MIXED PROBLEMS FOR LINEAR SYSTEMS OF FIRST ORDER EQUATIONS

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Introduction. A mixed problem in the theory of partial differential equations is an auxiliary data problem wherein conditions are assigned on two distinct surfaces having an intersection of lower dimension. Such problems have usually been formulated in connection with hyperbolic differential equations, with initial and boundary conditions prescribed. In this paper a study is made of the conditions appropriate to a system of R linear partial differential equations of first order, in R dependent and N independent variables. That such a system can be used to study a single linear equation of higher order, with one dependent variable, will be demonstrated in a later paper.

The method of analytical power series will be used, and applied to certain non-analytic problems by approximation procedures. However this study primarily reveals that a large class of analytic equations and systems can be treated in connection with mixed problems, and that the behaviour of solutions near the intersection of the two surfaces can be determined.

Mixed problems for normal hyperbolic linear equations of the second order have been treated by Krzyzanski and Schauder **(8)**, by Ladyzhenskaya **(9)**, and in **(3; 10; 11; 12)**. Systems of first order equations, restricted to the hyperbolic type, have been studied in the case of two independent variables by Campbell and Robinson **(2)** who establish mixed boundary and initial conditions by a Picard iteration process. The analytic systems treated here are not necessarily of hyperbolic type, although the existence of at least some characteristic surfaces is assumed.

The solution of a mixed problem for a hyperbolic equation or system is defined on a domain which is split up into two or more portions by certain characteristic surfaces. A reduction to standard form of such problems may be achieved by subtracting out a solution of a pure initial value problem with the given initial data. As the solution of the latter problem can be regarded as known **(10)** we shall employ this device. Thus the mixed problem is reduced to a problem wherein some of the data are given on a characteristic surface.

This fact has significance in a different connection. The basic existence theorem for analytic partial differential equations, the Cauchy-Kowalewsky theorem, contains the requirement that the system treated should be written in normal form **(4)**. This amounts to the condition that the datum surface be non-characteristic. Thus the above reduction of a mixed problem leads to an

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exceptional case of the Cauchy-Kowalewsky theorem. It is from this standpoint that we shall treat mixed problems involving a single characteristic surface, and consequently the theorem also constitutes a supplement to the Cauchy-Kowalewsky theorem. We remark that only for linear equations is a characteristic surface defined independently of solutions of the equations. For non-linear problems the data determine the characteristic surfaces, and a more complex situation arises.

Here is an outline of the detailed results. We first consider a characteristic surface of multiplicity μ and show that conditions given on a second surface are appropriate for an analytic solution. A certain algebraic condition appears in this result, and in the second part we study the case in which this condition fails and non-simple elementary divisors appear in certain coefficient matrices. It is shown that this case is not appropriate to a mixed problem.

The general problem in the analytic case, of an arbitrary number of characteristic surfaces, is then treated, with the assumption that each characteristic surface has simple elementary divisors. Series expansions for the solution functions are found in each of the several regions defined by the characteristic surfaces. Finally an extension of these results to the non-analytic case is made for symmetric hyperbolic systems, for which estimates of the Friedrichs-Lewy type are known (**6**; **7**).

1. The linear system. Consider the system of R linear partial differential equations of first order

$$a^i_{rs}\frac{\partial u_s}{\partial x^i} + b_{rs}u_s = f_r \qquad r, s = 1, \ldots, R, \quad i = 1, \ldots, N, \tag{1.1}$$

in R dependent variables u_r and N independent variables x^i. Here summation over the repeated indices s and i is understood. Since any linear system of partial differential equations can be reduced to a system of first order equations by taking suitable partial derivatives as new variables, (1.1) has considerable generality. We assume for the present that all coefficients, functions and solutions are real analytic functions of the variables x^i.

With matrix notation $\mathbf{A}^i = (a^i{}_{rs})$ for the coefficients and vector notation $\mathbf{u}$ for the unknowns we can write (1.1) as

$$\mathbf{Eu} = \mathbf{A}^i\frac{\partial \mathbf{u}}{\partial x^i} + \mathbf{Bu} = \mathbf{f}. \tag{1.2}$$

We note that the array $a^i{}_{rs}$ of the leading coefficients has two matrix indices r and s which will transform affinely under linear transformations of the dependent variables u_s, and one coordinate index i which can be taken to be contravariant under functional transformations of the coordinates x^i.

To apply the theorem of Cauchy and Kowalewsky to the system we choose a surface S: $\phi(x^i) = 0$ such that (1.1) can be written in normal form relative to S (**4**, p. 56). Thus if $t = \phi(x^i)$, the normal form is

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{E}_1\mathbf{u} + \mathbf{f}_1, \tag{1.3}$$

where derivatives with respect to t appear on the left only so that E_1 contains only differentiations with respect to $N - 1$ other coordinates. The theorem then asserts the existence of a unique analytic solution of (1.1) which assumes given analytic values on S. Thus R initial conditions are assigned. The condition of solvability for the transverse derivatives is readily computed and is found to be that the determinant

$$\left| \mathbf{A}^i \frac{\partial \phi}{\partial x^i} \right| \tag{1.4}$$

should not vanish (**14**, p. 30). Here a contraction over i is understood.

Let us consider the case when the determinant (1.4) does vanish on a surface G: $\phi(x^i) = 0$. Then the surface is, by definition, characteristic. If the matrix

$$\mathbf{A}^i \frac{\partial \phi}{\partial x^i} \tag{1.5}$$

has rank $R - \mu$, then μ will be called the multiplicity of G as a characteristic surface (**1**, p. 268). If G has multiplicity μ, it is possible to solve (1.1) for $R-\mu$ only of the derivatives $\partial u_r/\partial t$. We thus find $R - \mu$ equations of the form

$$\frac{\partial u_i}{\partial t} = L_i(uk) \qquad i = 1, \ldots, R - \mu, \tag{1.6}$$

together with μ further equations

$$L_j(u_r) = 0 \qquad j = R - \mu + 1, \ldots, R, \tag{1.7}$$

which contain no derivatives with respect to t. These latter are "inner" relations on G as they involve only the values and tangential derivatives of the u_r on G. Therefore they constitute necessary conditions for any set of values of the u_r on G. Thus it is to be expected that $R - \mu$ suitably chosen components u_r will determine on G the values of the remaining components, and so $R - \mu$ "initial" conditions are appropriate for G.

However, since (1.7) are differential equations on G, an equal number of initial conditions for them will be needed to determine uniquely all the initial values. We shall assign μ further conditions, subject to certain restrictions which will be stated below, on a second surface T: $\psi(x^i) = 0$ which we now introduce. Let T be not characteristic, and let G and T intersect in an edge C of $N - 2$ dimensions, which also will be analytic. We remark that if C is given then G may be determined as a characteristic surface passing through C, and composed of the characteristic curves of the characteristic equation

$$\left| \mathbf{A}^i \frac{\partial \phi}{\partial x^i} \right| = 0, \tag{1.8}$$

according to the theory of a single partial differential equation of the first order. These curves are the bicharacteristics of our system. Indeed we may

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suppose that $\phi(x^i)$ is constructed as a solution of (1.8), so that the family of surfaces $\phi(x^i)$ = const. are all characteristic.

2. Reduction to canonical form relative to a characteristic surface. In order to construct the solution described above, and to specify in detail the necessary conditions, we reduce the system (1.1) to an appropriate standard form relative to this problem. Let G and T have equations $\phi(x^i) = 0$, $\psi(x^i) = 0$ respectively, and set

$$t = x'^N = \phi(x^i), \qquad x = x'^{N-1} = \psi(x^i) \tag{2.1}$$

in a suitable new coordinate system x'^i. Dropping the primes we now let Greek indices ρ, σ run from 1 to $N - 2$ over the remaining coordinates $x^1, \ldots, x^{N-2}$.

Since $\phi(x^i) = x^N$, the matrix

$$A^N = \left(a^N_{rs} \right) = \left(a^i_{rs} \frac{\partial \phi}{\partial x^i} \right)$$

now has rank $N - \mu$. Let $y_r^{(\alpha)}$ and $z_s^{(\alpha)}$ denote the μ linearly independent left hand and right hand null vectors of this matrix. Thus

$$y_r^{(\alpha)} a^N_{rs} = 0, \qquad a^N_{rs} z_s^{(\alpha)} = 0, \qquad \alpha = 1, \ldots, \mu. \tag{2.2}$$

Multiplying (1.1) on the left by $y_r^{(\alpha)}$ we find

$$0 = -y_r^{(\alpha)} a^N_{rs} \frac{\partial u_s}{\partial x^N} = \sum_{\rho=1}^{N-1} y_r^{(\alpha)} a^\rho_{rs} \frac{\partial u_s}{\partial x^\rho} + \ldots,$$

which can be rewritten in the form

$$y_r^{(\alpha)} a^{N-1}_{rs} \frac{\partial u_s}{\partial x} = -\sum_{\rho=1}^{N-2} y_r^{(\alpha)} a^\rho_{rs} \frac{\partial u_s}{\partial x^\rho} + \ldots . \tag{2.3}$$

These μ equations are independent of derivatives with respect to t.

There are now $R - \mu$ equations containing derivatives with respect to t; we may write these in the form

$$\frac{\partial}{\partial t} \left(a^N_{rs} u_s \right) = -\sum_{\rho=1}^{N-1} a^\rho_{rs} \frac{\partial u_s}{\partial x^\rho} + \ldots,$$

where r varies from 1 to R. However as $a^N{}_{rs}$ has rank $R - \mu$, only that number of linearly independent combinations of the form

$$v_r = a^N_{rs} u_s \tag{2.4}$$

are generated. Let us number these independent combinations from 1 to $R - \mu$, and write the above equations as

$$\frac{\partial v_r}{\partial t} = \alpha_{rs} \frac{\partial u_s}{\partial x} + L_r(u_s), \qquad r = 1, \ldots, R - \mu. \tag{2.5}$$

The combinations v_r shall be called normal variables with respect to G. Here

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all the dependent variables are still denoted by u_s, a further μ linear combinations of the u_r remaining to be specified. $L_r(u_s)$ is a linear first order differential operator in the variables $x^\rho (\rho = 1, \ldots, N-2)$.

If we now define

$$w_{R-\alpha+1} = y_r^{(\alpha)} a_{rs}^{N-1} u_s, \qquad \alpha = 1, \ldots, \mu \tag{2.6}$$

we can write (2.3) in the form

$$\frac{\partial w_r}{\partial x} = L_r(u_s), \qquad r = R - \mu + 1, \ldots, R. \tag{2.7}$$

The linear combinations (2.6) shall be called null with respect to G, or more briefly, null. However we must show that the null quantities (2.6) are linearly independent of each other, and of the variables in (2.4), so that the combined new system (2.5) with (2.7) is equivalent to (1.1), being obtained from (1.1) by a non-singular linear transformation of the u_s into the v_s and w_s, together with linear combinations of a non-singular nature of the member equations of the system.

In the first place the combinations w_r $(r = R - \mu + 1, \ldots, R)$ of (2.6) are linearly independent. For otherwise, we should infer from a dependence

$$\sum_\alpha c_\alpha w_{R-\alpha+1} \equiv \sum_\alpha c_\alpha y_r^{(\alpha)} a_{rs}^{N-1} u_s \equiv 0$$

for all u_s, the relations

$$\sum_\alpha c_\alpha y_r^{(\alpha)} a_{rs}^{N-1} = 0.$$

However, since T was assumed non-characteristic, the matrix

$$(a_{rs}^{N-1})$$

is non-singular. It follows that

$$\sum c_\alpha y_r^{(\alpha)} = 0$$

which implies $c_\alpha = 0$ as the null vectors $y_r^{(\alpha)}$ are linearly independent.

