RATING OF LARGEST CLAIMS AND ECOMOR REINSURANCE TREATIES FOR LARGE PORTFOLIOS

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1. INTRODUCTION

In the present paper we deal with the problem of calculating a premium for the largest claims and ECOMOR reinsurance treaties. AMMETER derived already in 1964 formulas for calculating the premiums of the largest claims and ECOMOR reinsurance treaties (compare also SEAL (1969), THÉPAUT (1950)), which we will restate in the following Section 2. Lately BENKTANDER (1978) has established an interesting connection between the premiums of the largest claims and excess of loss reinsurance treaties. He proved that the net risk premium of the largest claims treaty covering the p largest claims is bounded by the risk premium of an excess of loss treaty plus p times its priority, which has to be determined such that the mean number of excess claims equals p. Furthermore Benktander showed in examples that the upper bound is quite good in case of the Poisson–Pareto risk process. Nevertheless he did not give a formal proof for the quality of the bound in the Poisson–Pareto case nor for other risk processes.

In the following note we take up this last point and prove that for general risk process Benktander's upper bound is equivalent to the premium of the largest claims reinsurance cover when the size of the collective approaches infinity. Consequently, for large portfolios the risk premium of the largest claims cover may be replaced by the upper bound, e.i., calculated from the premium of the corresponding excess of loss treaty. Moreover we state a similar result for the ECOMOR treaty.

2. PRELIMINARIES

Consider a collective K of risks of an insurance company (resp. of a special branch of the company), producing claims each year. Let N denote the random variable of claims number per year and X_i, $i = 1, \ldots, N$ the claim amounts. We arrange the claims in decreasing size:

$$X_{N:1} \geq \cdots \geq X_{N:N}.$$

In the following we investigate special reinsurance treaties, defining the reinsurer's claims amount by:

$$R = \sum_{i=1}^{N} f_i(X_{N:i})$$

where f_i, $i = 1, 2, \ldots$ are real-valued functions with:

$$f_i(x) \leq x, \quad \forall x, \quad \text{and} \quad \sum_{i=1}^{n} f_i(x_i) \geq 0, \quad \forall x_1 \geq \cdots \geq x_n > 0, \quad \forall n.$$

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We get with the definition:

(a) $$f_i(x) = \max(x-P, 0), \qquad i = 1, 2, \ldots$$

the excess of loss treaty ($XL(P)$) with priority P,

(b) $$f_i(x) = x, \qquad i = 1, \ldots, p$$
$$f_i(x) = 0, \qquad i = p+1, p+2, \ldots$$

the largest claims reinsurance treaty ($LC(p)$), covering the p largest claims,

(c) $$f_i(x) = x, \qquad i = 1, \ldots, p-1$$
$$f_p(x) = (1-p)x$$
$$f_i(x) = 0, \qquad i = p+1, p+2, \ldots$$

the ECOMOR reinsurance treaty (ECOMOR (p)), covering all claims excess the pth largest claim.

In the following we restrict on investigating the net risk premium

$$\mu = E(R)$$

and assume X_i, being i.i.d. random variables with distribution function F, and N independent of all X_i, $i = 1, 2, \ldots$. Then the net risk premium of the $XL(P)$-treaty is equal to:

$$\mu_{XL(P)} = E(N) \int_{[P,\infty)} (x-P)\, F(dx). \tag{2.1}$$

For the $LC(p)$ and ECOMOR (p) treaty the derivation of formulas for μ is more involved and easy tractable expressions can only be developed with additional assumptions on the distributions of N and X_i. Assume N being Poisson distributed with parameter $\lambda > 0$ and F being a Pareto distribution function with parameter $\alpha > 1$, e.i.,

$$F(x) = 1 - x^{-\alpha}, \qquad x \geqslant 1. \tag{2.2}$$

Then, according to AMMETER (1964) and BERLINER (1972), the net risk premium $\mu_{LC(p)}$ of the $LC(p)$-treaty can be approximated from above by

$$\hat{\mu}_{LC(p)} = \nu^{1/\alpha} \frac{\alpha}{\alpha-1} \frac{\Gamma(p+1-1/\alpha)}{\Gamma(p)} \tag{2.3}$$

where

$$\Gamma(y) = \int_0^\infty u^{y-1} \exp(-u)\, du.$$

CIMINELLI (1976) showed that even for negative binomial distributed N $\hat{\mu}_{LC(p)}$ is a quite good approximation of $\mu_{LC(p)}$. Finally Ammeter developed under the Poisson–Pareto-assumption an approximation for the net risk premium

