y \to -\infty} f(y)/y = 1 = \lim_{y \to \infty} f(y)/y.$$

Hence $f(y) \to \pm\infty$ according as $y \to \pm\infty$, and it follows from the continuity of f that $f(R_1) = R_1$.

(ii) If $[0, \infty) \subset S$ and $g(0) = 1$, then it follows as in (i) that $f(x) \to \infty$ as $x \to \infty$.

3.7 Suppose that $T = \{f(x) | x \in S\}$; then $T^2 \subset T$. For if $y_1, y_2 \in T$, there exist x_1, x_2 such that

$$y_1 y_2 = (1 \oplus x_1)(1 \oplus x_2) = 1 \oplus (x_1 \oplus x_2 \oplus x_1 x_2) \in T.$$

Also T, being the continuous image of a connected set, is an interval.

3.8 Suppose that $\theta = 1$ and T is as in § **3.7**. Then if $y \in T$, there is an x such that $1 \oplus x = y$. Hence

$$f(y) = 1 \oplus y = 1 \oplus (1 \oplus x) = (1 \oplus 1) \oplus x = 1 \oplus x = y.$$

3.9 We derive some results when there is a y in S with $f(y) = 1$.

LEMMA 2. *If $f(y) = 1$ for some y, then $f(y^n) = 1$ for all integers $n \geqq 1$. If, in addition, $\theta \neq 0$, then $f(\theta^m y^n) = 1$ for all integers $n \geqq 1$ and m.*

PROOF. We first prove that $f(y^n) = 1$ for all integers $n \geqq 1$. This is trivial if $n = 1$. If we assume it to be true for some $n \geqq 1$, then we see that

$$\begin{aligned} 1 = (1 \oplus y)(1 \oplus y^n) &= (1 \oplus y) \oplus y^n \oplus y^{n+1} \\ &= (1 \oplus y^n) \oplus y^{n+1} = 1 \oplus y^{n+1}. \end{aligned}$$

The result follows by induction.

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If $\theta \neq 0$, we now show that $f(\theta y') = f(\theta^{-1}y') = 1$ whenever $f(y') = 1$. For

$$1 \oplus \theta y' = 1 \oplus (y' \oplus y') = (1 \oplus y') \oplus y' = 1 \oplus y' = 1,$$

and

$$1 \oplus \theta^{-1}y' = \theta^{-1}(\theta \oplus y') = \theta^{-1}[1 \oplus (1 \oplus y')] = \theta^{-1}(1 \oplus 1) = 1.$$

The lemma follows by repeated application of these two results.

LEMMA 3. *If* $[0, \infty) \subset S$ *and* $f(0) = 1$, *then* $\theta \geqq 1$.

PROOF. Suppose, if possible, that $g(1) = \theta < 1$. Now, by §3.6, $g(x) \to \infty$ as $x \to \infty$; it follows from the continuity of g and Lemma 2 that there is a $w > 1$ with $g(w^n) = 1$ for all integers $n \geqq 1$. But $w^n \to \infty$ as $n \to \infty$ while $g(w^n)$ is bounded, which is a contradiction.

LEMMA 4. *If* $[0, \infty) \subset S$ *and* $\theta = 0$, *then* $F(x, y) = 0$ *for all* x, y *in* S.

PROOF. It follows from § 3.4 and Lemma 3 (and its dual) that $f(0) = g(0) = 0$. Hence, using §§ 3.3 and 3.4, we see that, for any x, y in S,

$$(x \oplus y)^2 = x^2 \oplus xy\theta \oplus y^2 = (x^2 \oplus 0) \oplus y^2 = 0 \oplus y^2 = 0.$$

Hence the lemma.

LEMMA 5. *If* $(0, \infty) \subset S$, $\theta \geqq 0$ *and there is a* $y \neq 0$ *with* $f(y) = 1$, *then* $\theta = 1$.

PROOF. We first observe from Lemma 4 that $\theta > 0$, and further, since it follows from Lemma 2 that $f(y^2) = 1$, we can assume that $y > 0$. Suppose, if possible, that $\theta \neq 1$. Then it is well known (see, for example, §11.1 of [4]) that there exist sequences $\{p_n\}$, $\{q_n\}$ of integers such that $q_n > 0$ for all n, $q_n \to \infty$ as $n \to \infty$, and

$$|\ln y/\ln \theta + p_n/q_n| \leqq q_n^{-2} \quad (\text{all } n).$$

i.e.

$$-|\ln \theta| q_n^{-1} \leqq q_n \ln y + p_n \ln \theta \leqq |\ln \theta| q_n^{-1}.$$

Hence, putting $\xi = \theta$ or θ^{-1} according as $\theta > 1$ or $\theta < 1$ respectively, we see that

$$\xi^{-1/q_n} \leqq y^{q_n}\theta^{p_n} \leqq \xi^{1/q_n}.$$

It is now clear that $y^{q_n}\theta^{p_n} \to 1$ as $n \to \infty$. But, by Lemma 2, $f(y^{q_n}\theta^{p_n}) = 1$ for all $n \geqq 1$, and so it follows from the continuity of f that $\theta = f(1) = 1$.

