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ON A SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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We consider a generalised Cahn–Hilliard system with elasticity based on constitutives laws proposed by Gurtin, with a logarithmic free energy. We obtain some results on the existence and uniqueness of solutions.

1. INTRODUCTION

The Cahn–Hilliard equation is central to materials science. It is a conservation law (in the sense that the average of the order parameter is conserved) and describes very important qualitative features of two-phase systems, namely the transport of atoms between unit cells (see [**3**, **4**] and the references therein). Some generalisations of this equation have been introduced by Gurtin in [**5**], which are based on constitutive equations that take into account the work of the internal microforces, the anisotropy and also the deformations of the material, which are essentially due to the displacement of atoms in the material. These derivations are based on belief that fundamental physical laws involving energy should account for the work associated with each kinematical process (the order parameter in our case). Assuming that the deformations are infinitesimal and that the displacement gradient is small, we can thus use the theory of linear elasticity. In this paper, we consider a model of these generalisations which have been derived in [**7**] and study the existence and uniqueness of solutions.

We set $\Omega = \prod_{i=1}^{n}]0, L_i[$, $L_i > 0$, $i = 1, \dots, n$, $n = 2$ or 3, and consider the following system:

$$
\begin{gathered}
\frac{\partial \rho}{\partial t} - a.\nabla \frac{\partial \rho}{\partial t} = \operatorname{div}(B\nabla \mu), \\
\mu - b.\nabla \mu = -\alpha \Delta \rho + f'(\rho) + \beta \frac{\partial \rho}{\partial t} - \frac{e}{2} \mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big) + e^2 \mathrm{TR}(\mathrm{CI})(\rho - \widetilde{\rho}_0), \\
\gamma \frac{\partial^2 u}{\partial t^2} - \frac{1}{2} \operatorname{div}\big(C[\nabla u + {}^t\nabla u]\big) + e\operatorname{div}(\rho\, \mathrm{CI}) = 0, \\
\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = u_1, \\
\rho \ \text{ and } \ u \ \text{ are } \ \Omega - \text{periodic};
\end{gathered}
\tag{1.1}
$$

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where $\alpha, \beta, \gamma, e > 0$, $\widetilde{\rho}_0$ is a constant, $a, b \in \mathbb{R}^n$, B is a symmetric positive definite tensor with constant coefficients (B is called the mobility tensor), C is the elasticity tensor with constant coefficients (we assume that it is a symmetric positive linear transformation which maps symmetric tensors onto symmetric tensors), μ is the chemical potential, ρ is the order parameter (corresponding to a density of atoms) and u is the displacement field. If y is a motion of Ω, then y is a field that associates with each material point x and time t a point $y(x,t) = x + u(x,t)$. $\mathrm{TR}(A)$ and ${}^tA$ are the trace and the transpose of A respectively. The free energy $f : [-1,1] \to \mathbb{R}$ is given by:

$$\begin{aligned} (1.2) \quad & f(s) = \frac{1}{2}(1-s^2) + \frac{\theta}{2}\Big[(1+s)\ln\Big(\frac{1+s}{2}\Big) + (1-s)\ln\Big(\frac{1-s}{2}\Big)\Big], \quad s \in]-1,1[, \\ & f(-1) = f(1) = 0, \end{aligned}$$

with $0 < \theta < 1$.

For the mathematical setting of the problem, we denote by $\|.\|$ and $(.,.)$ the usual norm and scalar product in $L^2(\Omega)$ (which are extended to $L^2(\Omega)^n$). For each $\rho \in L^1(\Omega)$, $m(\rho)$ stands for the average of ρ, that is, $m(\rho) = \big(1/|\Omega|\big)\int_\Omega \rho(x)\,dx$ (for a vector $u = (u_1, \ldots, u_n) \in L^1(\Omega)^n$, we have $m(u) = \big(m(u_1), \ldots, m(u_n)\big) \in \mathbb{R}^n$). For a space X, we denote by $\dot{X}$ the space $\{q \in X,\, m(q) = 0\}$, and by X' the dual space of X. We define by $N = -\mathrm{div}B\nabla$ a linear, self-adjoint, strictly positive operator with compact inverse N^{-1} on $\dot{H}^2_{\mathrm{per}}(\Omega)$. We set $\Omega_T = \Omega\times]0,T[$ and $\overline{q} = q - m(q)$. We endow $\big(\dot{H}^1_{\mathrm{per}}(\Omega)\big)'$ with the norm $\|.\|_{-1}$ defined by $\|\overline{q}\|_{-1} = \|N^{-1/2}\overline{q}\|$, $\forall q \in \big(H^1_{\mathrm{per}}(\Omega)\big)'$. Furthermore, there exist $c_1, c_2 > 0$ such that $\|\overline{q}\|_{-1} \leqslant c_1\|\overline{q}\| \leqslant c_2\|\nabla q\|$, $\forall q \in H^1_{\mathrm{per}}(\Omega)$. We finally note that N and $\dfrac{\partial}{\partial x_i}$, and thus N^{-1} and $\dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, commute.