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$\mu_{\text{ECOMOR}(p)}$ of the ECOMOR (p)-treaty:

$$\hat{\mu}_{\text{ECOMOR}(p)} = \nu^{1/\alpha} \frac{1}{\alpha - 1} \frac{\Gamma(p - 1/\alpha)}{\Gamma(p - 1)} \tag{2.4}$$

3. RATING FOR LARGE PORTFOLIOS

In order to derive results on the risk premium for large portfolios, we investigate growing collectives $K_k = \{R_i, i = 1, \ldots, k\}$ of risks R_i, e.g., let $k \to \infty$. We assume for claims number N_k

$$\lim_{k \to \infty} \nu_k = \infty, \qquad \lim_{k \to \infty} \frac{\sqrt{\text{Var}\,(N_k)}}{\nu_k} = 0 \tag{3.1}$$

with the abbreviation:

$$\nu_k = E(N_k).$$

Denote by $LC_k(p_k)$ resp. $\text{ECOMOR}_k\ (p_k)$ largest claims resp. ECOMOR reinsurance covers for collective K_k and suppose:

$$\lim_{k \to \infty} \frac{p_k}{\nu_k} = s \in (0, 1), \tag{3.2}$$

e.i., asymptotically our treaties cover the $s\nu_k$ largest claims. Now we can state our theorem:

Theorem

In addition to (3.1), (3.2) assume:

(1) the claim amounts X_i, $i = 1, 2, \ldots$ are identically distributed with the continuous distribution function F and existing first moment.
(2) N_k, X_i, $i = 1, 2, \ldots$ are independent
(3) $F(P_s - h) < F(P_s) < F(P_s + h)\ \forall h > 0$ at $P_s = F^{-1}(1 - s)$ and

$$\int_{[P_s, \infty)} (x - P_s) F(dx) > 0.$$

Then:

(a)
$$\lim_{k \to \infty} \frac{\mu_{LC_k(p_k)}}{\mu_{XL_k(P_k)} + p_k P_k} = 1,$$

(b)
$$\lim_{k \to \infty} \frac{\mu_{\text{ECOMOR}_k(p_k)}}{\mu_{XL_k(P_k)}} = 1,$$

where $XL_k(P_k)$ denotes the excess of loss treaty for collective K_k with priority:

$$P_k = F^{-1}\left(1 - \frac{p_k}{\nu_k}\right)$$

(with the usual convention $F^{-1}(u) = \inf\{x : F(x) \geq u\}$).

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Remark 1

One should notice that the above theorem holds under rather weak assumptions on the claims number and claims size distributions.

Proof

We restrict ourselves on proving (a), since the proof of (b) is similar. By Chebyshev's inequality one has:

$$P\left(\left| \frac{N_k}{\nu_k} - 1 \right| \geqslant \varepsilon \right) \leqslant \frac{\text{Var}\,(N_k)}{\varepsilon^2 \nu_k^2},$$

where the upper bound converges by (3.1), (3.2) to zero.
Consequently,

$$\lim_{k \to \infty} \frac{N_k}{\nu_k} = 1 \quad \text{in probability,}$$

implying with (3.2):

$$\lim_{k \to \infty} \frac{p_k}{N_k} = s \quad \text{in probability.}$$

According to theorem 19.6 in BAUER (1974), we may assume in our proof:

$$\lim_{k \to \infty} \frac{p_k}{N_k} = s \quad \text{almost surely} \tag{3.3}$$

and consequently (by (3.1), (3.2)):

$$\lim_{k \to \infty} N_k = \infty \quad \text{almost surely.} \tag{3.4}$$

With definitions:

$$T_k := \min\,(p_k, N_k)$$

$$Y_{k1} := \min\,(P_s, X_{N_k:T_k})$$

$$Y_{k2} := \max\,(P_s, X_{N_k:T_k})$$

$$r_k := E\left(\sum_{i=1}^{N_k} X_i \,\text{sign}\,(P_s - X_{N_k:T_k}) 1_{[Y_{k1},\, Y_{k2})}(X_i) \right)$$