3.10 We examine here the sets

$$Z = \{x | x \in S \text{ and } f(x) = 0\},$$
$$N = \{x | x \in S \text{ and } f(x) \neq 0\}.$$

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LEMMA 6. *If* $f(0) = g(0) = 0$, *then* $ZN^m \subset Z$ *for all integers* $m \geqq 1$.

PROOF. Suppose $z \in Z$ and $n \in N$. Then, using §§ 3.3 and 3.4, we see that

$$(1 \oplus n)(1 \oplus zn) = 1 \oplus n(1 \oplus z) \oplus zn^2 = (1 \oplus 0) \oplus zn^2 = 0 \oplus zn^2 = 0.$$

Because $1 \oplus n \neq 0$, we see that $1 \oplus zn = 0$, and so $ZN \subset Z$. The lemma follows by repeated application of this result.

LEMMA 7. *If* $[0, \infty) \subset S$ *and there is a* $z > 0$ *with* $f(z) = 0$, *then* $F(x, y) = 0$ *for all* x, y *in* S.

PROOF. We first show that $f(0) = g(0) = 0$. For suppose that $f(0) = 1$; then, by §3.6, $g(x) \to \infty$ as $x \to \infty$. But, by §3.5, $g(z^{-1}) = z^{-1}f(z) = 0$ and it follows from the continuity of g that there is a $w > z^{-1} > 0$ with $g(w) = 1$. Further, it follows from Lemma 3 that $\theta \geqq 1$. Hence we see from Lemma 5 that $\theta = 1$. But $f(z) = 0$ and it follows from § 3.8 that $f(0) = 0$, which is a contradiction. Similarly, by reversing the roles of f and g, we can show that $g(0) = 0$.

Suppose now that $\theta \neq 0$. Then it follows from the continuity of f that there is a $\delta > 0$ such that $[1-\delta, 1+\delta] \subset N$. From Lemma 6 it follows that

$$Z \supset ZN^m \supset z[1-\delta, 1+\delta]^m$$

for all integers $m \geqq 1$. But clearly $1 \in z[1-\delta, 1+\delta]^m$ for some integer m and so $\theta = f(1) = 0$, which is a contradiction. The lemma now follows from Lemma 4.

LEMMA 8. *If* $S = R_1$, $f(0) = g(0) = 0$ *and there is a* $z < 0$ *with* $f(z) = 0$, *then* $f(x) = 0$ *for all* $x \leqq 0$.

PROOF. If there is a $z_1 > 0$ with $f(z_1) = 0$, then the result follows from Lemma 7. Hence we can assume that $(0, \infty) \subset N$. Then, by Lemma 6,

$$Z \supset ZN \supset z(0, \infty) = (-\infty, 0),$$

and the lemma follows immediately.

4. (0, ∞)

We prove here the following theorem.

THEOREM 1. *All additions of semirings on* $(0, \infty)$ *with ordinary multiplication are given by the following five functions*:

$$F_1(x, y) = x; \qquad F_2(x, y) = y;$$
$$F_3(x, y) = \min(x, y); \qquad F_4(x, y) = \max(x, y);$$
$$F_5(x, y) = (x^c+y^c)^{1/c}, \qquad \text{where } c \neq 0.$$

It is easily verified that each of these functions gives such an addition. We prove in Lemmas 9 and 10 that these are the only such additions.

LEMMA 9. *The only additions on* $(0, \infty)$ *which have* $\theta = 1$ *are given by the functions* $F_1 - F_4$ *of Theorem* 1.

PROOF. If $T = \{f(x) | x > 0\}$, then it is clear that $1 \in T$. If follows from §3.7 that T is either $\{1\}$, $(0, \infty)$, $(0, 1]$ or $[1, \infty)$. Also, from §3.8, $f(x) = x$ for all x in T. Hence

(i) If $T = \{1\}$, then $f(x) = 1$ for all x.

(ii) If $T = (0, \infty)$, then $f(x) = x$ for all x.