We introduce a weak formulation of the problem:

Find

$$(\rho, \mu, u) : [0,T] \to H^1_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times \dot{H}^1_{\mathrm{per}}(\Omega)^n$$

such that $\rho(0) = \rho_0$, $u(0) = u_0$, $\dfrac{\partial u}{\partial t}(0) = u_1$, and for almost everywhere $t \in [0,T]$, $\forall T > 0$,

$$(1.3) \qquad \Big(\frac{\partial \rho}{\partial t}, q\Big) + \Big(\frac{\partial \rho}{\partial t}, a.\nabla q\Big) = \Big(N^{1/2}\mu, N^{1/2}q\Big), \quad \forall q \in H^1_{\mathrm{per}}(\Omega);$$

$$\begin{aligned} & (\mu, q) + (\mu, b.\nabla q) = \alpha(\nabla\rho, \nabla q) + (f'(\rho), q) + \beta\Big(\frac{\partial\rho}{\partial t}, q\Big) - \frac{e}{2}\Big(\mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big), q\Big) \\ (1.4) \qquad & \qquad + e^2\,\mathrm{TR}(\mathrm{CI})(\rho - \widetilde{\rho}_0, q), \quad \forall q \in H^1_{\mathrm{per}}(\Omega); \end{aligned}$$

$$(1.5) \qquad \gamma\Big(\frac{\partial^2 u}{\partial t^2}, \eta\Big) + \frac{1}{2}\big(C(\nabla u + {}^t\nabla u), \nabla\eta\big) - e(\rho(\mathrm{CI}), \nabla\eta) = 0, \quad \forall \eta \in \dot{H}^1_{\mathrm{per}}(\Omega)^n;$$

noting that $a.\nabla$ and $b.\nabla$ are antisymmetric on $H^1_{\mathrm{per}}(\Omega)$, that is, $(a.\nabla p, q) = -(p,\ a.\nabla q)$, $\forall p, q \in H^1_{\mathrm{per}}(\Omega)$. We take $q = 1$ in (1.3) and observe that the average of ρ is conserved:

$$(1.6) \qquad m\big(\rho(t)\big) = m(\rho_0), \quad \forall t \geqslant 0.$$

We now take $q = 1$ in (1.4) and obtain

$$m(\mu) = m\big(f'(\rho)\big) + e^2\,\mathrm{TR(CI)}(m(\rho) - \widetilde{\rho}_0). \tag{1.7}$$

Setting $q = N^{-1}\overline{q}$ in (1.3), we get $\mu = -N^{-1}\dfrac{\partial\rho}{\partial t} + a.\nabla N^{-1}\dfrac{\partial\rho}{\partial t} + m(\mu)$; and substituting in (1.4) by standard computations and noting that

$$b.\nabla(a.\nabla N^{-1}\frac{\partial\rho}{\partial t}) = \mathrm{div}\Big((a^tb + b^ta)/2\nabla N^{-1}\frac{\partial\rho}{\partial t}\Big),$$

we can reformulate problem (1.3)–(1.5) as follows:
We first look for $(\rho, u) : [0,T] \to H^1_{\text{per}}(\Omega) \times \dot{H}^1_{\text{per}}(\Omega)^n$ such that

$$\begin{aligned}
&\frac{d}{dt}\big[(N^{-1}\overline{\rho}, q) + (\widetilde{B}\nabla N^{-1}\overline{\rho}, \nabla q) + (N^{-1}\overline{\rho}, d.\nabla q)\big] + \alpha(\nabla\rho, \nabla q)\\
&\qquad - \frac{e}{2}\Big(\mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big), q\Big) + e^2\,\mathrm{TR(CI)}(\rho - \widetilde{\rho}_0, \overline{q}) + (f'(\rho), \overline{q}) = 0,\\
&\qquad\qquad \forall q \in H^1_{\text{per}}(\Omega);\\
&\gamma\Big(\frac{\partial^2 u}{\partial t^2}, \eta\Big) + \frac{1}{2}(C(\nabla u + {}^t\nabla u), \nabla\eta) - e\big(\rho(\mathrm{CI}), \nabla\eta\big) = 0, \quad \forall\eta \in \dot{H}^1_{\text{per}}(\Omega)^n;
\end{aligned} \tag{1.8}$$

and then set

$$\mu = -N^{-1}\frac{\partial\rho}{\partial t} + a.\nabla N^{-1}\frac{\partial\rho}{\partial t} + m\big(f'(\rho)\big) + e^2\,\mathrm{TR(CI)}\big(m(\rho) - \widetilde{\rho}_0\big); \tag{1.9}$$

where $d = a + b$ and $\widetilde{B} = \beta B - (a^tb + b^ta)/2$. We proved in [**2**] that $\widetilde{B}$ is a positive tensor, thanks to thermodynamical considerations:

$$\beta x^2 + d.yx + By.y \geqslant 0, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}^n$$

(see [**5**]).

Throughout this paper, the same letter c (and sometimes c_i, $i = 0, 1, 2, \ldots$) shall denote positive constants that may change from line to line.

2. A regularised problem

We denote by ψ and ϕ the functions

$$\psi(s) = \frac{\theta}{2}\Big[(1+s)\ln\Big(\frac{1+s}{2}\Big) + (1-s)\ln\Big(\frac{1-s}{2}\Big)\Big], \tag{2.1}$$

and $\phi(s) = \psi'(s)$, for $s \in]-1, 1[$. We then have $f(s) = (1 - s^2)/2 + \psi(s)$ and $f'(s) = -s + \phi(s)$.