(1_M denotes the indicator function of the set M) we can write:

$$\begin{aligned} \mu_{LC_k(p_k)} &= E\left(\sum_{i=1}^{N_k} X_i \, 1_{[X_{N_k:T_k}, \infty)}(X_i) \right) \\ &= E\left(\sum_{i=1}^{N_k} X_i \, 1_{[P_s, \infty)}(X_i) \right) + r_k \\ &= \mu_{XL_k(P_s)} + \nu_k P_s s + r_k. \end{aligned} \tag{3.5}$$

Obviously for r_k holds:

$$\frac{|r_k|}{\nu_k} \leq E\left(\nu_k^{-1} \sum_{i=1}^{N_k} |X_i| \, 1_{\{X_{N_k:T_k} \neq P_s\}}\right). \tag{3.6}$$

From (3.2)–(3.4) and the strong law of large numbers follows:

$$\lim_{k\to\infty} \nu_k^{-1} \sum_{i=1}^{N_k} |X_i| = E|X_i| \quad \text{almost surely.}$$

Since by (3.3):

$$\lim_{k\to\infty} \frac{T_k}{N_k} = s \quad \text{almost surely}$$

we get from a theorem in SERFLING (1980) (p. 75):

$$\lim_{k\to\infty} X_{N_k:T_k} = P_s \quad \text{almost surely.}$$

Consequently, the integrant in (3.6) converges almost surely to zero, implying with the theorem of dominated convergence (in the version as stated in LOÈVE (1963) on the bottom of page 162):

$$\lim_{k\to\infty} \frac{r_k}{\nu_k} = 0. \tag{3.7}$$

Since $\lim_{k\to\infty} P_k = P_s$, we have by (2.1):

$$\lim_{k\to\infty} \frac{\mu_{XL_k(P_k)}}{\nu_k} = \lim_{k\to\infty} \frac{\mu_{XL_k(P_s)}}{\nu_k} = \int_{[P_s,\infty)} (x - P_s) F(dx),$$

yielding with (3.2), (3.5), (3.7) statement (a).

Remark 2

Defining the expected total claims amount of collective K_k:

$$\mu_k = \nu_k E(X_1),$$

the statement of the theorem can be formulated equivalently for the premium rates as:

$$\lim_{k\to\infty} \left(\frac{\mu_{LC_k(p_k)}}{\mu_k} - \frac{\mu_{XL_k(P_k)}}{\mu_k} - \frac{p_k P_k}{\mu_k} \right) = 0 \tag{a'}$$

$$\lim_{k\to\infty} \left(\frac{\mu_{\text{ECOMOR}_k(p_k)}}{\mu_k} - \frac{\mu_{XL_k(P_k)}}{\mu_k} \right) = 0. \tag{b'}$$

According to this theorem, the net risk premium (more exactly the net premium rate) of the $LC(p)$ and ECOMOR (p) treaties may be calculated from the premium of the $XL(P)$ treaty with

$$P = F^{-1}\left(1 - \frac{p}{E(N)}\right),$$

if the expected claim number $E(N)$ is large. In practice one has to estimate P. Assume one can get information about the largest claims of the past years $i = 1, \ldots, I$ and that in the year i the claims number has been equal to m_i. If for the quotation year the expected claims number is estimated as being ν, then set:

$$t_i = \left[\frac{m_i p}{\nu}\right] + 1,$$

and estimate P by:

$$\hat{P} = \frac{1}{I} \sum_{i=1}^{I} X_{m_i : t_i}^{(i)},$$

where $X_{m_i:j}^{(i)}$ denotes the jth largest claim of the accident year i (assuming all claims being inflation- and IBNER-corrected).

Now let us compare our result with BENKTANDER (1978). Define $H_k(x)$, being the expected number of excess claims for the $XL_k(x)$ treaty in the collective K_k. For a solution $\tilde{P}_k$ of:

$$H_k(\tilde{P}_k) = p_k, \tag{3.8}$$

Benktander showed for a general risk process:

$$\mu_{LC_k(p_k)} \leqslant \mu_{XL_k(\tilde{P}_k)} + p_k \tilde{P}_k. \tag{3.9}$$

Since under our conditions:

$$H_k(x) = \nu_k(1 - F(x))$$

holds, a special solution of (3.8) is

$$\tilde{P}_k = F^{-1}\left(1 - \frac{p_k}{\nu_k}\right),$$

which is by (3.2) and condition (3) of our theorem even the unique solution for sufficiently large k.

Consequently, Benktander's upper bound (see (3.9)) is identical with the denominator of the ratio in part (a) of our theorem. So we have given a general proof that for large portfolio Benktander's bound is a good approximation to the pure risk premium.