(iii) If $T = (0, 1]$, then $f(x) = x$ for $0 < x \leqq 1$. Thus if $x > 1$, we see that $f(x^{-1}) = x^{-1}$ and so

$$1 = 1 \oplus xx^{-1} = 1 \oplus x(1 \oplus x^{-1}) = 1 \oplus x \oplus 1$$
$$= [1 \oplus x][1 \oplus (1 \oplus x)^{-1}] \leqq 1 \oplus x.$$

But $1 \oplus x \in (0, 1]$ and we conclude that $f(x) = 1$ for all $x > 1$.

(iv) If $T = [1, \infty)$, then $f(x) = x$ for all $x \geqq 1$. Thus if $0 < x < 1$, we see that $f(x^{-1}) = x^{-1}$ and so, as in (iii),

$$1 = [1 \oplus x][1 \oplus (1 \oplus x)^{-1}] \geqq 1 \oplus x.$$

But $1 \oplus x \in [1, \infty)$ and we conclude that $f(x) = 1$ for $0 < x < 1$.

In each of the four cases we have now fixed $f(x)$ for all x. If we proceed as in §3.1, we see that $\oplus$ is given by the functions $F_1 - F_4$ in the cases (i)—(iv) respectively.

LEMMA 10. *The only additions on* $(0, \infty)$ *with* $\theta \neq 1$ *are given by the function* F_5 *of Theorem* 1.

PROOF. (i) Suppose that $1 \oplus 1 = 2$. Then we show that $x \oplus y = x + y$ for all x, y.

If m is a positive integer, we denote by m^* the number which is the semiring sum of m ones. Then if m and n are positive integers with $m < n$, we see that

$$n^* = m^* \oplus (n-m)^* = m^* (1 \oplus (n-m)^*/m^*).$$

But as $\theta = f(1) > 1$, it follows from Lemma 5 that $f(x) > 1$ for all x. Thus $n^* > m^*$ if $n > m$. Also $(m^n)^* = (m^*)^n$ for any positive integers m and n; in particular, since $2^* = 1 \oplus 1 = 2$, it follows that $(2^n)^* = 2^n$.

Now let s be any integer $\geqq 3$. Then, as in §11.1 of [4], there exist sequences $\{p_n\}$, $\{q_n\}$ of positive integers such that $q_n \to \infty$ as $n \to \infty$ and

$$|\ln s / \ln 2 - p_n/q_n| < q_n^{-2} \quad (\text{all } n). \tag{1}$$

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Hence $p_n/q_n \to \ln s/\ln 2$ as $n \to \infty$, and we conclude that $2^{p_n/q_n} \to s$ as $n \to \infty$. On the other hand it follows from (1) that

$$2^{p_n-1} \leqq 2^{p_n-(1/q_n)} < s^{q_n} < 2^{p_n+(1/q_n)} \leqq 2^{p_n+1}.$$

Thus, from the properties of * deduced above,

$$2^{p_n-1} = (2^{p_n-1})^* < (s^{q_n})^* = (s^*)^{q_n} < (2^{p_n+1})^* = 2^{p_n+1}.$$

Therefore

$$2^{(p_n-1)/q_n} < s^* < 2^{(p_n+1)/q_n}.$$

If we allow $n \to \infty$, we see that

$$s^* = \lim_{n\to\infty} 2^{p_n/q_n} = s.$$

Now let $x = pq^{-1}$ and $y = mn^{-1}$, where p, q, m, n are positive integers, be any two rational numbers in $(0, \infty)$. Then it follows from the above that

$$\begin{aligned} x \oplus y = (qn)^{-1}(pn \oplus qm) &= (qn)^{-1}[(pn)^* \oplus (qm)^*] \\ &= (qn)^{-1}[(pn+qm)^*] \\ &= (qn)^{-1}(pn+qm) = x+y. \end{aligned}$$

Finally, if x and y are any two numbers in $(0, \infty)$, it follows from the continuity of $\oplus$ and $+$ that $x \oplus y = x+y$.

(ii) Suppose now that $\oplus$ is any addition with $\theta \neq 1$. Then the mapping $h(x) = x^{\ln 2/\ln \theta}$ maps $(0, \infty)$ homeomorphically onto itself and preserves ordinary multiplication. Hence, by Lemma 1, σ, defined by

$$x\sigma y = h[h^{-1}(x) \oplus h^{-1}(y)],$$

is an addition on $(0, \infty)$ with $1\sigma 1 = 2$. It follows from (i) that

$$x \oplus y = h^{-1}[h(x)\sigma h(y)] = h^{-1}[h(x)+h(y)].$$

Then, putting $c = \ln 2/\ln \theta$, we see that $F = F_5$.