The major difficulty in the study of problem (1.3)–(1.5) is that $\phi(s)$ is singular at $s = \pm 1$ and, therefore, has no meaning if $\rho = \pm 1$ in an open set of non-zero measure. To overcome this difficulty, we consider a regularised problem as in [**1**]. The logarithmic free energy $f(\rho)$ is replaced by the twice continuously differentiable function $f_\varepsilon(s) = (1-s^2)/2 + \psi_\varepsilon(s)$, where $\varepsilon \in]0,1[$, and

$$(2.2)\quad \psi_\varepsilon(s) = \begin{cases} \frac{\theta}{2}(1-s)\ln\Big[\frac{1-s}{2}\Big] + \frac{\theta}{4\varepsilon}(1+s)^2 + \frac{\theta}{2}(1+s)\ln\Big[\frac{\varepsilon}{2}\Big] - \frac{\theta\varepsilon}{4} & s \leqslant -1+\varepsilon, \\ \psi(s) & |s| \leqslant 1-\varepsilon, \\ \frac{\theta}{2}(1+s)\ln\Big[\frac{1+s}{2}\Big] + \frac{\theta}{4\varepsilon}(1-s)^2 + \frac{\theta}{2}(1-s)\ln\Big[\frac{\varepsilon}{2}\Big] - \frac{\theta\varepsilon}{4} & s \geqslant 1-\varepsilon. \end{cases}$$

The monotone function $\phi_\varepsilon = \psi_\varepsilon'$ has the following properties (see [**1**]),

$$(2.3)\qquad f_\varepsilon'(s)(r-s) \leqslant f_\varepsilon(r) - f_\varepsilon(s) + \frac{1}{2}(r-s)^2, \qquad \forall\, r,\, s$$

$$(2.4)\qquad \forall\,\varepsilon \leqslant \frac{1}{2}, \begin{cases} \theta(r-s)^2 \leqslant \big(\phi_\varepsilon(r) - \phi_\varepsilon(s)\big)(r-s), & \forall\, r,\, s, \\ \frac{\varepsilon}{\theta}\big(\phi_\varepsilon(r) - \phi_\varepsilon(s)\big)^2 \leqslant \big(\phi_\varepsilon(r) - \phi_\varepsilon(s)\big)(r-s), & \forall\, r,\, s; \end{cases}$$

for ε sufficiently small, for example, if $\varepsilon \leqslant \varepsilon_0 = \theta/8$, then

$$(2.5)\qquad f_\varepsilon(s) \geqslant \frac{\theta}{8\varepsilon}\big([s-1]_+^2 + [-1-s]_+^2\big) - 1 \geqslant -1 \quad \forall s,$$

where $[.]_+ = \max\{.,0\}$.

We now study the corresponding regularised problem:

Find $(\rho_\varepsilon, \mu_\varepsilon, u_\varepsilon) : [0,T] \to H^1_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega) \times \dot{H}^1_{\text{per}}(\Omega)^n$, such that $\rho_\varepsilon(0) = \rho_0$, $u_\varepsilon(0) = u_0$, $\frac{\partial u_\varepsilon}{\partial t}(0) = u_1$, and for almost everywhere $t \in [0,T]$, $\forall T > 0$,

$$(2.6)\qquad \begin{aligned} &\frac{d}{dt}\big[(N^{-1}\overline{\rho}_\varepsilon, q) + (\widetilde{B}\nabla N^{-1}\overline{\rho}_\varepsilon, \nabla q) + (N^{-1}\overline{\rho}_\varepsilon, d.\nabla q)\big] + \alpha(\nabla\rho_\varepsilon, \nabla q) \\ &\qquad - \frac{e}{2}\Big(\mathrm{TR}(C[\nabla u_\varepsilon + {}^t\nabla u_\varepsilon]), q\Big) + e^2\,\mathrm{TR}(\mathrm{CI})(\rho_\varepsilon - \widetilde{\rho}_0, \overline{q}) \\ &\qquad + (f_\varepsilon'(\rho_\varepsilon), \overline{q}) = 0, \quad \forall q \in H^1_{\text{per}}(\Omega); \end{aligned}$$

$$(2.7)\quad \gamma\Big(\frac{\partial^2 u_\varepsilon}{\partial t^2}, \eta\Big) + \frac{1}{2}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla\eta) - e\big(\rho_\varepsilon(\mathrm{CI}), \nabla\eta\big) = 0, \quad \forall \eta \in \dot{H}^1_{\text{per}}(\Omega)^n;$$

and

$$(2.8)\qquad \mu_\varepsilon = -N^{-1}\frac{\partial\rho_\varepsilon}{\partial t} + a.\nabla N^{-1}\frac{\partial\rho_\varepsilon}{\partial t} + m\big(f_\varepsilon'(\rho_\varepsilon)\big) + e^2\,\mathrm{TR}(\mathrm{CI})\big(m(\rho_\varepsilon) - \widetilde{\rho}_0\big).$$