We now ascertain the condition that the combined set (2.4) and (2.6) should be linearly independent. A linear relation of dependence takes the form

$$\sum_{r=1}^{R-\mu} c_r a_{rs}^N u_s + \sum_{\alpha=1}^{\mu} c_{R-\alpha+1} y_r^{(\alpha)} a_{rs}^{N-1} u_s \equiv 0,$$

and if this holds for all u_s we have

$$\sum_{r=1}^{R-\mu} c_r a_{rs}^N + \sum_{\alpha=1}^{\mu} c_{R-\alpha+1} y_r^{(\alpha)} a_{rs}^{N-1} = 0. \tag{2.8}$$

If a non-vanishing set of constants satisfies these equations, not all of either group may be zero. This has just been shown for the first group. For the second group, we refer to the definition (2.4) of the first group of the w_r and note that by a relabelling of columns we may assume that the $(R - \mu) \times (R - \mu)$

determinant $|a^N{}_{rs}|$ $(r, s = 1, \ldots, R - \mu)$ is not zero. Thus if the second term in (2.8) is assigned, the c_r in the first sum are determined by the $R - \mu$ equations with $s = 1, \ldots, R - \mu$.

Now the full determinant of (2.8) is

$$(2.9) \qquad \begin{vmatrix} a^N_{11}, \ldots, a^N_{R-\mu,1}, y_m{}^{(1)} a^{N-1}_{m1}, \ldots, y_m{}^{(\mu)} a^{N-1}_{m1} \\ \ldots \qquad\qquad \ldots \\ a^N_{1R}, \ldots, a^N_{R-\mu,R}, y_m{}^{(1)} a^{N-1}_{mR}, \ldots, y_m{}^{(\mu)} a^{N-1}_{mR} \end{vmatrix}.$$

Since the μ right null vectors $z_r{}^{(\alpha)}$ of $(a^N{}_{rs})$ are independent, we can multiply the sth row by $z_s{}^{(\alpha)}$ and subtract these multiples from the bottom μ rows in such a way as to make the $R - \mu \times \mu$ block in the lower left corner vanish. This can be effected by a choice of basis for the $z_s{}^{(\alpha)}$ such that

$$z^{(\alpha)}_{R-\mu+\beta} = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1, \ldots, \mu).$$

Such a choice of basis is possible since the $z_s{}^{(\alpha)}$ are independent and since $z_s{}^{(\alpha)} = 0$, $s > R - \mu$ implies

$$\sum_{s=1}^{R-\mu} a^N_{rs}\, z_s{}^{(\alpha)} = 0, \qquad r = 1, \ldots, R - \mu$$

which in turn implies $z_s{}^{(\alpha)} = 0$ as the determinant of this set of equations has been chosen different from zero. Thus if we multiply the sth row by $z_s{}^{(\beta)}$ and add these multiples to the $R - \mu + \beta$ row, the lower left hand $R - \mu \times \mu$ block will vanish. The determinant thus becomes the product

$$(2.10) \qquad |a^N_{rs}| \cdot |y_m{}^{(\alpha)} a^{N-1}_{mn} z_n{}^{(\beta)}|,$$

where $r, s = 1, \ldots, R - \mu$; $m, n = 1, \ldots, R$, and $\alpha, \beta = 1, \ldots, \mu$. The first factor is not zero. The null vectors $y_m{}^{(\alpha)}$ and $z_n{}^{(\beta)}$ depend on the indices m and n in such a way that the combination in the second determinant is invariant under linear transformations of the u's. Since

$$a^{N-1}_{rs} = a^i_{rs} \frac{\partial \psi}{\partial x^i},$$

in a general coordinate system, we see that the general invariant condition for the non-vanishing of (2.9) is that the determinant

$$(2.11) \qquad \left| y_m{}^{(\alpha)} a^i_{mn} \frac{\partial \psi}{\partial x^i} z_n{}^{(\beta)} \right| \neq 0.$$

Here $\alpha, \beta = 1, \ldots, \mu$; $i = 1, \ldots, N$, and $m, n = 1, \ldots, R$.

We now show that this condition is satisfied in the case when G has multiplicity one, provided that the edge $C = G \cap T$ is nowhere tangent to the bicharacteristic direction on G. If we define the contravariant vector

$$(2.11) \qquad h^i = y_m a^i_{mn} z_n \qquad (\alpha = \beta = 1),$$

then (2.10) implies

$$(2.12) \qquad h^i \frac{\partial \psi}{\partial x^i} \neq 0.$$

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The result will be established if we can show that h^i is parallel to the bicharacteristic direction defined by

(2.13) $$\frac{dx^i}{ds} = F_{p_i}, \qquad F = |a^i_{rs}\,p_i|\,.$$

We assume that not all of the F_{p_i} vanish, so that the bicharacteristic direction is well defined. On G we have $p_i = 0, i < N$, and $p_N = 1$ since the equation of G is $t \equiv x^N = 0$.

Now

(2.14) $$F_{p_i} = \sum_{r,s} a^i_{rs}\,M_{rs}(p),$$

where $M_{rs}(p)$ is the cofactor of the determinant F with respect to the r,s position. With $p_i = \delta_{iN}$, we find

(2.15) $$F_{p_i} = \sum_{r,s} a^i_{rs}\,m_{rs}$$

where now m_{rs} is the cofactor in $|a^N{}_{rs}|$. Since

(2.16) $$\sum_t a^N_{tr}\,m_{ts} = \delta_{rs}\,|\,a^N_{rs}\,| = 0$$

we see that for every s, $m_{r(s)}$ is a null vector on the left for the matrix $a^N{}_{rs}$ of rank $R - 1$. Thus $m_{r(s)} = y_{r(s)}$ in our previous notation for the null vectors, where, however, the bracketed index is inactive. Similarly, for any r, $m_{(r)s}$ is a null vector of $z_{s(r)}$ on the right. Since $a^N{}_{rs}$ has rank $R - 1$ the minors m_{rs} are not all zero; thus suppose $m_{ab} \neq 0$. Since there is only one independent null vector, we have

$$m_{rs} = c_s m_{rb},$$

where c_s is independent of r. Setting $r = a$, we find

$$c_s = \frac{m_{as}}{m_{ab}}$$

whence

$$m_{rs} = \frac{m_{as}\,m_{rb}}{m_{ab}}\,.$$

Thus

(2.17 $$\begin{aligned} F_{p_i} &= \sum_{r,s} a^i_{rs}\,\frac{m_{as}\,m_{rb}}{m_{ab}} \\ &= \frac{1}{m_{ab}} \sum_{r,s} m_{rb}\,a^i_{rs}\,m_{as}, \end{aligned}$$

and since $m_{rb} = k_b y_r$, $m_{as} = \bar{k}_a z_s$, where k_b, $\bar{k}_a$ depend only on the indicated suffixes, and are each different from zero since $m_{ab} \neq 0$, we find

(2.18) $$F_{p_i} = \frac{k_b\,\bar{k}_a}{m_{ab}} \sum_{rs} y_r\,a^i_{rs}\,z_s = \frac{k_b\,\bar{k}_a}{m_{ab}}\,h^i.$$

This proves that the vector h^i is parallel to the direction of the bicharacteristic displacement, as required.

An example of a case where this condition (2.12) holds for a simple characteristic surface ($\mu = 1$) is when the system (1.1) is symmetric, so that $a^{i}{}_{rs} = a^{i}{}_{sr}$, and when T is spacelike, which means, in effect, that $a^{N-1}{}_{rs}$ is positive definite **(6)**. The condition (2.12) is satisfied since $z_r = y_r$ and

$$(2.19) \qquad h^i \frac{\partial \psi}{\partial x^i} = y_r\, a^i_{rs} \frac{\partial \psi}{\partial x^i} z_s = a^{N-1}_{rs}\, y_r\, y_s > 0.$$

The auxiliary conditions to be applied on G and T will now be formulated, and will be referred to as initial and boundary conditions, respectively. For initial conditions (on G) we assign $R - \mu$ linear and independent combinations of the variables u_s in the form

$$(2.20) \qquad e^{\lambda}_r\, u_r = g^{\lambda}, \qquad \lambda = 1, \ldots, R - \mu.$$

An algebraic restriction necessary for our theorem is that it should be possible to solve these conditions for the normal group (2.4) of transformed variables $v_r (r = 1, \ldots, R - \mu)$. From (2.4) and the non-vanishing of $|a^{N}{}_{rs}|$ $(r, s = 1, \ldots, R - \mu)$, we have

$$u_s = A^{N}{}_{rs}\, v_r + \ldots$$

where $A^{N}{}_{rs}$ denotes an inverse matrix; and we shall therefore be able to solve (2.20) as required if on G the determinant

$$(2.21) \qquad |e^{\lambda}{}_s\, A^{N}{}_{sr}| \neq 0, \qquad \lambda, r, s = 1, \ldots, R - \mu,$$

which we now assume.

The form of the μ additional boundary conditions to be imposed on T is also linear:

$$(2.22) \qquad a^{\nu}_r\, u_r = g^{\nu} \qquad \nu = R - \mu + 1, \ldots, R.$$

We require that these equations be solvable for the null variables w_r of (2.7). To determine the condition necessary for this, we note that if (2.22) are not thus solvable, there will exist a linear dependence among the $R - \mu$ unknowns of the group (2.4) and the left side of (2.22) of the form

$$(2.23) \qquad \sum_{r=1}^{R-\mu} \alpha_r\, a^{N}{}_{rs}\, u_s + \sum_{\nu=R-\mu+1}^{R} c_{\nu}\, a^{\nu}_s\, u_s = 0.$$

Since the u_s are arbitrary, we find

$$(2.24) \qquad \sum_{r=1}^{R-\mu} \alpha_r\, a^{N}_{rs} + \sum_{\nu=R-\mu+1}^{R} c_{\nu}\, a^{\nu}_s = 0.$$

We shall require all solutions (α_r, c_{ν}) of these linear homogeneous equations to vanish, and thus assume that the determinant

$$(2.25) \qquad |a^{N}_{1s}, \ldots, a^{N}_{R-\mu, s}, a^{R-\mu+1}_s, \ldots, a^{R}_s| \neq 0.$$

Here the suffix s labels the R rows of the array. Multiplying the rows by components $z_s^{(\alpha)}$ of the right-hand null vectors as in (2.9), we can cause the lower left hand block of R $-\mu \times \mu$ terms to disappear. The determinant splits into two factors:

$$| a_{rs}^{N} |_{r,s=1,\ldots,R-\mu} \cdot | a_s^{\lambda} z_s^{(\nu)} |_{\lambda,\nu=R-\mu+1,\ldots,R}$$

of which the first is not zero. The necessary condition is therefore the non-vanishing of the determinant of order $R - \mu$:

$$| a_s^{\lambda} z_s^{(\nu)} | \neq 0. \tag{2.26}$$

Here s is summed from 1 to R in each element, and this condition will apply on the surface T.

On the edge of intersection C both of these sets of conditions should apply. We assume that taken together (2.20) and (2.22) shall determine the values of all R dependent variables uniquely on C. The compatibility as well as the uniqueness of such values will be assured if we suppose that the $R \times R$ determinant

$$| e_r^1, \ldots, e_r^{R-\mu}, a_r^1, \ldots, a_r^{\mu} | \neq 0. \tag{2.27}$$

3. Construction of the solution. The preceding calculations lead to a standard form for the differential system, and for the auxiliary conditions, which we shall now employ. The differential equations are a group in normal form

$$\frac{\partial v_r}{\partial t} = \alpha_{rs} \frac{\partial v_s}{\partial x} + L_r(v_s, w_s) + f_r, \qquad r = 1, \ldots, R - \mu, \tag{3.1}$$

and a group in which derivatives with respect to x of the null variables appear:

$$\frac{\partial w_r}{\partial x} = L_r(v_s, w_s) + f_r, \qquad r = R - \mu + 1, \ldots, R. \tag{3.2}$$

The operators $L_r(v_s, w_s)$ of the first order contain no differentiations with respect to t or to x. The initial conditions are given by values of the normal group as linear combinations of the null group:

$$v_r = c_{r\lambda} w_\lambda + g_r \qquad \begin{array}{l} r = 1, \ldots, R - \mu \\ \lambda = R - \mu + 1, \ldots, R. \end{array} \tag{3.3}$$

These hold for $t = 0$. The boundary conditions on T are of the opposite type:

$$w_\lambda = \tilde{c}_{\lambda r} v_r + g_\lambda \qquad \begin{array}{l} \lambda = R - \mu + 1, \ldots, R \\ r = 1, \ldots, R - \mu, \end{array} \tag{3.4}$$

and this seems to prevent the ordering used by Riquier **(16)**.