5. $[0, \infty)$

THEOREM 2. *All additions of semirings on* $[0, \infty)$ *with ordinary multiplication are given by the following seven functions*:

$F_1(x, y) = 0$; $\quad F_2(x, y) = x$; $\quad F_3(x, y) = y$;

$F_4(x, y) = \min(x, y)$; $\quad F_5(x, y) = \max(x, y)$;

$F_6(x, y) = (x^c+y^c)^{1/c}$, *where* $c > 0$;

$$F_7(x, y) = \begin{cases} (x^c+y^c)^{1/c} & (x > 0 \text{ and } y > 0), \\ 0 & (x = 0 \text{ or } y = 0), \textit{ where } c < 0. \end{cases}$$

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PROOF. It is easily verified that each of these functions gives an addition on $[0, \infty)$. To show that these are the only ones, we consider two cases.

(i) If there is a $z > 0$ with $f(z) = 0$, then it follows from Lemma 7 that $F = F_1$.

(ii) If $f(x) > 0$ for all $x > 0$, then, for all $x, y > 0$,

$$x \oplus y = xf(yx^{-1}) > 0.$$

Hence F restricted to $(0, \infty) \times (0, \infty)$ is one of the functions listed in Theorem 1. Because F is continuous on $[0, \infty) \times [0, \infty)$, it follows that F is one of the functions F_2—F_7.

6. R_1

In this section we prove the following theorem.

THEOREM 3. *All additions of semirings on R_1 with ordinary multiplication are given by the following ten functions*:

$F_1(x, y) = 0;\qquad F_2(x, y) = x;\qquad F_3(x, y) = y;$

$F_4(x, y) = \operatorname{sgn} x \,.\, \min(|x|, |y|);$

$F_5(x, y) = \operatorname{sgn} y \,.\, \min(|x|, |y|);$

$F_6(x, y) = \frac{1}{2}(\operatorname{sgn} x + \operatorname{sgn} y) \,.\, \min(|x|, |y|);$

$F_7(x, y) = \operatorname{sgn}\{\operatorname{sgn} x \,.\, |x|^c + \operatorname{sgn} y \,.\, |y|^c\} \cdot |\operatorname{sgn} x \,.\, |x|^c + \operatorname{sgn} y \,.\, |y|^c|^{1/c}$,

where $c > 0$;

$$F_8(x, y) = \begin{cases} \operatorname{sgn} x \,.\, (|x|^c + |y|^c)^{1/c} & (x \neq 0,\ y \neq 0), \\ 0 & (x = 0 \text{ or } y = 0), \end{cases}$$

where $c < 0$;

$$F_9(x, y) = \begin{cases} \operatorname{sgn} y \,.\, (|x|^c + |y|^c)^{1/c} & (x \neq 0,\ y \neq 0), \\ 0 & (x = 0 \text{ or } y = 0), \end{cases}$$

where $c < 0$;

$$F_{10}(x, y) = \begin{cases} \frac{1}{2}(\operatorname{sgn} x + \operatorname{sgn} y) \,.\, (|x|^c + |y|^c)^{1/c} & (x \neq 0,\ y \neq 0), \\ 0 & (x = 0 \text{ or } y = 0), \end{cases}$$

where $c < 0$.

It is easily verified that each of these functions satisfies the requirements of an addition on R_1 with ordinary multiplication. We prove by a succession of lemmas that these are the only such additions.

LEMMA 11. *The only additions on R_1 which have $f(0) \neq g(0)$ are given by the functions F_2 and F_3 of Theorem* 3.

PROOF. It follows from § 3.4 that either $f(0) = 1$ and $g(0) = 0$, in which case, for any x, y,

$$x \oplus y = (x \oplus 0) \oplus y = x \oplus (0 \oplus y) = x \oplus 0 = x,$$

or $f(0) = 0$ and $g(0) = 1$, in which case it is similarly shown that $x \oplus y = y$ for all x, y.

Accordingly we can assume from now on that $f(0) = g(0)$.

LEMMA 12. *The only additions on R_1 with $f(0) = g(0) = 1$ are given by the function F_7 of Theorem 3.*

PROOF. It follows from Lemma 3 that $\theta \geqq 1$. Further, it follows from § 3.6 that $0 \in f(R_1)$. Thus $\theta \neq 1$, for otherwise it follows from § 3.8 that $f(0) = 0$, and we conclude that $\theta > 1$.