Lemma 2.1. *We assume that $(\rho_0, u_0, u_1) \in H^1_{\text{per}}(\Omega) \times \dot{H}^1_{\text{per}}(\Omega)^n \times \dot{L}^2(\Omega)^n$, with $\|\rho_0\|_{L^\infty(\Omega)} \leqslant 1$, and that $|m(\rho_0)| \leqslant 1-\delta$, $\delta \in]0,1[$. Then, for all ε, there exists a unique trio of functions $(\rho_\varepsilon, \mu_\varepsilon, u_\varepsilon)$ solution of (2.6)–(2.8) such that*

$$\rho_\varepsilon \in L^\infty\big(0,T; H^1_{\text{per}}(\Omega)\big) \cap L^2\big(0,T; H^2_{\text{per}}(\Omega)\big) \cap \mathcal{C}\big([0,T]; L^2(\Omega)\big), \;\; \mu_\varepsilon \in L^2\big(0,T; H^1_{\text{per}}(\Omega)\big),$$

and

$$u_\varepsilon \in L^\infty(0,T; \dot{H}^1_{\text{per}}(\Omega)^n) \cap \mathcal{C}\big([0,T]; \dot{L}^2(\Omega)^n\big),$$

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with $\dfrac{\partial \rho_\varepsilon}{\partial t} \in L^2(\Omega_T)$ *and* $\dfrac{\partial u_\varepsilon}{\partial t} \in L^\infty(0,T;\dot{L}^2(\Omega)^n)$. *Furthermore, we obtain uniform estimates with respect to* ε*, when* $\varepsilon \leqslant \varepsilon_0$*, for a sufficiently small* ε_0.

PROOF: (i) Existence. For a fixed ε, the existence of solution $(\rho_\varepsilon, \mu_\varepsilon)$ of (2.6)–(2.7) follows from standard arguments using Galerkin approximations and then passing to the limit. In order to derive a priori estimates, we formally take $q = \dfrac{\partial \rho_\varepsilon}{\partial t}$ in (2.6). We note that

$$(d.\nabla N^{-1}q, q) = 0, \ \ (-\mathrm{div}(\widetilde{B}\nabla N^{-1}q), q) = \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}q\|^2, \ \forall q \in \dot{L}^2(\Omega).$$

Furthermore, the mapping $q \mapsto \|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}q\|$ defines a norm in $\dot{L}^2(\Omega)$ that is equivalent to the usual $L^2(\Omega)-$norm (see [**2**]). Therefore,

$$\begin{aligned}(2.9)\quad &\frac{1}{2}\frac{d}{dt}\Big(\alpha\|\nabla\rho_\varepsilon\|^2 + e^2\,\mathrm{TR(CI)}\|\rho_\varepsilon\|^2 + 2\int_\Omega f_\varepsilon(\rho_\varepsilon)\,dx\Big)\\ &\qquad + \Big\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho_\varepsilon}{\partial t}\Big\|^2 + \Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|_{-1}^2 - \frac{e}{2}\Big(\mathrm{TR}\big[C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon)\big], \frac{\partial\rho_\varepsilon}{\partial t}\Big) = 0;\end{aligned}$$

and then

$$(2.10)\qquad \frac{d}{dt}\Big(\alpha\|\nabla\rho_\varepsilon\|^2 + e^2\,\mathrm{TR(CI)}\|\rho_\varepsilon\|^2 + 2\int_\Omega f_\varepsilon(\rho_\varepsilon)\,dx\Big) + c_1\Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|^2 \leqslant c_2\|\nabla u_\varepsilon\|^2.$$

We now take $\eta = u_\varepsilon$ in (2.7) and obtain

$$(2.11)\qquad \gamma\Big(\frac{\partial^2 u_\varepsilon}{\partial t^2}, u_\varepsilon\Big) + \frac{1}{2}\big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon\big) = e\big(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon\big).$$

We have $\big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon\big) = \big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon\big)/2$ and thanks to Korn's inequality, there exists a positive constant c_0 such that

$$(2.12)\qquad \big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon\big) \geqslant c_0\|\nabla u_\varepsilon\|^2;$$

and therefore

$$(2.13)\qquad \gamma\frac{d}{dt}\Big(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\Big) + c_1\|\nabla u_\varepsilon\|^2 \leqslant \gamma\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2 + c_2\|\rho_\varepsilon\|^2.$$

We finally take $\eta = \dfrac{\partial u_\varepsilon}{\partial t}$ in (2.7) and obtain

$$(2.14)\qquad \frac{\gamma}{2}\frac{d}{dt}\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2 + \frac{1}{2}\big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla\frac{\partial u_\varepsilon}{\partial t}\big) = e\Big(\rho_\varepsilon(\mathrm{CI}), \nabla\frac{\partial u_\varepsilon}{\partial t}\Big).$$

We have

$$\begin{aligned}\Big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla\frac{\partial u_\varepsilon}{\partial t}\Big) &= \frac{1}{2}\Big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla\frac{\partial u_\varepsilon}{\partial t} + {}^t\nabla\frac{\partial u_\varepsilon}{\partial t}\Big)\\ (2.15)\qquad &= \frac{1}{4}\frac{d}{dt}\big(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon\big);\end{aligned}$$

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and

$$\left(\rho_\varepsilon(\mathrm{CI}), \nabla\frac{\partial u_\varepsilon}{\partial t}\right) = \frac{d}{dt}(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon) - \left((\mathrm{CI})\frac{\partial \rho_\varepsilon}{\partial t}, \nabla u_\varepsilon\right). \tag{2.16}$$

Therefore,

$$\frac{d}{dt}\left(\gamma\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 + \frac{1}{2}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon) - 2e\big(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon\big)\right) \leqslant \tau\left\|\frac{\partial \rho_\varepsilon}{\partial t}\right\|^2 + c(\tau)\|\nabla u_\varepsilon\|^2, \tag{2.17}$$

for any positive real τ.