Now (3.2), (3.3) and (3.4) enable us to determine initial values for all of the w_r on G. Consider the null group of differential equations on the $N - 1$ dimensional surface G and note that relative to the edge C and the set w_r of unknowns they are in Cauchy-Kowalewsky normal form. The v_r are to be replaced by their values (3.3) on the right side of (3.2), so that a self-contained system for the w_r is established. On the edge C the initial values for the w_r are assigned, and it follows from the Cauchy-Kowalewsky theorem that a unique analytic

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system of values for the w_r on G exists. This process determines initial values on G for all of the variables.

Let the unknowns now be expanded in a series of powers of t: thus

$$w_r = \sum_{n=0}^{\infty} w_{r(n)}\,(x, x_\rho)\,t^n. \tag{3.5}$$

The coefficients, functions, and operators appearing in the differential equations or auxiliary conditions shall also be expanded in power series of t, the coefficient of t^n of such a function f_r being denoted by $f_{r(n)}$. We have to show that the coefficients in (3.5) can be determined recursively.

Suppose known all coefficients of index less than $n + 1$, and let us calculate the $w_{r(n+1)}$. To do this, let us expand (3.1), (3.2) and (3.4) in powers of t and equate coefficients of equal powers on both sides. It is found that

$$n\,v_{r(n+1)} = \sum_{\nu=0}^{n} \alpha_{rs(n-\nu)} \frac{\partial v_{s(\nu)}}{\partial x} + \sum_{\nu=0}^{n} L_{r(n-\nu)}(v_\nu, w_\nu) + f_{r(n)}, \qquad r = 1, \ldots, R - \mu, \tag{3.6}$$

$$\frac{\partial w_{r(n+1)}}{\partial x} = \bar{L}_{r(0)}(v_{s(n+1)}) + \tilde{L}_{r(0)}(w_{s(n+1)}) + \sum_{\nu=0}^{n-1} L_{r(n-\nu)}(v_{s(\nu)}, w_{s(\nu)}) + f_{r(n+1)} \qquad r = R - \mu + 1, \ldots, R, \tag{3.7}$$

and

$$w_{r(n+1)} = \sum_{\nu=0}^{n+1} \tilde{c}_{ra(n-\nu)}\, w_{a(\nu)} + g_{r(n+1)}, \qquad x = 0. \tag{3.8}$$

Here $\bar{L}$ and $\tilde{L}$ denote the terms in L which contain the v_r and the w_r respectively.

Assuming known all coefficients of index not exceeding n, we can calculate from (3.7) the values of the $v_{r(n+1)}$ $(r = 1, \ldots, R - \mu)$. Then all terms on the right of (3.7) except the first or L term are known, and (3.7) can be regarded as a system of differential equations for the determination of the $w_{r(n+1)}$. Together with (3.8) as initial conditions, these equations are in normal form relative to the variety $x = 0$. Thus, by the Cauchy-Kowalewsky theorem, we conclude that the $w_{r(n+1)}$ exist and are uniquely determined by a power series in x and x_ρ convergent for sufficiently small values of these variables. This completes the step of the recursive construction, and shows that the series (3.5) can be formed term by term.

4. Convergence of the series expansion. To complete this existence theorem it is necessary to show that the formal power series (3.5) converges for some interval of values of t. For this purpose we shall consider that the $v_{r(n)}(x, x_\rho)$ have been expanded in a multiple power series and that this has been substituted in (3.5). If a non-trivial domain of convergence for the resulting multiple series is established, this will show, by absolute convergence, the result desired. The convergence will be established by means of dominant series constructed from a similar but dominant differential problem. From (3.1), (3.2), (3.4) and the computations of the Cauchy-Kowalewsky solution of (3.7), for each value of n, all of which involve only additions and multiplications, it is clear that if we find dominant series for all coefficients on the right sides of (3.1), (3.2), (3.4) and for the initial values $v_{r(0)}$, $w_{r(0)}$, then the series solution so determined will dominate the original one. By subtracting $v_{r(0)}$ or $w_{r(0)}$ from each of the unknowns we can assume that the initial values are zero.

All of the normal unknowns v_r of the first group shall be dominated by a single function V, and all those of the null group w_n by a second function W. Let $y = x^1 + \ldots + x^{N-2}$ and set

$$z = y + \frac{x}{\alpha} + \frac{t}{\alpha^2}, \qquad 0 < \alpha < 1, \tag{4.1}$$

where α is left undetermined for the present. The dominant system consists of two differential equations, so chosen as to dominate the right sides of (3.1) and (3.2) respectively. For a suitable choice of M, M_1, F, F_1 and ρ, these equations can be written as

$$\frac{\partial V}{\partial t} = \frac{M}{1-(x+y+t)/\rho}\left[\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + V + W + F\right] \tag{4.2}$$

and

$$\frac{\partial W}{\partial x} = \frac{M_1}{1-(x+y+t)/\rho}\left[\frac{\partial V}{\partial y} + \frac{\partial W}{\partial y} + V + W + F\right]. \tag{4.3}$$

Here a single dominant series is selected for all coefficients on the right side of (3.1) and (3.2) respectively. A separate choice of constants F and F_1 is made for the non-homogeneous terms which, after the reduction of initial values to zero, will depend upon the given initial data. If V and W dominate the v_r and w_r respectively, then the above right hand sides will dominate the corresponding members of (3.1) and (3.2).

Dominant forms of the auxiliary conditions are

$$V \gg 0, \qquad t = 0, \tag{4.4}$$

for the initial conditions, and

$$W \gg \frac{M_2}{1-(t+y)/\rho}[V + F_2], \qquad x = 0, \tag{4.5}$$

for boundary conditions. It is the direction of the dominating relations which is significant here.

Since we are free to increase any of the constants in the dominating series, we may suppose M_1 so large that

$$M_1 > 2MM_2. \tag{4.6}$$

We now assume, for the purpose of finding a convergent solution of this system, that V and W are functions of the single variable z only. By writing

$1 - z/\alpha^2\rho$ in place of $1 - (x + y + t)/\rho$ in certain denominators we do not decrease any coefficients in the series expansions, and so maintain the necessary domination. Thus we find (derivatives with respect to z being denoted by primes)

$$\frac{1}{\alpha^2} V' = \frac{M}{1 - z/\alpha^2\rho}\left[\left(\frac{1}{\alpha} + 1\right) V' + \left(\frac{1}{\alpha} + 1\right) W' + V + W + F\right] \tag{4.7}$$

and

$$\frac{1}{\alpha} W' = \frac{M_1}{1 - z/\alpha^2\rho}[V' + W' + V + W + F_1]. \tag{4.8}$$

To this system we adjoin the boundary condition

$$W \gg \frac{M_2}{1 - z/\alpha^2\rho}[V + F_2], \tag{4.9}$$

to replace (4.5). If (4.9) holds, and if also $V \gg 0$ (that is, V has positive coefficients) then the relation (4.5) will be maintained for $x = 0$.

Rearranging terms in (4.7) and (4.8), which are ordinary differential equations with variable z, we find

$$\left[\frac{1}{\alpha} - \frac{z}{\alpha^3\rho} - M(1 + \alpha)\right]V' - (1 + \alpha)MW' = M\alpha[V + W + F] \tag{4.10}$$

and

$$\left[\frac{1}{\alpha} - \frac{z}{\alpha^3\rho} - M_1\right]W' - MV' = \bar{M}_1[V + W + F]. \tag{4.11}$$

On the right side of (4.11) we have introduced a new constant $\bar{M}_1 \geqslant M_1$ to replace M_1 in that position: this is a permissible alteration.

We have to show that the system has a convergent series solution with positive coefficients such that (4.9) holds, and we will be able to assign initial values $U(0)$ and $V(0)$ at will. The choice of α is still open. Let us begin by showing that the coefficients in any such series are positive, provided that U(0) and V(0) are positive. Set

$$V(z) = \sum_{n=0}^{\infty} v_n z^n, \qquad W(z) = \sum_{n=0}^{\infty} w_n z^n. \tag{4.12}$$

Then (4.10) and (4.11) yield the recursion formulae

$$\begin{aligned} \left(\frac{1}{\alpha} - M(1 + \alpha)\right)(n + 1)v_{n+1} - M(1 + \alpha)(n + 1)w_{n+1} \\ = \left(\frac{n}{\alpha^3\rho} + M\alpha\right)v_n + M\alpha w_n + F\delta_{0n}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} - M_1(n + 1)v_{n+1} + \left(\frac{1}{\alpha} - M_1\right)(n + 1)w_{n+1} \\ = \bar{M}_1 v_n + \left(\frac{n}{\alpha^3\rho} + \bar{M}_1\right)w_n + F_1\delta_{0n}. \end{aligned} \tag{4.14}$$

Here δ_{0n} indicates the value one for $n = 0$, and zero otherwise. These equations have the form, after division by $n+1$,

$$(4.15)\qquad \begin{aligned} Av_{n+1} + Bw_{n+1} &= F_n, \\ Cv_{n+1} + Dw_{n+1} &= G_n, \end{aligned}$$

where $F_n > 0$, $G_n > 0$ if we suppose u_n, v_n both positive. Now for sufficiently small positive α,

$$(4.16)\qquad \begin{aligned} A &= \frac{1}{\alpha} - M(1+\alpha) > 0, & B &= -M(1+\alpha) < 0, \\ C &= -M_1 < 0 \quad , & D &= \frac{1}{\alpha} - M_1 > 0. \end{aligned}$$

The solutions of (4.15) are given by

$$(4.17)\qquad \begin{aligned} (AD - BC)v_{n+1} &= F_n D - G_n B > 0, \\ (AD - BC)w_{n+1} &= G_n A - F_n C > 0. \end{aligned}$$

Thus u_{n+1} and v_{n+1} will be positive provided that the determinant

$$\begin{aligned} AD - BC &= \frac{1}{\alpha^2}[1 - \alpha(1+\alpha)M - \alpha M_1] \\ &> \frac{1}{\alpha^2}[1 - \alpha(2M + M_1)] \end{aligned}$$

is positive. This condition, as well as (4.16), can be achieved if we set

$$(4.18)\qquad \alpha = \frac{1}{2M + M_1}.$$

Now the boundary condition (4.9) will hold if

$$(4.19)\qquad \left(1 - \frac{z}{\alpha^2 \rho_2}\right) W \gg M_2[V + F_2], \qquad 0 < \rho_2 \leqslant \rho,$$

and we will show that this relation follows from (4.11) provided only that

$$(4.20)\qquad W(0) > M_2 V(0) + F_2,$$

a condition which is clearly necessary in any case. Dividing (4.11) by a certain constant, we have

$$(4.21)\qquad \begin{aligned} \left(1 - \frac{z}{\alpha^2 \rho(1 - \alpha M_1)}\right) W' - \frac{\alpha M_1}{1 - \alpha M_1} V' - \bar{M}_1 W \\ = \bar{M}_1[V + F_1] \gg 0. \end{aligned}$$

Simplifying the coefficient of V' by means of (4.18), we find

$$(4.22)\qquad \left(1 - \frac{z}{\alpha^2 \rho(1 - \alpha M_1)}\right) W' - \frac{M_1}{2M} V' - \bar{M}_1 W \gg 0.$$

Now the derivative of (4.19) is

$$(4.23)\qquad \left(1 - \frac{z}{\alpha^2 \rho_2}\right) W' - M_2 V' - \frac{1}{\alpha^2 \rho_2} W \gg 0,$$

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and we wish to show that by proper adjustment of constants this will follow from (4.22). We can choose for ρ_2 any value less than ρ, and the dominance of the boundary condition will persist. Thus let us choose

$$\rho_2 = \rho(1 - \alpha M_1), \qquad \bar{M}_1 > \frac{1}{\alpha^2 \rho_2}.$$

Then, in view of (4.6), and the fact that V and W are series with positive coefficients, (4.22) will imply (4.23). Assuming now that (4.23) holds, as well as (4.20), we see that (4.19) will be valid. Multiplying each side of (4.19) by the series with positive coefficients which is the reciprocal of $1 - z/\alpha^2\rho_2$, we find (4.9). This establishes the dominance of the boundary conditions.