Because $f(0) = 1$, we see from Lemma 7 that $f(x) > 0$ for all $x > 0$. Hence, if $x, y > 0$, then

$$x \oplus y = xf(yx^{-1}) > 0.$$

Therefore F restricted to $[0, \infty) \times [0, \infty)$ gives an addition of a semiring on $[0, \infty)$ and so is one of the functions in Theorem 2. In particulaı, if $\theta = 2$, it follows thar $x \oplus y = x+y$ for all $x, y \geqq 0$.

From §3.6, we see that there is a $y \neq 0$ with $f(y) = -1$. Hence

$$\begin{aligned} f\{yf(-1)\} = 1 \oplus y(1 \oplus -1) &= (1 \oplus y) \oplus -y \\ &= -1 \oplus -y = -(1 \oplus y) = 1 \end{aligned}$$

and, since $\theta > 1$, we conclude from Lemma 5 that $f(-1) = 0$. Then, for any x,

$$x \oplus -x = xf(-1) = 0, \tag{1}$$

and

$$-x \oplus x = -(x \oplus -x) = 0. \tag{2}$$

We can now show that if $\theta = 2$, then $x \oplus y = x+y$ for all x, y in R_1. We consider four cases.

(a) If $x, y \geqq 0$, we have already shown this.

(b) If $x, y < 0$, then it follows from (a) that

$$x \oplus y = -|x| \oplus -|y| = -(|x| \oplus |y|) = -(|x|+|y|) = x+y.$$

(c) If $x < 0$ and $y \geqq 0$, then either $x+y \geqq 0$, in which case, from (a), (1) and §3.4,

$$\begin{aligned} x \oplus y = x \oplus [-x+(x+y)] = x \oplus [-x \oplus (x+y)] &= (x \oplus -x) \oplus (x+y) \\ &= 0 \oplus (x+y) = x+y, \end{aligned}$$

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or $x+y < 0$, in which case, by (b), (2) and §3.4,

$$\begin{aligned} x \oplus y = [(x+y)+-y] \oplus y &= [(x+y) \oplus -y] \oplus y \\ &= (x+y) \oplus (-y \oplus y) \\ &= (x+y) \oplus 0 = x+y. \end{aligned}$$

(d) If $x \geqq 0$ and $y < 0$, the result follows by arguments similar to those used in (c).

Suppose again that $\oplus$ is any addition on R_1 with $f(0) = g(0) = 1$. Then, because $\theta > 1$, the mapping

$$h(x) = \operatorname{sgn} x \, . \, |x|^{\ln 2/\ln \theta},$$

with inverse

$$h^{-1}(x) = \operatorname{sgn} x \, . \, |x|^{\ln \theta/\ln 2},$$

maps R_1 homeomorphically onto itself and preserves ordinary multiplication. By Lemma 1, σ, defined by

$$x\sigma y = h[h^{-1}(x) \oplus h^{-1}(y)],$$

is therefore an addition on R_1 with $1\sigma 1 = 2$. It follows that

$$x \oplus y = h^{-1}[h(x)\sigma h(y)] = h^{-1}[h(x)+h(y)].$$

If we put $c = \ln 2/\ln \theta$ (> 0), it now follows that $F = F_7$.

We now consider the possibilities when $f(0) = g(0) = 0$, and show that F_1, F_4–F_6 and F_8–F_{10} are the only resulting additions.

LEMMA 13. *If* $f(0) = g(0) = 0$, *then either*

(i) $\theta = 0$, *when* $F(x, y) = 0$ *for all* x, y *in* R_1, *or*

(ii) $\theta = 1$, *when* $F(x, y) = \min(x, y)$ *for* $x, y \geqq 0$, $f((0, \infty)) = (0, 1]$ *and* $f(x) \to 1$ *as* $x \to \infty$, *or*

(iii) $0 < \theta < 1$, *when there is a* $c < 0$ *with* $F(x, y) = (x^c+y^c)^{1/c}$ *for* $x, y > 0$, $F(x, y) = 0$ *for* $x = 0$ *or* $y = 0$, $f((0, \infty)) = (0, 1)$ *and* $f(x) \to 1$ *as* $x \to \infty$.

PROOF. Suppose, if possible, that $\theta < 0$. Then, using §3.5, we see that

$$f(\theta) = 1 \oplus (1 \oplus 1) = (1 \oplus 1) \oplus 1 = g(\theta) = \theta f(\theta^{-1}).$$

It follows that there is a $z < 0$ (and between θ and θ^{-1}) with $f(z) = 0$; hence $f(\theta) = 0$, by Lemma 8. But then

$$\theta^2 = 1 \oplus 1 \oplus 1 \oplus 1 = 1 \oplus (1 \oplus \theta) = 1 \oplus 0 = 0,$$

which is a contradiction. We conclude that $\theta \geqq 0$.