We now add (2.10) and (2.17) with a proper τ, and obtain an estimate of the form

$$\frac{dE_1}{dt} + c_1\left\|\frac{\partial \rho_\varepsilon}{\partial t}\right\|^2 \leqslant c_2\|\nabla u_\varepsilon\|^2, \tag{2.18}$$

where,

$$E_1(t) = \gamma\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 + \frac{1}{2}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon) - 2e(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon) + \alpha\|\nabla\rho_\varepsilon\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho_\varepsilon\|^2 + 2\int_\Omega f_\varepsilon(\rho_\varepsilon)\,dx. \tag{2.19}$$

We now combine δ(2.13), σ(2.18) and (2.10), (δ, $\sigma > 0$) and then obtain

$$\frac{dE_2}{dt} + c_1\left\|\frac{\partial \rho_\varepsilon}{\partial t}\right\|^2 + c_2\|\nabla u_\varepsilon\|^2 \leqslant c_3\left(\|\nabla u_\varepsilon\|^2 + \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 + \|\rho_\varepsilon\|^2\right), \tag{2.20}$$

where,

$$E_2(t) = \gamma\delta\Big(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\Big) + \frac{\sigma}{2}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon) + \gamma\sigma\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 - 2e\sigma\big(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon\big) + \alpha(\sigma+1)\|\nabla\rho_\varepsilon\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})(\sigma+1)\|\rho_\varepsilon\|^2 + 2(\sigma+1)\int_\Omega f_\varepsilon(\rho_\varepsilon)\,dx. \tag{2.21}$$

We first fix σ such that

$$\frac{\sigma}{4}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon) - 2e\sigma\big(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon\big) + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho_\varepsilon\|^2 \geqslant c_1\big(\|\nabla u_\varepsilon\|^2 + \|\rho_\varepsilon\|^2\big), \tag{2.22}$$

where $c_1 > 0$; and we then fix δ such that

$$\gamma\sigma\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 + \frac{\sigma}{4}(C(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \nabla u_\varepsilon + {}^t\nabla u_\varepsilon) + \gamma\delta\Big(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\Big) \geqslant c_2\left(\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2 + \|\nabla u_\varepsilon\|^2\right), \tag{2.23}$$

where $c_2 > 0$. Estimates (2.22) and (2.23) come from the fact that, using Poincaré and Young inequalities, we have

$$-\big(\rho_\varepsilon(\mathrm{CI}), \nabla u_\varepsilon\big) \geqslant -\tau_1/2\|\rho_\varepsilon\|^2 - 1/2\tau_1\Big\|\nabla u_\varepsilon\Big\|^2,$$

and

$$\Big(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\Big) \geqslant -\tau_2/2\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2 - 1/2\tau_2\|\nabla u_\varepsilon\|^2,$$

for any strictly positive reals τ_1 and τ_2.

Thanks to (2.21), (2.22) and (2.23), together with Poincaré's inequality, there exists c_1 and c_2 such that

$$E_2(t) \geqslant c_1\left(\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2 + \|\rho_\varepsilon\|^2_{H^1_{\mathrm{per}}(\Omega)} + \|u_\varepsilon\|^2_{\dot{H}^1_{\mathrm{per}}(\Omega)^n} + \int_\Omega f_\varepsilon(\rho_\varepsilon)\,dx\right) - c_2. \tag{2.24}$$

We finally obtain from (2.20) an inequality of the form

$$\frac{dE_2}{dt} + c_1\|\frac{\partial \rho_\varepsilon}{\partial t}\|^2 + c_2\|u_\varepsilon\|^2_{\dot{H}^1_{\mathrm{per}}(\Omega)^n} \leqslant c_3E_2 + c_4. \tag{2.25}$$

Using Gronwall's lemma and noting that $f_\varepsilon(\rho_0) \leqslant f(\rho_0)$ for $\varepsilon \leqslant \varepsilon_0$, ε_0 sufficiently small, we deduce the following estimates:

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\rho_\varepsilon\|_{H^1_{\mathrm{per}}(\Omega)} + \Big\|\frac{\partial \rho_\varepsilon}{\partial t}\Big\|_{L^2(\Omega_T)} \leqslant c; \tag{2.26}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega \big([\rho_\varepsilon - 1]_+^2 + [-1-\rho_\varepsilon]_+^2\big)\,dx \leqslant c\varepsilon; \tag{2.27}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]}\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\| + \operatorname*{ess\,sup}_{t\in[0,T]} \|u_\varepsilon\|_{\dot{H}^1_{\mathrm{per}}(\Omega)^n} \leqslant c; \tag{2.28}$$

where c is independent of ε. The existence of ρ_ε and u_ε are deduced from (2.26) and (2.28) using classical compactness results (see for instance [**6**] or [**8**]). The existence of μ_ε is deduced from (2.8).