4. ASYMPTOTIC PREMIUM RATES FOR SPECIAL RISK PROCESSES

Our theorem of Section 3 was derived under quite general assumptions on the claim size distribution F and the claim number distribution. Completing our investigation, we now consider two special models for the risk process.

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Example 1

Assume F being a Pareto distribution function with parameter $\alpha > 1$ (see (2.2)). Then one has with the notations of Sections 2 and 3:

$$P_s = s^{-1/\alpha}$$

$$\mu_{XL(P_s)} = E(N)\frac{1}{\alpha-1}s^{1-1/\alpha}.$$

Denote by μ the expected total claims amount of colletive K, e.i.,

$$\mu = E(N)\frac{\alpha}{\alpha-1}.$$

Now one easily derives with remark 2 the handy approximations:

$$\frac{\mu_{LC(p)}}{\mu} \approx \left(\frac{p}{E(N)}\right)^{1-1/\alpha} \tag{4.1}$$

$$\frac{\mu_{ECOMOR(p)}}{\mu} \approx \frac{1}{\alpha}\left(\frac{p}{E(N)}\right)^{1-1/\alpha} \tag{4.2}$$

for large $E(N)$. In addition assume, N being Poisson (or negative binomial) distributed. Then we can calculate the premium rates $\mu_{LC(p)}/\mu$, $\mu_{ECOMOR(p)}/\mu$ with formulas (2.3), (2.4). The following tables contain the resulting values for various α and $s = p/E(N)$. The approximations (4.1), (4.2) are written in the last column.

(1) *LC*(p)*-treaty*:

	$E(N)$			
$\alpha = 1.5$	100	200	400	∞
$s = 0{\cdot}01$	19.2%	20.4%	21.0%	21.5%
0.02	25.7%	26.4%	26.8%	27.1%
0.03	29.9%	30.5%	30.8%	31.1%
0.04	33.2%	33.7%	34.0%	34.2%
0.05	36.0%	36.4%	36.6%	36.8%

	$E(N)$			
$\alpha = 2.0$	100	200	400	∞
$s = 0.01$	8.7%	9.4%	9.7%	10.0%
0.02	13.3%	13.7%	13.9%	14.1%
0.03	16.6%	17.0%	17.1%	17.3%
0.04	19.4%	19.7%	19.8%	20.0%
0.05	21.8%	22.1%	22.2%	22.4%

$\alpha = 2.5$	$E(N)$ 100	200	400	∞
$s = 0.01$	5.6%	5.9%	6.1%	6.3%
0.02	9.0%	9.3%	9.4%	9.6%
0.03	11.7%	12.0%	12.1%	12.2%
0.04	14.1%	14.3%	14.4%	14.5%
0.05	16.2%	16.4%	16.5%	16.6%

$\alpha = 3.0$	$E(N)$ 100	200	400	∞
$s = 0.01$	4.2%	4.4%	4.5%	4.6%
0.02	7.0%	7.2%	7.3%	7.4%
0.03	9.3%	9.5%	9.6%	9.7%
0.04	11.4%	11.5%	11.6%	11.7%
0.05	13.3%	13.4%	13.5%	13.6%

(2) ECOMOR (p)-*treaty*:

$\alpha = 1.5$	$E(N)$ 100	200	400	∞
$s = 0.02$	12.8%	15.8%	17.0%	18.1%
0.03	17.1%	19.1%	19.9%	20.7%
0.04	20.0%	21.5%	22.2%	22.8%
0.05	22.2%	23.4%	24.0%	24.6%
0.06	24.0%	25.1%	25.7%	26.1%

$\alpha = 2.0$	$E(N)$ 100	200	400	∞
$s = 0.02$	4.4%	5.9%	6.5%	7.1%
0.03	6.7%	7.7%	8.2%	8.7%
0.04	8.3%	9.2%	9.6%	10.0%
0.05	9.7%	10.5%	10.8%	11.2%
0.06	10.9%	11.6%	11.9%	12.2%

$\alpha = 2.5$	$E(N)$ 100	200	400	∞
$s = 0.02$	2.3%	3.1%	3.5%	3.8%
0.03	3.6%	4.3%	4.6%	4.9%
0.04	4.7%	5.3%	5.5%	5.8%
0.05	5.6%	6.1%	6.4%	6.6%
0.06	6.5%	6.9%	7.2%	7.4%