If $\theta = 0$, then (i) follows from Lemma 4.

If $\theta > 0$, it follows from Lemma 7 that $f(x) > 0$ for all $x > 0$. Hence

$x \oplus y > 0$ for all $x, y > 0$ and so F restricted to $[0, \infty) \times [0, \infty)$ is one of the functions listed in Theorem 2. Parts (ii) and (iii) of the lemma now follow.

LEMMA 14. *If* $f(0) = g(0) = 0$ *and* $f((-\infty, 0)) \subset (-\infty, 0)$, *then the only additions are given by* F_5 *and* F_9 *of Theorem* 3.

PROOF. Let $S_1 = f((0, \infty))$ and $S_2 = f((-\infty, 0))$, and suppose y_1, $y_2 \in S_2$. Then there exist $z, w < 0$ such that

$$y_1 y_2 = (1 \oplus z)(1 \oplus w) = f(z \oplus w \oplus zw).$$

Now

$$z \oplus w \oplus zw = z[1 \oplus wz^{-1}(1 \oplus z)] = zf[wz^{-1}f(z)].$$

Hence, since $f((-\infty, 0)) \subset (-\infty, 0)$, it follows that $z \oplus w \oplus zw > 0$, and so $y_1 y_2 \in S_1$. Therefore, $S_2 \subset S_1$. It follows from Lemma 13 that either $\theta = 1$ and $S_2 \subset [-1, 0)$, or $0 < \theta < 1$ and $S_2 \subset (-1, 0)$.

Suppose that $x < y < 0$; then $0 < yx^{-1} < 1$ and it follows from Lemma 13 that there is a $w > 0$ with $yx^{-1} = 1 \oplus w$. Hence

$$1 \oplus y = 1 \oplus x \oplus xw = [1 \oplus x][1 \oplus xw(1 \oplus x)^{-1}].$$

Then, since $xw(1 \oplus x)^{-1} > 0$, we see that $f(x) \leqq f(y)$ when $\theta = 1$ and $f(x) < f(y)$ when $0 < \theta < 1$.

It now follows that if $d = \inf S_2$, then $-1 \leqq d < 0$ and $\lim_{x \to -\infty} f(x) = d$. Hence

$$d^2 = \lim_{x \to -\infty} (f(x))^2 = \lim_{x \to -\infty} [1 \oplus \theta x(1 \oplus x\theta^{-1})].$$

Now as $x \to -\infty$, $1 \oplus x\theta^{-1} \to d$ and so $\theta x(1 \oplus x\theta^{-1}) \to \infty$. Therefore we see from Lemma 13 that $1 \oplus \theta x(1 \oplus x\theta^{-1}) \to 1$. That is, $d^2 = 1$ and so $d = -1$. We conclude that $(-1, 0) \subset S_2$.

If $\theta = 1$, it follows from §3.8 that $f(x) = x$ whenever $-1 < x < 0$. Further f is increasing and $f(x) \geqq -1$ for all $x < 0$. We conclude that $f(x) = -1$ for all $x \leqq -1$. It follows as in §3.1 that $F = F_5$.

If $0 < \theta < 1$, we show that $f(x) = -f(-x)$ for all $x < 0$. For suppose there exists a $y < 0$ for which this is not true. Then either

$$-1 < f(y)/f(-y) < 0 \quad \text{or} \quad -1 < f(-y)/f(y) < 0.$$

Thus, since $S_2 = (-1, 0)$, there is a $w < 0$ with

$$f(y) = f(-y)f(w) = f(-y \oplus w \oplus -yw)$$

or

$$f(-y) = f(y)f(w) = f(y \oplus w \oplus yw)$$

respectively. But it follows from Lemma 13 and the second paragraph of this proof that f is strictly increasing, which implies that

$$y = -y \oplus w \oplus -yw \quad \text{or} \quad -y = y \oplus w \oplus yw$$

respectively. That is,

$$-1 = 1 \oplus -wy^{-1} \oplus w \quad \text{or} \quad -1 = 1 \oplus wy^{-1} \oplus w$$

respectively, each of which contradicts the fact that $f(x) > -1$ for all x. Thus we have fixed $f(x)$ for any x; if we proceed as in §3.1, we see that $F = F_9$.