We further obtain some uniform estimates in ε, for $\varepsilon \leqslant \varepsilon_0$, ε_0 sufficiently small. We note that

$$\Big\|\nabla N^{-1}\frac{\partial \rho_\varepsilon}{\partial t}\Big\| \leqslant c\Big\|\frac{\partial \rho_\varepsilon}{\partial t}\Big\|_{-1} \leqslant c\Big\|\frac{\partial \rho_\varepsilon}{\partial t}\Big\|,$$

and

$$\Big\|\nabla\Big(a.\nabla N^{-1}\frac{\partial \rho_\varepsilon}{\partial t}\Big)\Big\| \leqslant c\Big\|\frac{\partial \rho_\varepsilon}{\partial t}\Big\|$$

and therefore

$$\|\nabla \mu_\varepsilon\| \leqslant c\Big\|\frac{\partial \rho_\varepsilon}{\partial t}\Big\|;$$

and thanks to Poincaré's inequality and the regularised counterparts of (1.6) and (1.7), we obtain

$$\|\mu_\varepsilon\|^2_{L^2(0,T;H^1_{\mathrm{per}}(\Omega))} \leqslant c\Big(1 + \Big\|m\big(f'_\varepsilon(\rho_\varepsilon)\big)\Big\|^2_{L^2(\Omega_T)}\Big). \tag{2.29}$$

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We now want to prove that $\left\|m\big(f_\varepsilon'(\rho_\varepsilon)\big)\right\|^2_{L^2(\Omega_T)}$ is bounded independently of ε. We formally take $q = \overline{\rho}_\varepsilon$ in the regularised counterpart of (1.4), and obtain

$$\text{(2.30)}\quad \alpha\|\nabla\rho_\varepsilon\|^2 + (f_\varepsilon'(\rho_\varepsilon), \overline{\rho}_\varepsilon) = \Big(\mu_\varepsilon - b.\nabla\mu_\varepsilon - \beta\frac{\partial\rho_\varepsilon}{\partial t} + \frac{e}{2}\mathrm{TR}\big(C[\nabla u_\varepsilon + {}^t\nabla u_\varepsilon]\big) - e^2\,\mathrm{TR}(\mathrm{CI})(\rho_\varepsilon - \widetilde{\rho}_0), \overline{\rho}_\varepsilon\Big).$$

Noting that

$$\big(f_\varepsilon'(\rho_\varepsilon), \overline{\rho}_\varepsilon\big) = \big(f_\varepsilon'(\rho_\varepsilon), \rho_\varepsilon - \lambda\big) + \big(f_\varepsilon'(\rho_\varepsilon), \lambda - m(\rho_\varepsilon)\big), \quad \forall\lambda\in\mathbb{R},$$

and using (2.3), it follows that

$$\text{(2.31)}\quad \big(f_\varepsilon'(\rho_\varepsilon), \lambda - m(\rho_\varepsilon)\big) \leqslant \big(f_\varepsilon(\lambda) - f_\varepsilon(\rho_\varepsilon), 1\big) + \frac{1}{2}\|\rho_\varepsilon - \lambda\|^2 + c_1\|\nabla\rho_\varepsilon\|^2 + c_2\Big(\|\mu_\varepsilon\| + \|\nabla\mu_\varepsilon\| + \|\nabla u_\varepsilon\| + \Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|\Big)\|\nabla\rho_\varepsilon\|.$$

Choosing $\lambda = \pm 1$ and using (2.5) and the assumptions on ρ_0, we deduce

$$\text{(2.32)}\qquad \delta\,|\Omega|\,\Big|m\big(f_\varepsilon'(\rho_\varepsilon)\big)\Big| \leqslant c\Big[1 + \|\nabla\rho_\varepsilon\|^2 + \Big(\|\nabla\mu_\varepsilon\| + \|\nabla u_\varepsilon\| + \Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|\Big)\|\nabla\rho_\varepsilon\|\Big],$$

and, therefore,

$$\text{(2.33)}\qquad \Big\|m\big(f_\varepsilon'(\rho_\varepsilon)\big)\Big\|^2 \leqslant c\Big[1 + \|\nabla\rho_\varepsilon\|^4 + \Big(\|\nabla\mu_\varepsilon\|^2 + \|\nabla u_\varepsilon\|^2 + \Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|^2\Big)\|\nabla\rho_\varepsilon\|^2\Big];$$

hence

$$\text{(2.34)}\qquad \Big\|m\big(f_\varepsilon'(\rho_\varepsilon)\big)\Big\|^2_{L^2(\Omega_T)} \leqslant c.$$

Taking now $q = \phi_\varepsilon(\rho_\varepsilon) - m\big(\phi_\varepsilon(\rho_\varepsilon)\big)$ in the regularised counterpart of (1.4), we obtain

$$\text{(2.35)}\quad \alpha\big(\phi_\varepsilon'(\rho_\varepsilon)\nabla\rho_\varepsilon, \nabla\rho_\varepsilon\big) + \Big(\phi_\varepsilon(\rho_\varepsilon) - \rho_\varepsilon, \phi_\varepsilon(\rho_\varepsilon) - m\big(\phi_\varepsilon(\rho_\varepsilon)\big)\Big) = \Big(\mu_\varepsilon - b.\nabla\mu_\varepsilon - \beta\frac{\partial\rho_\varepsilon}{\partial t} + \frac{e}{2}\mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big) - e^2\,\mathrm{TR}(\mathrm{CI})(\rho_\varepsilon - \widetilde{\rho}_0), \phi_\varepsilon(\rho_\varepsilon) - m\big(\phi_\varepsilon(\rho_\varepsilon)\big)\Big).$$