The pair of linear ordinary first order differential equations (4.10) and (4.11) have the origin $z = 0$ as an ordinary point in view of (4.18). Consequently there exists a convergent series solution satisfying (4.20), for example with $V(0) = 1$, $W(0) = 2M_2 + 2F_2$, and the radius of convergence of these series is determined by the singular points of (4.10) and (4.11). Thus this radius depends on M, M_1, M_2 and ρ, but not on F, F_1 or F_2. By the substitution (4.1), dominant multiple series having a positive radius of convergence (independent of the initial or boundary data) are found for the series solutions of the original problem. This completes the proof that the latter series converge for sufficiently small values of the coordinate variables.

The origin of coordinates can be chosen at will on the edge C and by analytic continuation a solution will exist in a region containing any given compact portion of C. If uniform hypotheses regarding the coefficients of the original problem are made, this local solution can be extended to large intervals of the x and t coordinates by analytic continuation as is usual for analytic linear differential equations.

To sum up, we have

THEOREM I. *Let $G : \phi(x^i) = 0$ be a characteristic surface of multiplicity μ relative to the analytic linear system*

$$\mathbf{A}^i \frac{\partial \mathbf{u}}{\partial x^i} + \mathbf{B}\mathbf{u} = \mathbf{f}, \tag{1.2}$$

of R first order equations. Let $T : \psi(x^i) = 0$ intersect G in an edge C such that, as in (2.11),

$$\left| y_m^{(\alpha)} a^i_{mn} \frac{\partial \phi}{\partial x^i} z_n^{(\beta)} \right| \neq 0.$$

Then there exists a unique analytic solution which satisfies $R - \mu$ initial conditions of type (2.20), (2.21) *on G and μ boundary conditions of type* (2.22), (2.26) *on T.*

In order to bring this result into relation with a mixed problem, let us note that if the non-homogeneous terms in (3.2) and (3.3) are zero on G, and if the data g_λ in (3.4) vanish on C, then the values $w_r(0)$ of all components of the

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solution on G will be zero. Now let Cauchy data (values of all components) be assigned on a non-characteristic surface S, and let T be a second non-characteristic surface meeting S in an edge C. Let G, a characteristic surface of multiplicity μ, pass through C and divide into two regions R_S and R_T one of the four regions defined by S and T. Suppose that the necessary condition (2.11) is satisfied, and let us determine a solution analytic in R_S and in R_T, and continuous across G, which takes the Cauchy values on S and satisfies boundary conditions of type (2.22), (2.26) on T. Subtracting away the Cauchy-Kowalewsky solution of the initial value problem on S, we are left with a homogeneous system and homogeneous auxiliary conditions on G. By the remark above, we can define a solution analytic in R_T, and vanishing on G, provided that the functions g_λ in (3.4) vanish on C. Thus a piecewise analytic solution is found for the mixed problem by adding (in R_T) this solution of the characteristic problem. The restriction on the data g_λ is a compatibility condition of the first order. This construction includes as special cases the mixed problems for linear second order equations treated in **(3, 8)**, but we shall not state it as a separate theorem since a more general problem is treated below.

5. Case of non-simple elementary divisors. If the basic condition (2.11), which permits reduction of the differential equations to the standard forms (3.1) and (3.2), is not satisfied, a somewhat different proof is required. It turns out that the theorem still holds in very much the same form, but that the solution on the characteristic surface is affected by the boundary data on the whole surface T, not just the edge C. This has the consequence that the result is not directly applicable to any mixed problem.

The earlier calculations and reductions have all been made essentially as if the number of independent variables were two; this is an advantage of the analytic case made possible by the generality of the Cauchy-Kowalewsky theorem. We shall now employ the general standard form for a system of first order equations in two variables, as presented by Petrowsky **(14**, p. 54**)** for example, where the Jordan normal form of $\mathbf{A}^N$ relative to $\mathbf{A}^{N-1}$ is used **(17**, p. 137**)**. Let $G : \phi(x^i) \equiv t = 0$ be a characteristic surface of multiplicity μ and let $T : \psi(x^i) \equiv x = 0$ by a non-characteristic boundary meeting G in the edge C as before. We select the two variables t and x and perform the reduction to canonical form with respect to them. Since T is non-characteristic, $\mathbf{A}^{N-1}$ is non-singular, and can be brought to unit matrix form. If then $\mathbf{A}^N$ is reduced to Jordan normal form by linear transformations of the u_r, the process is completed. That G is a characteristic surface of multiplicity μ signifies that μ of the characteristic roots of the coefficient matrix $\mathbf{A}^N$ relative to $\mathbf{A}^{N-1}$ are zero. With this simplification we can write the first μ of the equations in the form

$$(5.1) \qquad \begin{aligned} \frac{\partial w_1}{\partial x} &= L_1(v_r, w_s), \\ \frac{\partial w_r}{\partial x} &= \alpha_{r-1}\frac{\partial w_{r-1}}{\partial t} + L_r(v_n, w_s) \qquad r = 2,3,\dots,\mu. \end{aligned}$$

Here the v_r and w_s denote appropriate linear combinations of the original dependent variables, while the coefficients α_{r-1} and differential operators L_r depend on the coordinates t, x, and $x_\rho(\rho = 1, \ldots, N-2)$. The L_r contain only differentiations with respect to the x_ρ.

The second set of equations will contain derivatives with respect to t and x of the v_s only. Since all other characteristic roots of $\mathbf{A}^N$ differ from zero, we can write these equations in a form solved for the derivatives with respect to t. Thus

$$(5.2)\qquad \begin{aligned}\frac{\partial v_1}{\partial t} &= \beta_1 \frac{\partial v_1}{\partial x} + L_{\mu+1}(v_n, w_s),\\ \frac{\partial v_s}{\partial t} &= \beta_s \frac{\partial v_s}{\partial x} + \gamma_{s-1}\frac{\partial v_{s-1}}{\partial x} + L_{\mu+s}(v_n, w_s).\end{aligned}$$

The operators $L_{\mu+s}$ are again independent of $\partial/\partial x$ and $\partial/\partial t$.

We remark that the difficulties of this particular problem arise from the presence of the coefficients α_{r-1} and γ_{r-1}, which appear in the canonical forms of the original coefficient matrices because of certain non-simple elementary divisors (**17**, p. 137). The variables v_s in (5.2) may again be termed normal with respect to the characteristic surface G. We shall again refer to the w_r as the null variables proper to the characteristic value $\lambda = 0$, that is, to the characteristic surface G. Indeed, the reduction to canonical form shows that these null variables are obtained from the u_s by contraction with a suitable characteristic, or proper, vector of the coefficient matrix $\mathbf{A}^N$ (**14**, pp. 54-58).

For simplicity we assign auxiliary conditions of a more restricted type: values of the $w_r(r = 1, \ldots, \mu)$ on T and values of the $v_s(s = 1, \ldots, R-\mu)$ on G.

Power series expansions in the two variables t and x are required: thus

$$(5.3)\qquad \begin{aligned} v_r &= \sum v_{r(m,n)} x^m t^n \\ w_s &= \sum w_{s(m,n)} x^m t^n.\end{aligned}$$

Inserting these series developments in (5.1) and (5.2) and equating coefficients of like powers of x and t, we find recursion formulae

$$(5.4)\qquad \begin{aligned}(m+1)w_{r(m+1,n)} = (n+1)&w_{r-1(m,n+1)}\alpha_{r-1(0,0)} + \\ &\ldots + L_{r(0,0)}(v_{k(m,n)} w_{k(m,n)})\end{aligned}$$

and

$$(5.5)\qquad \begin{aligned}(n+1)v_{s(m,n+1)} &= (m+1)v_{s(m+1,n)}\beta_{s(0,0)} + \ldots \\ &+ (m+1)v_{s-1(m+1,n)}\gamma_{s-1(0,0)} + \ldots \\ &+ L_{\mu+s(0,0)}(v_{k(m,n)}, w_{l(m,n)}) + \ldots\end{aligned}$$

Here all terms omitted contain coefficients of powers of x less than $m+1$ and powers of t less than $n+1$. Also we shall understand that $\alpha_{-1} \equiv 0$ and $\gamma_{-1} \equiv 0$, so that the first equation of each set (5.1), (5.2) need not be written separately.

The boundary conditions determine the coefficients $v_{r(m,0)}$ and $w_{s(0,m)}$.

To determine the coefficients recursively, let us suppose that all $v_{r(m,n)}$ and $w_{r(m,n)}$ with $m + n < k$ are known, and let us determine those for which $m + n = k$. From the boundary conditions $w_{r(0,k)}$ is known; from (5.4) we may obtain in succession $w_{r(1,k-1)}, w_{r(2,k-2)}, \ldots w_{r(3,k-3)}, \ldots, w_{r(k,0)}$. Likewise, we have $v_{s(k,0)}$ from the boundary conditions, and from (5.5) we find successively $v_{s(k-1,1)}, v_{s(k-2,2)}, \ldots, v_{s(0,k)}$, for $s = 1, \ldots, R - \mu$ in this order at each step. This completes the proof of induction on the recursive construction of the coefficients.

The formal power series so formed is clearly unique and to complete the solution we must show that it converges. This will be done in the next section. However we remark here that even if all non-homogeneous terms are zero except the $v_{r(0,n)}$ for $n > 0$, the values of the $w_{r(m,0)}$, and so the values of the w_r on G, will not be zero in general. The coefficients of the α_{r-1} carry these non-zero $w_{r(0,n)}$ through the steps of the recursion. Thus, values on G are affected by those on T when non-simple divisors are present.

6. Convergence of the double series. The recursion formulae are so arranged that if the right sides of the differential equations are dominated by certain series, then the coefficients calculated from the corresponding recursion formulae will be increased. By an easy preliminary transformation we can reduce the auxiliary conditions to homogeneous ones, and we therefore assume that the $v_{r(m,0)}$ and $w_{r(0,n)}$ of the dominating problem are zero.