LEMMA 15. *The only additions on R_1 with $f(0) = g(0) = 0$ are given by the functions F_1, F_4—F_6, F_8—F_{10} of Theorem* 3.

PROOF. If $\theta = 0$, we have shown in Lemma 4 that $F = F_1$. If $\theta \neq 0$, it follows from Lemma 8 that there are only the following three possibilities.

(i) If $f((-\infty, 0)) = \{0\}$, then $f(x) = 0$ for all $x < 0$. It follows from Lemma 13 and §3.1 that $F = F_6$ or F_{10}.

(ii) If $f((-\infty, 0)) \subset (-\infty, 0)$, then, by Lemma 14, $F = F_5$ or F_9.

(iii) If $f((-\infty, 0)) \subset (0, \infty)$, then, since $f(x) = xg(x^{-1})$, it follows that $g((-\infty, 0)) \subset (-\infty, 0)$. Then, using the duality between f and g, it can be shown as in Lemma 14 that $F = F_4$ or F_8.

This completes all possible cases; Theorem 3 follows immediately from Lemmas 11, 12 and 15.

7. Intervals of types VI, VII and VIII

Let J_1, J_2, J_3 be intervals of types VI, VII, VIII respectively (where these roman numerals refer to those in the Introduction), and let K_1, K_2, K_3 be $(0, \infty)$, $[0, \infty)$, R_1 respectively. Further, put $d = \frac{1}{2}b$ in the cases of J_1 and J_2, and, in the case of J_3, put $d = \frac{1}{2}\min(|a|, |b|)$ (where a and b refer again to the Introduction).

If $1 \leqq i \leqq 3$, and if $\oplus$ is a binary operation defined on J_i into K_i, then we extend $\oplus$ to a binary operation σ on K_i into K_i as follows:

$$(1) \qquad x\sigma y = \begin{cases} x \oplus y & (x, y \in J_i), \\ |x|d^{-1}(d \operatorname{sgn} x \oplus yd|x|^{-1}) & (x \in K_i \sim J_i, y \in J_i), \\ |y|d^{-1}(xd|y|^{-1} \oplus d \operatorname{sgn} y) & (x \in J_i, y \in K_i \sim J_i), \\ xyd^{-2}(d^2y^{-1} \oplus d^2x^{-1}) & (x, y \in K_i \sim J_i). \end{cases}$$

It can be readily checked that this definition only assumes that $\oplus$ is defined on J_i, and also that σ takes its values in K_i.

LEMMA 16. *If $1 \leqq i \leqq 3$, and $\oplus$ is a binary operation on J_i into K_i such that*

(2) $$x(y \oplus z) = xy \oplus xz$$

for all x, y, z *in* J_i, *and if* σ *is defined as in* (1), *then*

(3) $$x(y\sigma z) = (xy)\sigma(xz)$$

for all x, y, z *in* K_i.

We first prove the following lemma.

LEMMA 17. *If* $\oplus$ *and* σ *are as in Lemma* 16, *then*

(4) $$u(v\sigma w) = (uv)\sigma(uw)$$

for all u, v, w *in* K_i *such that each of* $|u|, |uv|, |uw|, |uvw|$ *is at most* d.

PROOF. It is clear that each of uv, uw is in J_i. Hence, by (1), we see that

(5) $$(uv)\sigma(uw) = uv \oplus uw.$$

We consider four cases.

(i) If $v, w \in J_i$, then (4) follows from (2) because $\oplus$ and σ agree on J_i.

(ii) If $v \in K_i \sim J_i$ and $w \in J_i$, then, by (1),

$$u(v\sigma w) = u[|v|d^{-1}(d \operatorname{sgn} v \oplus wd|v|^{-1})] = d^{-1}[(u|v|)(d \operatorname{sgn} v \oplus wd|v|^{-1})].$$

Now $|uv| \leqq d$ and hence $u|v|$, $uv \in J_i$; also uw, $d \in J_i$. Thus, using in turn, (2), (2), (5), we see that

$$u(v\sigma w) = d^{-1}[duv \oplus duw] = d^{-1}d(uv \oplus uw) = (uv)\sigma(uw).$$

(iii) If $v \in J_i$ and $w \in K_i \sim J_i$, the proof is similar to that in (ii).

(iv) If $v, w \in K_i \sim J_i$, then, by (1),

$$u(v\sigma w) = u[vwd^{-2}(d^2w^{-1} \oplus d^2v^{-1})] = d^{-2}[(uvw)(d^2w^{-1} \oplus d^2v^{-1})].$$

Now each of uvw, uv, uw, d^2 is in J_i. Hence, using in turn (2), (2), (5), we see that

$$u(v\sigma w) = d^{-2}[d^2uv \oplus d^2uw] = d^{-2}d^2(uv \oplus uw) = (uv)\sigma(uw).$$

This completes the proof of Lemma 17.