We note that $\phi_\varepsilon'(\rho_\varepsilon) \geqslant \theta$ (which follows from (2.4)), and then deduce the following estimate

$$\text{(2.36)}\quad \alpha\theta\|\nabla\rho_\varepsilon\|^2 + \frac{1}{2}\Big\|\phi_\varepsilon(\rho_\varepsilon) - m\big(\phi_\varepsilon(\rho_\varepsilon)\big)\Big\|^2 \leqslant c_1\Big(1 + \|\mu_\varepsilon\|^2 + \|\nabla\mu_\varepsilon\|^2 + \|\nabla\rho_\varepsilon\|^2 + \|\nabla u_\varepsilon\|^2 + \Big\|\frac{\partial\rho_\varepsilon}{\partial t}\Big\|^2\Big),$$

which yields $\left\|\phi_\varepsilon(\rho_\varepsilon) - m\big(\phi_\varepsilon(\rho_\varepsilon)\big)\right\|^2_{L^2(\Omega_T)} \leqslant c$ and, therefore,

$$\|\phi_\varepsilon(\rho_\varepsilon)\|_{L^2(\Omega_T)} \leqslant c. \tag{2.37}$$

Finally, the uniform estimate on ρ_ε in $L^2\big(0,T;H^2_{\text{per}}(\Omega)\big)$ follows from the estimate

$$\begin{aligned}\|\Delta\rho_\varepsilon\|^2_{L^2(\Omega_T)} \leqslant c_1\Bigg(1 + \|\mu_\varepsilon\|^2_{L^2(\Omega_T)} + \|\nabla\mu_\varepsilon\|^2_{L^2(\Omega_T)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_T)} \\ + \big\|\phi_\varepsilon(\rho_\varepsilon)\big\|^2_{L^2(\Omega_T)} + \left\|\frac{\partial\rho_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega_T)}\Bigg) \leqslant c_2;\end{aligned} \tag{2.38}$$

which follows from the second line of the regularised counterpart of (1.1) and the fact that the following regular inequality

$$\Big\|\rho_\varepsilon - m(\rho_0)\Big\|^2_{L^2(0,T;H^2_{\text{per}}(\Omega))} \leqslant c\|\Delta\rho_\varepsilon\|^2_{L^2(\Omega_T)}$$

is held.

(ii) Uniqueness. Let (ρ_1,μ_1,u_1) and (ρ_2,μ_2,u_2) be two solutions of (2.6)–(2.8) with the same initial data. Setting $\rho = \rho_1 - \rho_2$ and $u = u_1 - u_2$, we have $\rho(0) = 0$, $u(0) = 0$, $\dfrac{\partial u}{\partial t}(0) = 0$ and

$$\begin{aligned}&\frac{d}{dt}\big[(N^{-1}\rho,q) + (\widetilde{B}\nabla N^{-1}\rho,\nabla q) + (N^{-1}\rho, d.\nabla q)\big] + \alpha(\nabla\rho,\nabla q)\\ &\qquad - \frac{e}{2}\Big(\mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big), q\Big) + e^2\,\mathrm{TR}(\mathrm{CI})(\rho,\overline{q})\\ &\qquad + \big(f'_\varepsilon(\rho_1) - f'_\varepsilon(\rho_2),\overline{q}\big) = 0, \ \ \forall q \in H^1_{\text{per}}(\Omega);\end{aligned} \tag{2.39}$$

$$\gamma\Big(\frac{\partial^2 u}{\partial t^2},\eta\Big) + \frac{1}{2}\big(C(\nabla u + {}^t\nabla u),\nabla\eta\big) - e\big(\rho(\mathrm{CI}),\nabla\eta\big) = 0, \ \ \forall\eta \in \dot{H}^1_{\text{per}}(\Omega)^n. \tag{2.40}$$

We take $q = \rho$ in (2.39), and noting that $m(\rho) = 0$, we obtain

$$\begin{aligned}&\frac{d}{dt}\big[\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\rho\|^2 + \|\rho\|^2_{-1}\big] + \alpha\|\nabla\rho\|^2 - \frac{e}{2}\Big(\mathrm{TR}\big(C[\nabla u + {}^t\nabla u]\big),\rho\Big)\\ &\qquad + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2 + \big(\phi_\varepsilon(\rho_1) - \phi_\varepsilon(\rho_2),\rho\big) = \|\rho\|^2 - \Big(\frac{\partial\rho}{\partial t}, d.\nabla N^{-1}\rho\Big).\end{aligned} \tag{2.41}$$

We have $\big(\phi_\varepsilon(\rho_1) - \phi_\varepsilon(\rho_2),\rho\big) \geqslant \theta\|\rho\|^2$ (which comes from (2.4)) and $\|\rho\|_{H^1_{\text{per}}(\Omega)} \leqslant c\|\nabla\rho\|$ (Poincaré's inequality). Therefore,

$$\frac{d}{dt}\big(\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\rho\|^2 + \|\rho\|^2_{-1}\big) + c_1\|\rho\|^2_{H^1_{\text{per}}(\Omega)} \leqslant c_2\bigg(\|\rho\|^2 + \|\nabla u\|^2 + \left\|\frac{\partial\rho}{\partial t}\right\|^2\bigg). \tag{2.42}$$