Let the dependent variables of the dominant problem which we shall set up be denoted by capitals V_r, W_s. It is necessary to distinguish the terms containing each of the V_r on the right side of each equation, and we denote by $\bar{L}_{r,m}(V_m)$ an operator with coefficients majorizing those in (5.1) or (5.2), and containing only first derivatives of V_m. Terms containing no derivatives will be expressed by majorizing operators $\bar{M}_r(V,W)$, while $\bar{L}_{\mu+s}(V)$ shall denote a similar expression containing first derivatives of the $V_s (s = 1, \ldots, R - \mu)$ with respect to the $x^\rho (\rho = 1, 2, \ldots, N - 2)$. We denote by A_{r-1}, B_s and Γ_{s-1} functions whose series expansions dominate those of α_{r-1}, β_s and γ_{s-1}, respectively. With these preliminaries, we can write the dominant equations in the form

$$
(6.1)\quad
\begin{aligned}
\frac{\partial W_1}{\partial x} &= \sum_m \bar{L}_{1m}(W_m) + \bar{L}_1(V) + \bar{M}_1(V, W),\\
\frac{\partial W_r}{\partial x} &= A_{r-1}\frac{\partial W_{r-1}}{\partial t} + \sum_m \bar{L}_{rm}(W_m) + \bar{L}_r(V) + \bar{M}_r(V, W),\\
\frac{\partial V_1}{\partial t} &= B_1\frac{\partial V_1}{\partial x} + \sum_m \bar{L}_{\mu+1,m}(V_m) + L_{\mu+1}(W) + \bar{M}_{\mu+1}(V, W),\\
\frac{\partial V_s}{\partial t} &= B_s\frac{\partial V_s}{\partial x} + \Gamma_{s-1}\frac{\partial V_{s-1}}{\partial x} + \sum_m \bar{L}_{\mu+s,m}(V_m) + \bar{L}_{\mu+s}(W)\\
&\qquad + \bar{M}_{\mu+s}(V, W).
\end{aligned}
$$

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Solutions which are functions of a single combination $t/\alpha + x/\beta + \Sigma_\rho x^\rho$ of the independent variables will be sought. However, in order to avoid the vicious circle which results from application of the usual reduction technique to this system, it is necessary to introduce changes of scale of the dependent variables v_r as well as of the independent variables t, x and x^ρ. Let us denote new variables, both dependent and independent, by a bar, and replace

$$W_r,\ x, \text{ and } x^\rho \tag{6.2}$$

by

$$a_r\bar{W}_r,\ l\bar{x},\ \delta\bar{x}^\rho, \tag{6.3}$$

respectively, where a_r, l and δ are $\mu + 2$ undetermined constants. In terms of the new quantities we find that (6.1) becomes

$$\begin{aligned}
\frac{\partial\bar{W}_1}{\partial x} &= \frac{l}{\delta a_1}\left[\sum_m a_m\bar{L}_{1m}(\bar{W}_m) + \bar{L}_1(V)\right] + \frac{l}{a_1}\bar{M}_1(V, \bar{W}),\\
\frac{\partial\bar{W}_r}{\partial x} &= A_{r-1}\frac{l\,a_{r-1}}{a_r}\frac{\partial\bar{W}_{r-1}}{\partial t} + \frac{l}{\delta a_r}\left[\sum_m a_m\bar{L}_{rm}(\bar{W}_m) + \bar{L}_r(V)\right]\\
&\qquad + \frac{l}{a_r}\bar{M}_r(V, \bar{W}),\\
\frac{\partial V_1}{\partial t} &= \frac{B_1}{l}\frac{\partial V_1}{\partial\bar{x}} + \frac{1}{\delta}\left[\sum_m a_m\bar{L}_{\mu+1,m}(\bar{W}_m) + L_{\mu+1}(V)\right]\\
&\qquad + \bar{M}_{\mu+1}(V, \bar{W}),\\
\frac{\partial V_s}{\partial t} &= \frac{B_s}{l}\frac{\partial V_s}{\partial\bar{x}} + \frac{\Gamma_{s-1}}{l}\frac{\partial V_{s-1}}{\partial\bar{x}} + \frac{1}{\delta}\left[\sum_m a_m\bar{L}_{\mu+s,m}(\bar{W}_m) + \bar{L}_{\mu+s}(V)\right]\\
&\qquad + \bar{M}_{\mu+s}(V, \bar{W}).
\end{aligned} \tag{6.4}$$

Here the indices r, m range from 1 to μ while s ranges from 1 to $R - \mu$.

The undetermined constants will now be chosen so that every term on the right side of these equations which contains a derivative will have a factor, not larger in magnitude than a given ϵ, multiplying it. Thus the following combinations must all be made no larger than ϵ:

$$\frac{l}{a_1\delta},\ \frac{la_r}{a_1\delta},\ \frac{la_{r-1}}{a_r},\ \frac{la_m}{\delta a_r},\ \frac{l}{\delta a_r},\ \frac{1}{l},\ \frac{a_m}{\delta},\ \frac{1}{\delta}. \tag{6.5}$$

Of these all but the third and sixth contain the factor $1/\delta$. The sixth shows that $l \geqslant 1/\epsilon$, and the third, that the a_r must increase with r as a geometric series of ratio $1/\epsilon^2$. These conditions are all satisfied if we choose

$$l = \frac{1}{\epsilon},\ a_r = a^r,\ a = \frac{1}{\epsilon^2},\ \delta = \frac{1}{\epsilon^{2\mu+2}}. \tag{6.6}$$

Now let a dominant series

$$\frac{M}{1 - (t + \bar{x} + \bar{y})/\rho},\qquad M \geqslant 1,\ \bar{y} \equiv \textstyle\sum_\rho \bar{x}^\rho,$$

be chosen for the sum of all coefficients on the right side of (6.4), assuming that the above special factors are not included in these coefficients. Then it is seen that the first group of (6.4) are in turn dominated by a single equation for new variables $\bar{W}$ and V;

$$\frac{\partial \bar{W}}{\partial x} = \frac{M}{1 - (t + \bar{x} + \bar{y})/\rho}\left[\epsilon\frac{\partial \bar{W}}{\partial t} + \epsilon\frac{\partial \bar{W}}{\partial \bar{y}} + \epsilon\frac{\partial V}{\partial \bar{y}} + \frac{\bar{W} + V + F}{\epsilon^{2\mu+1}}\right], \tag{6.7}$$

and that the second group are likewise dominated by a corresponding equation for V:

$$\frac{\partial V}{\partial t} = \frac{M}{1 - (t + \bar{x} + \bar{y})/\rho}\left[\epsilon\frac{\partial V}{\partial \bar{x}} + \epsilon\frac{\partial \bar{W}}{\partial \bar{y}} + \epsilon\frac{\partial V}{\partial \bar{y}} + \frac{\bar{W} + V + F}{\epsilon^{2\mu+1}}\right]. \tag{6.8}$$

The factor $\epsilon^{2\mu+1}$ in the last term of this equation is inserted for convenience, on the assumption $\epsilon < 1$.

Let us show that these equations have a convergent series solution with positive coefficients. Set $z = t + \bar{x} + \bar{y}$, and suppose $\bar{W}$ and V depend only on z. If derivatives with respect to z are indicated by a prime, we have, after rearranging,

$$\left(1 - \frac{z}{\rho} - 2\epsilon M\right)\bar{W}' - \epsilon M V' = \frac{M}{\epsilon^{2\mu+1}}[\bar{W} + V + F],$$

and

$$\left(1 - \frac{z}{\rho} - 2\epsilon M\right)V' - \epsilon M \bar{W}' = \frac{M}{\epsilon^{2\mu+1}}[\bar{W} + V + F],$$

respectively. Now let $Y(z)$ be a solution of

$$\left(1 - 3\epsilon M - \frac{z}{\rho}\right)Y' = \frac{M}{\epsilon^{2\mu+1}}[2Y + F], \tag{6.9}$$

and take $\bar{W} = V = Y$. In order that W should have positive coefficients, we shall choose

$$\epsilon < \frac{1}{6M}. \tag{6.10}$$

Then

$$Y' = \frac{432M^4[2Y + F]}{1 - 2z/\rho},$$

an equation in which the right side, after expansion, has positive coefficients. Thus if $W(0) = 1$, a solution with positive coefficients and radius of convergence $\frac{1}{2}\rho$, independently of F, is secured.

Retracing the steps of this reduction, we conclude that (6.1) has a convergent solution set with positive coefficients, and hence that the formal expansions (5.3) converge. This completes the local existence proof for the problem formulated in §5. The domain can be extended as in other linear problems.

THEOREM II. *Let $G : \phi(x^i) = 0$ be a characteristic surface of multiplicity μ for the system*

$$\mathbf{A}^i \frac{\partial \mathbf{u}}{\partial x^i} + \mathbf{B}\mathbf{u} = \mathbf{f}, \tag{1.2}$$

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and let $T : \psi(x^i) = 0$ *be non-characteristic. Then there exists a unique analytic solution of* (1.2) *with assigned values on* T *for the* μ *null variables* w_r *relative to* G, *and assigned values on* G *for the remainder.*

From the recursion formula (5.4) it is evident that the values of the proper variables w_r on G depend on the data given on T. The values of the coefficients $w_{r(0,n)}$ enter the solutions of the difference equations (5.4) by means of the terms with coefficient $\alpha_{r-1(m,n)}$ of which the first is indicated explicitly.

On the other hand, if the elementary divisors relative to the eigenvalue $\lambda = 0$ are all simple, then (5.1) has the form (3.2) since all coefficients α_{r-1} vanish. We have shown that in this case the values of the solution on G depend only on the values of the data assigned on G.

7. The general mixed problem. Let an initial surface S: $\phi(x^i) = 0$, and a boundary surface $T : \psi(x^i) = 0$, both non-characteristic, meet in an edge C. There will in general be a number of characteristic surfaces of (1.1) which pass through C, as the characteristic equation (1.8) has degree R. For the present we assume that each has multiplicity one. We select as domain D one of the four "quadrants" defined by S and T, and choose any $k_0(1 \leqslant k_0 \leqslant R)$ of these characteristic surfaces $G_i(i = 1, \ldots, k_0)$ which lie in that quadrant D. A solution of the differential equations is sought in D, which is analytic except on G_i $(i \leqslant k_0)$ and continuous there, which takes given Cauchy data on S, and satisfies k_0 suitable boundary conditions on T.

An analytic solution taking the Cauchy values on S can be constructed by the Cauchy-Kowalewsky theorem. Supposing this done, we subtract away this solution and so have a reduced problem with zero Cauchy data and homo-

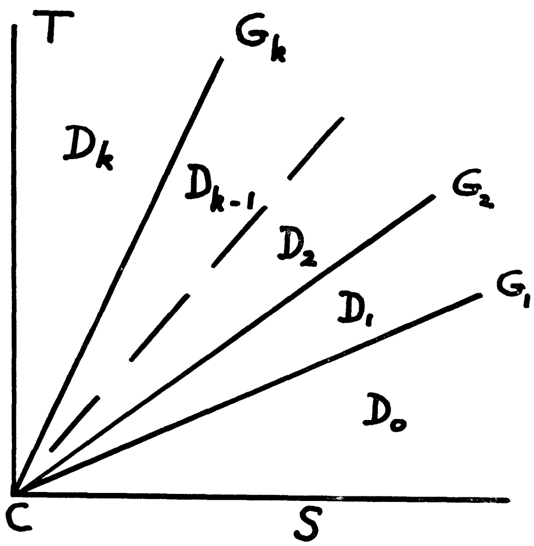

Fig. 1

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geneous differential equations. The selected characteristic surfaces G_i, which we shall suppose do not intersect except on C, divide D into $k_0 + 1$ domains D_i ($i = 0, 1, \ldots, k_0$) such that D_i lies between G_i and G_{i+1} as in Fig. 1. In each of these domains we shall construct a power series solution $u_{r(i)}$ which shall be defined in every D_j, $j \geqslant i$. The final solution will take the form

$$u_r = \sum_{i=1}^{h} u_{r(i)}$$

in D_h, and so will be analytic except on the G_i.

It is a well-known property of hyperbolic equations that discontinuities of derivatives of solutions are confined to characteristic surfaces. Indeed the magnitudes of transverse discontinuities of this type satisfy ordinary differential equations along the bicharacteristics. Our series expansions will be determined in the light of these facts. In the reduced problem the only non-homogeneous terms are the boundary data and these may be said to generate the whole solution of the reduced problem. Since the Cauchy solution is zero in D_0 the solution is, so to speak, built up step by step in the D_i from the power series in $t - t_j(x, x^\rho)$ determined by the discontinuities of higher order encountered in traversing the G_j ($j \leqslant i$).