PROOF OF LEMMA 16. Suppose $x, y, z \in K_i$ and let

$$m = d^{-2}\max(|x|, |xy|, |xz|, |xyz|).$$

Now (3) follows from Lemma 17 if $m \leqq d^{-1}$; hence we assume that $m > d^{-1}$. It is clear that if we substitute $m^{-2}x, y, z$ for u, v, w respectively, then the conditions of Lemma 17 are satisfied. Hence

(6) $$m^{-2}[x(y\sigma z)] = (m^{-2}x)(y\sigma z) = (m^{-2}xy)\sigma(m^{-2}xz).$$

If we now substitute m^{-2}, xy, xz for u, v, w respectively, the conditions of Lemma 17 are again satisfied. Thus

$$(m^{-2}xy)\sigma(m^{-2}xz) = m^{-2}[(xy)\sigma(xz)]. \tag{7}$$

The lemma now follows from (6) and (7).

LEMMA 18. *If $1 \leqq i \leqq 3$, let $\oplus$ be the addition of a semiring on J_i with ordinary multiplication, and define σ as in* (1). *Then σ is an addition of a semiring on K_i with ordinary multiplication.*

PROOF. We have shown above that σ takes its values in K_i and satisfies the distributive law. It remains to show that σ is continuous and associative.

Let $x, y, z, w \in K_i$. Then we must show that

$$(x\sigma y)\sigma z = x\sigma(y\sigma z) \tag{8}$$

and

$$\lim_{(x,y)\to(z,w)} (x\sigma y) = z\sigma w. \tag{9}$$

It is clear that there is a constant $K > 0$ such that each of the relevant quantities, viz $x, y, z, w, x\sigma y, y\sigma z, z\sigma w$, has modulus at most dK. Further, because of Lemma 16, it is clear that (8) and (9) are equivalent to

$$[(K^{-1}x)\sigma(K^{-1}y)]\sigma(K^{-1}z) = (K^{-1}x)\sigma[(K^{-1}y)\sigma(K^{-1}z)] \tag{10}$$

and

$$\lim_{(K^{-1}x,K^{-1}y)\to(K^{-1}z,K^{-1}w)} [(K^{-1}x)\sigma(K^{-1}y)] = (K^{-1}z)\sigma(K^{-1}w) \tag{11}$$

respectively. But we have chosen K such that all the relevant quantities in (10) and (11) are in J_i. Also $\oplus$ and σ agree on J_i, and $\oplus$ is continuous and associative on J_i. Hence (10) and (11) follow and the lemma is proved.

It follows from Lemma 18 that if $1 \leqq i \leqq 3$, then any addition $\oplus$ on J_i is given by the restriction to $J_i \times J_i$ of one of the functions which gives an addition on K_i. It is also clear that an addition σ on K_i gives an addition on J_i when so restricted if and only if $x\sigma y$ is in J_i for all x, y in J_i. The following three theorems now follow immediately from Theorems 1,2 and 3 respectively.

THEOREM 4. *All additions of semirings on an interval J_1, of type* VI, *with ordinary multiplication are given by the restrictions to $J_1 \times J_1$ of the functions $F_1 - F_4$ and F_5, when $c < 0$, defined in Theorem* 1.

THEOREM 5. *All additions of semirings on an interval J_2, of type* VII, *with ordinary multiplication are given by the restrictions to $J_2 \times J_2$ of the functions $F_1 - F_5$ and F_7 defined in Theorem* 2.

THEOREM 6. *All additions of semirings on an interval J_3, of type* VIII, *with ordinary multiplication are given by the restrictions to $J_3 \times J_3$ of the functions F_1-F_6 and F_8-F_{10} defined in Theorem* 3.

8. Intervals of type IX

THEOREM 7. *All additions of semirings on an interval J, of type* IX, *with ordinary multiplication are given by the restrictions to $J \times J$ of the functions F_1-F_4 and F_5, when $c > 0$, defined in Theorem* 1.

PROOF. The mapping $h(x) = x^{-1}$ maps J homeomorphically onto an interval J_1 of type VI, and preserves ordinary multiplication. It follows from Lemma 1 that any addition $\oplus$ on J is given by

$$x \oplus y = h^{-1}[h(x)\sigma h(y)]$$

where σ is an addition on J_1. The theorem now follows from Theorem 4.

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