$\alpha = 3.0$	$E(N)$ 100	200	400	∞
$s = 0.02$	1.4%	2.0%	2.2%	2.5%
0.03	2.3%	2.8%	3.0%	3.2%
0.04	3.1%	3.5%	3.7%	3.9%
0.05	3.8%	4.2%	4.3%	4.5%
0.06	4.4%	4.8%	4.9%	5.1%

Remark 3

A referee pointed out that the formulas (4.1), (4.2) could also be derived directly with the Stirling approximation from (2.3), (2.4). This conjecture is only partly true, since (2.3), (2.4) were deduced with the assumption of Poisson (or negative binomial) distributed claim numbers, whereas with our general theorem (4.1), (4.2) easily follows for arbitrary claim number processes which only satisfy (3.1).

Example 2.

Now assume F being an exponential distribution function. For comparison with example 1, we choose F such that its range and its mean value are identical with those of the Pareto distribution, e.i.,

$$F(x) = 1 - \exp(-(\alpha - 1)(x - 1)), \qquad x \geqslant 1.$$

We get:

$$P_s = 1 - \frac{1}{\alpha - 1} \ln(s)$$

$$\mu_{XL(P_s)} = E(N) \frac{s}{\alpha - 1}$$

and the expected total claims amount:

$$\mu = E(N) \frac{\alpha}{\alpha - 1}$$

yielding with our remark 2 the approximations:

$$\text{(4.3)} \qquad \frac{\mu_{LC(p)}}{\mu} \approx \frac{p}{E(N)} \left(1 - \frac{1}{\alpha} \ln\left(\frac{p}{E(N)}\right)\right)$$

$$\text{(4.4)} \qquad \frac{\mu_{\mathrm{ECOMOR}(p)}}{\mu} \approx \frac{1}{\alpha} \frac{p}{E(N)}$$

for large $E(N)$. Values for the approximations (4.3), (4.4) are given in the following tables (with $s = p/E(N)$):

LC(p)-treaty

	$\alpha = 1.5$	2.0	2.5	3.0
$s = 0.01$	4.1%	3.3%	2.8%	2.5%
0.02	7.2%	5.9%	5.1%	4.6%
0.03	10.0%	8.3%	7.2%	6.5%
0.04	11·6%	10.4%	9.2%	8.3%
0.05	15.0%	12.5%	11.0%	10.0%

ECOMOR *(p)-treaty*

	$\alpha = 1.5$	2.0	2.5	3.0
$s = 0.02$	1.3%	1.0%	0.8%	0.7%
0.03	2.0%	1.5%	1.2%	1.0%
0.04	2.7%	2.0%	1.6%	1.3%
0.05	3.3%	2.5%	2.0%	1.7%
0.06	4.0%	3.0%	2.4%	2.0%

Obviously for small α and small s, the asymptotic premium rates are much smaller than in the Pareto case (compare last columns of the tables in example 1), a result which has already been mentioned by KUPPER (1971) for the $LC(p)$-cover with $p = 1$, $E(N) = 100$.

REFERENCES

AMMETER, H. (1964). The rating of "Largest Claim" reinsurance covers. Quarterly letter from the Algemeene Reinsurance Companies Jubilee, Number 2.

BAUER, H. (1974). *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie.* De Gruyter: Berlin.

BENKTANDER, G. (1978). Largest claims reinsurance (LCR). A quick method to calculate LCR-risk rates from excess of loss risk rates. *ASTIN BULLETIN,* **10**, 54–58.

BERLINER, B. (1972). Correlations between excess of loss reinsurance covers and reinsurance of the *n* largest claims. *ASTIN BULLETIN,* **6**, 260–275.

CIMINELLI, E. (1976). On the distribution of the highest claims and its application to the automobile insurance liability. *Transactions of the* 20th *International Congress of Actuaries,* 501–517.

KUPPER, J. (1971). Contributions to the theory of the largest claim cover. *ASTIN BULLETIN,* **6**, 134–146.

LOÈVE, M. (1963). *Probability Theory.* Van Nostrand: New York.

SEAL, H. L. (1969): *Stochastic Theory of a Risk Business.* Wiley & Sons: New York.

SERFLING, R. J. (1980): *Approximation Theorems of Mathematical Statistics.* Wiley and Sons: New York.

THÉPAUT, A. (1950): Le traité d'excédent du coût moyen relatif. *Bulletin Trimestriel de l'Institut des Actuares Francais,* No. 192.