With $S : \phi(x^i) \equiv t = 0$ and $T : \psi(x^i) \equiv x = 0$, as before, we may write the homogeneous system in canonical form:

$$\frac{\partial u_r}{\partial t} = \lambda_r \frac{\partial u_r}{\partial x} + L_r(u_s), \qquad r = 1, \ldots, R. \tag{7.1}$$

Here all elementary divisors are simple, by hypothesis, and the derivatives with respect to t and x of u_r appear only in the rth equation. We take $\lambda_r \neq \lambda_s$ ($r \neq s$) for the present and note that the λ_r need not be real. Those variables u_r which are null with respect to one of the k_0 selected characteristic surfaces G_r appear in the differential equation with eigenvalue λ_r. To maintain our previous notations we distinguish the selected null variables u_r by the symbol w_r ($r = 1, \ldots, k_0$). The remaining $R - k_0$ variables are denoted by v_r ($r = 1, \ldots, R - k_0$). Thus the differential system appears as:

$$\begin{aligned} \frac{\partial w_r}{\partial t} &= \lambda_r \frac{\partial w_r}{\partial x} + L_r(v_s, w_s) \qquad & r = 1, \ldots, k_0, \\ \frac{\partial v_s}{\partial t} &= \lambda_s \frac{\partial v_s}{\partial x} + L_s(v_m, w_m) \qquad & s = 1, \ldots, R - k_0. \end{aligned} \tag{7.2}$$

Here the L_r contain the transverse derivatives $\partial/\partial x_p$ only.

The k_0 boundary conditions shall take the form

$$w_r = \sum_s c_{rs} v_s + g_r \qquad r = 1, \ldots, k_0. \tag{7.3}$$

These are linear conditions solved for the proper or null variables (which are in this instance the same). The datum functions $g_r(t, x_\rho)$ are real analytic on T, and since our solution is to be continuous we postulate

$$g_r(0, x_\rho) = 0. \tag{7.4}$$

8. The discontinuity expansion. We shall calculate the discontinuities across G_i of the successive derivatives with respect to t of each of the unknowns v_s, w_r. At each stage we must consider the jump of each component across each of the selected characteristic surfaces. It turns out that most of these quantities can be calculated directly, but that at each stage there are k_0 which must be found by solving a differential equation on each of the G_i.

Let a discontinuity across G_i of the nth time derivative of a function u be denoted by

$$(u^{(n)})_i. \tag{8.1}$$

We define parameters s_i on G_i, measured from the edge $C = S \cap T$, such that

$$\frac{\partial}{\partial s_i} = \frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}, \qquad i = 1, \ldots, k_0. \tag{8.2}$$

Since all discontinuities to be considered are finite, and analytic along the G_i, the total discontinuity $(u)_0$ taken across C of a function defined on T is the sum of the limits of its jumps across the G_i:

$$(u)_0 = \sum_{i=1}^{k_0} (u)_i \Big|_{s_i=0}. \tag{8.3}$$

Replacing derivatives with respect to x by derivatives with respect to s_i (i being fixed), by (8.2), we have

$$(\lambda_r - \lambda_i) \frac{\partial w_r}{\partial t} = -\lambda_r \frac{\partial w_r}{\partial s_i} + \lambda_i L_r \qquad r = 1, \ldots, k_0, \tag{8.4}$$

and

$$(\lambda_s - \lambda_i) \frac{\partial v_s}{\partial t} = -\lambda_s \frac{\partial v_s}{\partial s_i} + \lambda_i L_s \qquad s = 1, \ldots, R - k_0. \tag{8.5}$$

Let us suppose that only g_i in (7.3) does not vanish; there is no loss of generality as the equations are linear. The coefficients of g_i, expanded in a series of powers of t, will be denoted by $g_{i(n)}$, and we shall assume, for the exposition, that $g_{i(1)} \not\equiv 0$. Then we calculate the first order jumps $(w_r^{(1)})_i$, $(v_s^{(1)})_i$ as follows.

First take $r \neq i$ in (8.4), and take the discontinuity across G_i. We get

$$(\lambda_r - \lambda_i)(w_r^{(1)})_i = 0 \qquad r \neq i, \tag{8.6}$$

since the other terms are continuous by hypothesis. Similarly

$$(\lambda_s - \lambda_i)(v_s^{(1)})_i = 0. \tag{8.7}$$

Thus all first order jumps vanish except possibly $(w_i^{(1)})_i$. To find this quantity, we differentiate (8.4i) with respect to t and take the jump across G_i. Since the above left side is zero we get

$$\begin{aligned} \frac{\partial}{\partial s_i}(w_i^{(1)})_i &= \left(\frac{\partial L_i}{\partial t}(v, w)\right)_i \\ &= a_{ii}^{\rho} \frac{\partial}{\partial x_\rho}(w_i^{(1)})_i + b_{ii}(w_i^{(1)})_i. \end{aligned} \tag{8.8}$$

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Here the appropriate coefficients in L_i have been exhibited. All other terms, being continuous, drop out when the jump operator is applied. (No summation over repeated Latin indices is intended.) This equation has the Cauchy-Kowalewsky normal form with respect to the edge C, in the variables s_i and x^ρ ($\rho = 1, \ldots, N-2$), since C has the equation $s_i = 0$ in the surface G_i. An initial condition on C for (8.8) is now to be found. Using (8.3), and differentiating (7.3) with respect to t and taking jumps, we have

$$(8.9)\qquad \begin{aligned} (w_i^{(1)})_i \Big|_{s_i=0} &= (w_i^{(1)})_0 - \sum_{j\neq i} (w_i^{(1)})_j \Big|_{s_j=0} \\ &= (w_i^{(1)})_0 \\ &= \sum_s c_{is}(v_s^{(1)})_0 + g_{i1} \\ &= g_{i1}\,. \end{aligned}$$

With this initial condition the single partial differential equation (8.8) has a unique solution on G_i. This completes the calculation of the first-order jumps and it may be noted that the non-homogeneous term g_{i1} induces a first order contribution only from the corresponding proper variable w_i over the corresponding surface G_i.

If the first non-zero term in g_i is of a higher order n, the only non-zero nth order discontinuity is of the same kind as that just mentioned.

The discontinuities of higher orders are found in succession by this process. Suppose known all jumps of order $n-1$ or less, and let us find those of order n. Differentiating (8.4) and (8.5) $n-1$ times with respect to t, and taking jumps over G_i, we have

$$(8.10)\qquad \begin{aligned} (\lambda_r - \lambda_i)(w_r^{(n)})_i &= -\lambda_r \frac{\partial}{\partial s_i}(w_r^{(n-1)})_i + \lambda_i\left(\frac{\partial^{n-1}}{\partial t^{n-1}} L_r\right)_i + \ldots, \\ (\lambda_s - \lambda_i)(v_s^{(n)})_i &= -\lambda_s \frac{\partial}{\partial s_i}(v_s^{(n-1)})_i + \lambda_i\left(\frac{\partial^{n-1}}{\partial t^{n-1}} L_s\right)_i + \ldots. \end{aligned}$$

Now the right hand sides are all known in terms of the discontinuities of order $\leqslant n-1$ already calculated. Again, provided $r \neq i$ in the first group, we obtain the values of the $(w_r^{(n)})_i$ and $(v_s^{(n)})_i$ along G_i.

To find the remaining quantity $(w_i^{(n)})_i$ we differentiate (8.4i) n times with respect to t and then take the discontinuity across G_i. The result is

$$(8.11)\qquad \begin{aligned} \frac{\partial}{\partial s_i}(w_i^{(n)})_i &= \left(\frac{\partial^n}{\partial t^n} L_i(v, w)\right)_i \\ &= \sum_r a_{ir}^\rho \frac{\partial}{\partial x^\rho}(w_r^{(n)})_i + \sum_r b_{ir}(w_r^{(n)})_i \\ &\quad + \sum_s a_{is}^\rho \frac{\partial}{\partial x^\rho}(v_s^{(n)})_i + \sum_s b_{is}(v_s^{(n)})_i, \end{aligned}$$

where the terms omitted are of discontinuity order less than n. However all jumps present except that of $w_i^{(n)}$ are known and we obtain the non-homogeneous differential equation

$$\frac{\partial}{\partial s_i}(w_i^{(n)})_i = a_{ii}^{\rho}\frac{\partial}{\partial x^{\rho}}(w_i^{(n)})_i + b_{ii}(w_i^{(n)})_i + K, \tag{8.12}$$

where K stands for a known expression. The initial condition is now found from (7.3) by differentiating n times with respect to t and taking jumps. Thus from (8.3)

$$(w_i^{(n)})_i\Big|_{s_i=0} = (w_i^{(n)})_0 - \sum_{j\neq i}^{k_0}(w_i^{(n)})_j\Big|_{s_j=0}, \tag{8.13}$$

and by Leibnitz' formula used in connection with (7.3),

$$(w_i^{(n)})_0 = \sum_{s,\ m=0}^{m=n} c_{ism}\binom{n}{m}(v_s^{(m)})_0 + g_{in}. \tag{8.14}$$

The right hand sides are known and the initial value determined. Since (8.12) is a non-homogeneous version of (8.8) the existence of an analytic solution on G_i follows as in the first order case. This completes the calculations for the nth order.

It may be noted that the interaction of the calculations for the various G_i ($i = 1, \ldots, k_0$) is brought about by the terms in the sum on the right side of (8.13). The negative sign appearing there will have no special effect in the proof of convergence.

The recursive construction being complete, both for the v_r and the w_s, we define the series of which the solution functions are composed. Let the ith characteristic surface G_i have the (analytic) equation $t = t_i(x,x^{\rho})$. The series $u_{r(i)}$ is now given by

$$u_{r(i)} = \sum_{n=0}^{\infty}(u_r^{(n)})_i\,(t - t_i(x, x^{\rho}))^n, \tag{8.15}$$

where u_r stands for any one of the variables v_r, w_s. Then, as indicated previously, the final formal solution is

$$u_r = \sum_{i=1}^{h} u_{r(i)} \qquad \text{in } D_h,\ h = 0,1,\ldots,k_0. \tag{8.16}$$

To complete our existence proof we must show that these series have a common domain of convergence.

9. Convergence of the discontinuity expansion. We will show that each of the series (8.15) is dominated by the solution of a certain problem wherein only one characteristic surface G appears, and one boundary condition is present. The solution of this simplified problem will follow from Theorem I. We shall find expressions which dominate the various terms $(u_r^{(n)})_i$ by requiring that the coefficients on the right sides of all of the differential equations and recursive relations used in the construction of the solution should be simultaneously majorized. Thus let $G = G(t,x^{\rho})$ denote a series with positive terms which dominates every one of the datum functions $g_i(t, x^{\rho})$, on T, and vanishes for $t = 0$; and let $K = K(t, x^{\rho})$ dominate all of the coefficients c_{rs} of (7.3).

The dominating series in the differential operators are constructed as follows: for $r \neq i$, divide (8.4) by $\lambda_r - \lambda_i$; and divide (8.5) by $\lambda_s - \lambda_i$. Then equations of the form

$$\frac{\partial w_r}{\partial t} = \mathscr{L}_r(v, w), \frac{\partial v_s}{\partial t} = \mathscr{L}_s(v, w) \tag{9.1}$$

are formed, where $\mathscr{L}_r$ is a linear operator in the x^ρ and in the tangential variable s_i. Let every one of these operators be expanded in power series about each of the characteristic surfaces G_i; that is, let the coefficients be written as power series in each of the k_0 sets of variables $t - t_j$, s_j, and x^ρ. We can now select series which dominate all of these series for every operator, and for every value of the index j, and for every value of the index i. The variables $t - t_i$ and s shall be replaced by two common variables t and s. Let us also attempt to majorize all expansions of the w_r about $G_i(i \neq r)$ by a single series W, and all expansions of the v_r by a single series V. To dominate the development of w_i about G_i, for each selected i, we take a third series Z. Thus we construct an operator $L(V, W, Z)$ which will dominate the right sides of (9.1) provided that V, W, and Z dominate v_r, $w_r(r \neq i)$ and w_i, respectively.

Similarly we consider the single equation (8.4i) for each i, and it has the form

$$\frac{\partial w_i}{\partial s_i} = \mathscr{L}_i(v, w). \tag{9.2}$$

Let $L_1(V, W, Z)$ dominate the right side of (9.2), for all i, when its arguments dominate those of $\mathscr{L}_i$.

Now consider the system, already in canonical form in the variables t, s and x^ρ,

$$\begin{aligned} \frac{\partial V}{\partial t} &= L(V, W, Z), \\ \frac{\partial W}{\partial t} &= L(V, W, Z), \\ \frac{\partial Z}{\partial s} &= L_1(V, W, Z). \end{aligned} \tag{9.3}$$

By Theorem I, the appropriate auxiliary conditions include two for $t = 0$, viz.

$$V = W = 0, \tag{9.4}$$

and one for $s = 0$, which we take as

$$Z = RK(t, x^\rho)V + RW + G(t, x^\rho). \tag{9.5}$$

This system satisfies the conditions of Theorem I and the existence of a convergent power series solution follows.

We now show by induction on n that the coefficients $V_{(n)}, W_{(n)}$ and $Z_{(n)}$, in the expansion of this solution in powers of t, dominate the series for the discontinuity terms of order n of $w_r(r \neq i)$, v_s and w_i across G_i. The solution of (9.3)—(9.5), the existence of which is guaranteed by Theorem I, could itself be equally well regarded as a discontinuity expansion relative to the characteristic surface $t = 0$. In general, therefore, the computations based upon it will lead to series dominating the original discontinuity expansion.

To verify this in detail we begin with the first order jumps. Since $L_1(V, W, Z)$ has coefficient functions with positive coefficients in their expansions, we find for Z on $G : t = 0$ a series with positive coefficients. Thus $Z_0 \gg 0$. For V_1 and W_1 we also get expressions with positive coefficients in view of the choice of the operator $L(V, W, Z)$; and these certainly dominate the $(w_r{}^{(1)})_i$ and $(v_s{}^{(1)})_i$ which are all zero. For Z_1 we have a differential equation found by differentiating the third of (9.3) with respect to t and setting $t = 0$; the operator on the right side of this equation certainly dominates that in (8.8). To complete the demonstration for the first order terms we must show the dominance of the initial value for Z_1 when $s = 0$, namely

$$(9.6) \qquad Z_1 \Big|_{s=t=0} = RK(0, x^\rho)V_1 + RK_t(0, x^\rho)V + RW_1 + G_1,$$

as is seen by differentiating (9.5) with respect to t and then setting $t = 0$. A comparison with (8.9) leads to the desired conclusion since G dominates g_i. Therefore the dominance holds for first order terms.

Proceeding by induction for higher orders, we shall assume that the dominance holds for all orders less than n. By the definition of the operator $L(V, W, Z)$, and by comparison with (8.4), (8.5) and (8.10), we see that

$$(9.7) \qquad (w_r^{(n)})_i \ll W_n \qquad (r \neq i)$$

and

$$(9.8) \qquad (v_s^{(n)})_i \ll V_n.$$

In the differential equation for Z_n, namely

$$(9.9) \qquad \frac{\partial Z_n}{\partial s} = \frac{d^n}{dt^n} L_1(V, W, Z)\Big|_{t=0},$$

every coefficient of a derivative of Z_n, and the coefficient of Z_n, will dominate the corresponding terms in (8.11), and, moreover, the non-homogeneous terms (jumps of lower order) will each dominate the corresponding items on the right side of (8.11). Thus (9.9) dominates (8.12), in a formal sense. The Cauchy-Kowalewsky solution of (9.9) is so constructed that the dominance will then hold for the solutions if it holds for the initial conditions at $s = 0$.

From (8.13) and (8.14) we find

$$(9.10) \qquad (w_i^{(n)})_i\Big|_{s_i=0} = \sum_{s,m=0}^{n} c_{ism}\, a_{mn} (v_s^{(m)})_0 + g_{in} - \sum_{j\neq 1} (w_i^{(n)})_j \Big|_{s_j=0}$$

where the a_{mn} are a set of positive numerical coefficients of the type of combination symbols. This is dominated by

$$(9.11) \qquad R\sum_{m=0}^{n} k_{im}\, a_{mn}\, V_n + G_n + RW_n,$$

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where the k_{im} are symbols corresponding to the c_{ism} but obtained by differentiation from the dominant function $K(\mathrm{t}, x^\rho)$. Here we have made use of (9.7) and (9.8), and have replaced the summation over s and over j by the factor R. However, by (9.5), the expression (9.11) is exactly the initial value which should be computed by the jump process, for the quantity Z_n, at $s = 0$. Therefore

$$(w_i^{(n)})_i \Big|_{s_i=0} \ll Z_n \Big|_{s=0},$$

a dominating relation in the variables x^ρ, $(\rho = 1, \ldots, N-2)$ which completes the calculations for order n. Thus the induction is complete and one of the series U, W, Z dominates each of the series (8.15).

As the series (8.15) converge for $t - t_i$ sufficiently small, and for suitably small values of the remaining variables x and x^ρ, they will all converge for sufficiently small positive values of t and x, since $t_i(x, x^\rho)$ tends to zero with x. This establishes convergence of the solution in a neighbourhood of the origin, which has been chosen, in effect, at a typical point of the edge C. Extension of the domain is now possible by conventional methods, and will not be pursued here, though we remark that analytic continuation must be pursued separately for each sector domain $D_i (i = i, \ldots, k_0)$.

Before stating our result as a theorem, we make two minor extensions connected with the eigenvalues λ_i. First, it is permissible that the selected λ_i should be a multiple eigenvalue of multiplicity μ, say, provided that the corresponding elementary divisors of $\mathbf{A}^N$, with respect to $\mathbf{A}^{N-1}$, are simple. Then the Cauchy differential equations (8.8) and (8.11) become systems of order μ in μ proper variables v_r, but the formal structure of the calculations is not affected. The number of boundary conditions is then the sum of the multiplicities of the characteristic roots λ_i.

Secondly, the non-select characteristic roots may have larger multiplicity without restriction on the elementary divisors. A comparison with Theorem I shows that the additional terms present with non-simple divisors do not disrupt the calculations. However, the variables proper to each eigenvalue must generally be treated in a fixed order at every stage.

THEOREM III. *Let non-characteristic surfaces S and T relative to the analytic system*

$$\mathbf{A}^i \frac{\partial \mathbf{u}}{\partial x^i} + \mathbf{B}\mathbf{u} = \mathbf{f}$$

intersect in an edge C from which issue into a quadrant at least k_0 distinct characteristic surfaces G_i. Let the elementary divisors referring to the eigenvalues λ_i be simple. Then there exists a solution, continuous in the quadrant and analytic except across the G_i, which takes given Cauchy data on S, and for which the variables w_i null with respect to the G_i take values on T determined by linear boundary conditions.

Let us remark, in conclusion, that the only restriction on the reality of the eigenvalues is that the select λ_i be real. They correspond to real characteristic surfaces. The remaining roots may be complex, and the theorem is thus applicable to systems which are only partly "hyperbolic" in nature. This freedom of "type" should accompany a theorem based on the Cauchy-Kowalewsky theorem, which is quite independent of such restrictions.

10. Uniqueness of the series solution. The expansions of Theorem III imply that the solutions are analytic not only in each sector domain D_h, but also on the closure of D_h. This is a stronger condition than that of being piecewise analytic—for instance, e^{-1/x^2} is piecewise analytic but is not analytic for $x = 0$. We can therefore only assert, in general, that the series solution found above is unique in the class of vector functions u_r having this strong piecewise analyticity. That it is unique in this class follows from the well-defined nature of the construction of the solution.

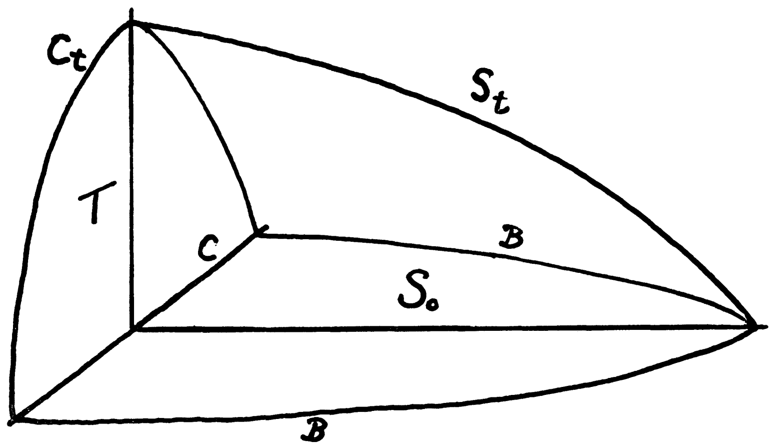

Fig. 2

One case in which uniqueness of the solution in a wider class of real vector functions can be shown is the case when all roots are real and different from zero, and all positive roots are select. By a modification of Holmgren's theorem (**14**, p. 34) we can prove uniqueness within the class of once continuously differentiable vector functions. It is sufficient to prove such a uniqueness theorem locally, and we therefore consider a region R defined as follows. Let S and T meet in C as before, and let S_1 be a surface nearly parallel to S, meeting S and T in an edge B_1 which intersects C, and such that S, T and S_1 enclose a region R which is a half of a lens-shaped region (Fig. 2). Let an analytic family S_t of surfaces t = const. fill R in such fashion that $S = S_0$, and $S_1 = S_{t=1}$ Let the real characteristic roots β_i of the matrix $- a^{N-1}{}_{rs}$ with respect to $a^N{}_{rs}$ not vanish, change sign or become complex as t varies from 0 to 1. This will happen if S_1 is sufficiently near and parallel to S_0.

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Then we may write

(10.1) $$L_r(u) = \frac{\partial u_r}{\partial t} + \beta_r \frac{\partial u_r}{\partial x} + \sum_{\rho=1}^{N-2} a^{\rho}_{rs} \frac{\partial u_s}{\partial x^{\rho}} + b_{rs} u_s,$$

and we define the adjoint operator

(10.2) $$M_r(v) = \frac{\partial v_r}{\partial t} + \frac{\partial}{\partial x} (\beta_r v_r) + \sum_{\rho=1}^{N-2} \frac{\partial}{\partial x^{\rho}} (a^{\rho}_{sr} v_s) - b_{sr} v_s.$$

Consider the mixed problem for $M_r(v) = 0$ with "initial" surface S_1 and boundary T, the solution being defined in R. The number of characteristic surfaces issuing from C_2 into R is equal to the number $R - k$ of negative roots β_r; and we may suppose that all are select. By Theorem III, we can construct a solution of the adjoint system with analytic "initial" values on S_1, and satisfying $R - k$ suitable conditions on T.

Now suppose that u_r vanishes on S, and that the k select components $u_s(\beta_s > 0)$ satisfy homogeneous boundary conditions

(10.3) $$u_s = \sum_n c_{sn} u_n \qquad \begin{matrix} s = 1, \dots, k, \\ n = k+1, \dots, R. \end{matrix}$$

Supposing that $L_r(u) = 0$ and that the u_r are C^1, we wish to show $u_r \equiv 0$ in R.

Let the values u_r on S_1 be approximated by analytic values for v_r such that

(10.4) $$|u_r - v_r| < \epsilon \qquad \text{on } S_1.$$

Then let v_r denote the piecewise analytic solution of $M_r(v) = 0$ with these "initial" values on S_1, which satisfies the $R - k$ adjoint homogeneous boundary conditions

(10.5) $$\beta_n v_n = - \sum_{s=1}^{k} \beta_s c_{sn} v_s, \qquad n = k+1, \dots, R,$$

on T. Applying Green's formula, which is in this case

(10.6) $$\int_R (v_r L_r(u) + u_r M_r(v)) dV = \int_{S_1 - S_0} \sum_r u_r v_r dS + \int_T \sum_r \beta_r u_r v_r dS,$$

we see that the volume integral on the left vanishes. The surface integral over T becomes

(10.7) $$\begin{aligned} &\int_T \left(\sum_{s=1}^{k} \beta_s u_s v_s + \sum_{n=k+1}^{R} \beta_n u_n v_n \right) dS \\ &= \int_T \left(\sum_{s=1}^{k} \sum_{n=k+1}^{R} \beta_s c_{sn} u_n v_s + \sum_{n=k+1}^{R} \beta_n u_n v_n \right) dS \\ &= \int_T \sum_{n=k+1}^{R} u_n \left(\beta_n v_n + \sum_{s=1}^{k} \beta_s c_{sn} v_s \right) dS = 0. \end{aligned}$$

Thus, as the integral over S_0 is zero since $u_r = 0$ there, we find from (10.4)

(10.8) $$\int_{S_1} \sum u_r v_r dS = \int_{S_1} \sum u_r^2 dS + 0(\epsilon) = 0.$$

Letting $\epsilon \to 0$ we see that the integral over S_1 is zero and hence that $u_r \equiv 0$ on S_1. It follows that $u_r \equiv 0$ in a region R sufficiently near S_0, and this proves the uniqueness theorem.

We may note that unless the datum functions $g_i(x^\rho, t)$ satisfy compatibility conditions of the first order with respect to the initial data, the analytic solution of Theorem III may not be C^1 across the G_i. However by subtracting away the solution of an auxiliary problem in which the $g_i(x^\rho, t)$ are linear functions of t, we can cause the discontinuities of first order to vanish, and for simplicity we shall suppose that this has been done. We recall that all elementary divisors are assumed to be simple.

If all characteristic roots are real and different from zero, and all positive roots are select, the piecewise analytic solution of Theorem III is the only C^1 solution of the problem.

An instance where uniqueness will hold in the stronger sense is that of symmetric hyperbolic systems, which we now consider. The estimates to be found for these systems also imply uniqueness in the C^1 class.

11. Symmetric hyperbolic systems. Theorem III may be used, in combination with estimates of the Friedrichs-Lewy type and Sobolev's lemma, to establish a mixed initial and boundary value theorem for symmetric hyperbolic systems having a finite order of differentiability. A system (1.1) is called symmetric hyperbolic **(6)** if the coefficient matrices are symmetric: $a^i{}_{rs} = a^i{}_{sr}$, and if there exists a covariant vector $\xi_i{}^0$ such that $\xi_i{}^0 a^i{}_{rs}$ is a positive definite matrix. We note that for a symmetric system all elementary divisors are simple.

By a suitable transformation of coordinates we may suppose that $a^N{}_{rs}$ is positive definite. The surface $S : t = x^N =$ const. is then said to be spacelike, and we assume that the initial surface, carrying Cauchy data, has this property. Let the boundary surface $T : x = x^{N-1} = 0$ meet S in the edge C of $N - 2$ dimensions. We mark off on S an initial region S_0 having as boundary part of C and also a variety B of $N - 2$ dimensions, which will be held fixed in the following calculations. Let S^i be a *spacelike* surface which meets T in a locus C_t such that the boundary bC_t lies in $B \cap C$, as in Fig. 2. The surfaces C_t of dimension $N - 2$ shall lie in T, having t-intercepts increasing with t in an obvious sense, except that the boundaries bC_t are fixed. Thus the C_t cover, for $0 \leqslant t \leqslant t_1$, a lens-shaped portion of T, having the base C. The t-intercepts of the family of spacelike surfaces S_t shall also be increasing with t, except for the fixed portion B of bS_t. The region of space covered by the S_t is a half of a lens-shaped region. All these surfaces are assumed to have a certain degree of differentiability.

Since $A^N{}_{rs}$ is positive definite we may write the system (1.1) in normal form relative to S_t: $t =$ const. and we may then apply the reduction to standard form, relative to x as second variable, given in **(14**, p. 53**)**. The equations then take the form

$$\frac{\partial u_r}{\partial t} + \beta_r \frac{\partial u_r}{\partial x} + L_r(u) + f_r = 0, \qquad r = 1, \ldots, R. \tag{11.1}$$

Here $L_r(u)$ is a symmetric operator in the remaining variables, and the smoothness of the coefficients in (11.1) is unchanged by the transformation.

Reduction to this canonical form might equally be attained by the simultaneous reduction of the pair of quadratic forms

$$a^N_{rs}\, u_r\, u_s, \qquad a^{N-1}_{rs}\, u_r\, u_s \tag{11.2}$$

to the standard forms (**17**, p. 148)

$$\delta_{rs} u_r u_s, \qquad \beta_r \delta_{rs} u_r u_s. \tag{11.3}$$

To each of the necessarily real characteristic roots β_r there corresponds a characteristic surface G_r containing C. If β_r is positive this characteristic surface issues from C into the domain V wherein our solution is to be constructed. Suppose that k of the R roots β_r are positive, and that the multiplicity of each root is constant in V. The remaining roots β_r are negative since the surfaces S_t are not characteristic.

The variables u_r corresponding in (11.1) to the positive β_r are the null variables of the k characteristic surfaces G_r lying in V. We assign k linear boundary conditions, expressible as

$$u_r = g_r \qquad r = 1, \ldots, k. \tag{11.4}$$

Here the data g_r are assumed to have the same degree of differentiability as the differential equations and auxiliary surfaces.

Thus R functions—the Cauchy data—are given on S, and k on T. We seek a solution u_r of (11.1) in V, which is continuously differentiable in V except across the G_r where it need only be continuous. Since the Cauchy initial value problem can be regarded as solved (**10**) we subtract away the solution and so find zero Cauchy data for the reduced problem. Then the data g_r in (11.4), to be compatible with the above conditions, must vanish to the first order on C. We shall suppose that they vanish, together with their derivatives of order $\leqslant l - 1$, on C.

12. Estimates. We derive the Hilbert space estimates from a certain differential identity, which contains the essential property of a symmetric hyperbolic system (**6**). Let summation over all values of i, r, s, m be understood in the following equations. Writing the system as

$$L_r(u) = a^i_{rs} \frac{\partial u_s}{\partial x^i} + b_{rs}\, u_s + f_r = 0, \tag{12.1}$$

we have

$$\begin{aligned} 2u_r L_r(u) &= 2a^i_{rs} u_r \frac{\partial u_s}{\partial x^i} + b_{rs} u_r u_s + f_r u_r \\ &= \frac{\partial}{\partial x^i}(a^i_{rs} u_r u_s) + q_{rs} u_r u_s + f_r u_r. \end{aligned} \tag{12.2}$$

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Integrating over the domain V_t, we have, by the divergence formula,

$$0 = \int_{V_t} 2u_r L_r(u) dV_t = \int_{S_t - S_0 + T_t} a^i_{rs} u_r u_s n_i dS + \int_{V_t} Q(u, f) dV, \tag{12.3}$$

where n_i denotes the covariant surface normal, and where $Q(u, f)$ is quadratic in the u_r and linear in the f_r.

Now let us isolate the integral over S_t:

$$\int_{S_t} \sum_r u_r^2 dS \equiv \int_{S_t} a^N_{rs} u_r u_s dS = -\int_{V_t} Q(u, f) dV + \int_{S_0} \sum_r u_r^2 dS + \int_{T_t} a^{N-1}_{rs} u_r u_s dS. \tag{12.4}$$

Here we have used $n_N = 1$, $n_{N-1} = -1$. From (11.1) we find

$$a^{N-1}_{rs} u_r u_s = \sum_{r=1}^{k} \beta_r u_r^2 + \sum_{r=k+1}^{R} \beta_r u_r^2; \tag{12.5}$$

and the boundary conditions have been chosen so that this form is bounded above. Indeed, in view of (11.2) and the negative values of all β_r for $r > k$, we have

$$a^{N-1}_{rs} u_r u_s \leqslant \sum_{r=1}^{k} \beta_r u_r^2. \tag{12.6}$$

Let us denote by $||u||_{S_t}{}^2$, $||u||_{T_t}{}^2$ the square integrals of $\Sigma_r u_r^2$ over S_t and T_t respectively. Then

$$||u||^2_{V_t} \equiv \int_{V_t} \sum{}_r u_r^2 dV \leqslant K_1 \int_0^t ||u||^2_{S_t} dt, \tag{12.7}$$

and from (12.4) we find, by conventional majorizations,

$$||u||^2_{S_t} \leqslant K\,[||u||^2_{S_0} + ||g||^2_{T_t}] + K_1 \int_0^t ||u||^2_{S_t} dt + K\,||f||^2_{V_t}. \tag{12.8}$$

By iteration of this inequality we obtain

$$||u||^2_{S_t} \leqslant K[||u||^2_{S_0} + ||g||^2_{T_t} + ||f||^2_{V_t}] e^{K_1 t}, \tag{12.9}$$

and upon integration with respect to t,

$$||u||^2_{V_t} \leqslant K[||u||^2_{S_0} + ||g||^2_{T_t} + ||f||^2_{V_t}] \frac{e^{K_1 t}}{K_1}. \tag{12.10}$$

This is the Friedrichs-Lewy estimate for the components u_r.

By differentiation of the system we can show that all first derivatives, except those with respect to x, satisfy a similar system of first order equations and boundary conditions. Repeating the above argument, we can show that these derivatives satisfy similar estimates in which the derivatives of the data appear. From the system (12.1) we then find corresponding estimates for the derivatives with respect to x. Estimates of all higher order derivatives can be found by repetition of this type of calculation; and we omit details.

The existence of a solution is now established by a sequence of analytic approximations, based on Theorem III. Let $u_r(n)$ be the piecewise analytic solution of an approximating analytic problem in which all coefficients and functions together with their derivatives up to an order $[\frac{1}{2}N] + h + 1$ approximate in the square integral norm the corresponding quantities of (11.1). Then the norms

$$||D_{h'}\, u_r^{(n)}||^2_{V_t}$$

are uniformly bounded, where $D_{h'}$ denotes a derivative of order

$$h' \leqslant [\tfrac{1}{2}N] + h + 1.$$

By Sobolev's Lemma **(15)** the functions

$$D_{h'}\, u_r^{(n)}$$

are then uniformly bounded, and by Ascoli's theorem **(5**, p. 122) we can select a subsequence which converges, together with all derivatives of order $\leqslant h$, to a limit u_r. This limit is a solution of the non-analytic problem. For this result we shall assume that the given system, surfaces, and boundary data are of class $C^{[\frac{1}{2}N]+h+1}$, and that the data g_r of (11.4) satisfy on C compatibility conditions of order l. Then the approximations $u_r^{(n)}$ are of class C^l in V, as is easily seen by examining the series expansions of Theorem III. Thus for $l \leqslant h$, the final solution u_r is C^h in V except across the characteristic surfaces issuing into V from C, where u_r is C^l.

We remark that the number of boundary conditions is determined by the signature of the second quadratic form (11.2).

THEOREM IV. *A symmetric hyperbolic system* (12.1) *of differentiability class* $[\frac{1}{2}N] + h + 1$ *has a unique solution in a domain V bounded in part by a space-like initial surface S and a boundary surface T, which*

(a) *assumes given Cauchy data on S,*

(b) *satisfies k boundary conditions* (11.4) *on T, where k is the number of characteristic surfaces issuing from $C = T \cap S$ into V,*

(c) *is of class C^h in V except across these characteristic surfaces where it is of class C^l.*

Extension of the domain has been treated for similar problems in **(3**, **8**, **10)** and will not be pursued here. The above method of estimation will apply to boundary conditions of the form

$$u_r = g_r + \epsilon \sum_{s=k+1}^{R} c_{rs}\, u_s\,,$$

provided that $|\epsilon|$ is sufficiently small.

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