We now take $q = \dfrac{\partial\rho}{\partial t}$ in (2.39) and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\alpha\|\nabla\rho\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2\big) + \left\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\frac{\partial\rho}{\partial t}\right\|^2 + \left\|\frac{\partial\rho}{\partial t}\right\|^2_{-1}\\ &\qquad - \frac{e}{2}\Big(\mathrm{TR}\big[C(\nabla u + {}^t\nabla u)\big],\frac{\partial\rho}{\partial t}\Big) + \Big(\phi_\varepsilon(\rho_1) - \phi_\varepsilon(\rho_2),\frac{\partial\rho}{\partial t}\Big) = \Big(\rho,\frac{\partial\rho}{\partial t}\Big).\end{aligned} \tag{2.43}$$

We note that $\phi_\varepsilon(\rho_1) - \phi_\varepsilon(\rho_2) = \rho\phi'_\varepsilon(\zeta\rho_1 + (1-\zeta)\rho_2)$, with $\zeta \in [0,1]$; and that $\phi'_\varepsilon(s) \leqslant \theta\varepsilon^{-1}$, $\forall s$, (which follows from (2.4)). We thus have

$$\Big(\phi_\varepsilon(\rho_1) - \phi_\varepsilon(\rho_2), \frac{\partial\rho}{\partial t}\Big) \leqslant \eta\Big\|\frac{\partial\rho}{\partial t}\Big\|^2 + c\varepsilon^{-2}\|\rho\|,\ \forall\eta > 0,$$

and

$$\frac{d}{dt}\big(\alpha\|\nabla\rho\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2\big) + c_1\Big\|\frac{\partial\rho}{\partial t}\Big\|^2 \leqslant c_2(1+\varepsilon^{-2})\|\rho\|^2 + c_3\|\nabla u\|^2. \tag{2.44}$$

We combine $\sigma(2.42)$ and (2.44) with a proper positive σ and obtain

$$\begin{aligned}\frac{d}{dt}\big[\sigma\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\rho\|^2 + \sigma\|\rho\|_{-1}^2 + \alpha\|\nabla\rho\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2\big]&\\ + c_1\|\rho\|^2_{H^1_{\mathrm{per}}(\Omega)} + c_1\Big\|\frac{\partial\rho}{\partial t}\Big\|^2 \leqslant c_2(\varepsilon)\big(\|\rho\|^2 + \|\nabla u\|^2\big).&\end{aligned} \tag{2.45}$$

On the other hand we take $\eta = u$ in (2.40). Proceeding as above, we obtain

$$\gamma\frac{d}{dt}\Big(\frac{\partial u}{\partial t}, u\Big) + c_1\|\nabla u\|^2 \leqslant \gamma\Big\|\frac{\partial u}{\partial t}\Big\|^2 + c_2\|\rho\|^2. \tag{2.46}$$

We finally take $\eta = \dfrac{\partial u}{\partial t}$ in (2.40) and noting that $\Big(\rho(\mathrm{CI}), \nabla\dfrac{\partial u}{\partial t}\Big) = -\Big((\mathrm{CI})\nabla\rho, \dfrac{\partial u}{\partial t}\Big)$

$$\frac{d}{dt}\Big(\gamma\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}(C(\nabla u + {}^t\nabla u), \nabla u + {}^t\nabla u)\Big) \leqslant \tau\|\nabla\rho\|^2 + c\tau^{-1}\Big\|\frac{\partial u}{\partial t}\Big\|^2, \tag{2.47}$$

for any strictly positive τ.

We combine (2.45), δ(2.46) and (2.47) with a suitable τ, and then obtain

$$\begin{aligned}\frac{d}{dt}\Big[\sigma\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\rho\|^2 + \sigma\|\rho\|_{-1}^2 + \alpha\|\nabla\rho\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2&\\ + \gamma\delta\Big(\frac{\partial u}{\partial t}, u\Big) + \gamma\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}(C(\nabla u + {}^t\nabla u), \nabla u + {}^t\nabla u)\Big]&\\ + \|\rho\|^2_{H^1(\Omega)} + c_1\Big\|\frac{\partial\rho}{\partial t}\Big\|^2 \leqslant c\Big\|\frac{\partial u}{\partial t}\Big\|^2 + c\|\nabla\rho\|^2 + c\|\nabla u\|^2.&\end{aligned} \tag{2.48}$$

We now fix δ such that

$$\gamma\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}(C(\nabla u + {}^t\nabla u), \nabla u + {}^t\nabla u) + \gamma\delta\Big(\frac{\partial u}{\partial t}, u\Big) \geqslant c_2\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \|\nabla u\|^2\Big), \tag{2.49}$$

where $c_2 > 0$.

Setting

$$\begin{aligned}E_3(t) = \sigma\|\widetilde{B}^{1/2}\nabla B^{1/2}\nabla N^{-1}\rho\|^2 + \sigma\|\rho\|_{-1}^2 + \alpha\|\nabla\rho\|^2 + e^2\,\mathrm{TR}(\mathrm{CI})\|\rho\|^2&\\ + \gamma\delta\Big(\frac{\partial u}{\partial t}, u\Big) + \gamma\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}(C(\nabla u + {}^t\nabla u), \nabla u + {}^t\nabla u));&\end{aligned} \tag{2.50}$$

we find

$$E_3(t) \geqslant c_1\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla\rho\|^2 + \|\nabla u\|^2\right).$$

And therefore, we deduce an estimate of the form

$$\frac{dE_3}{dt} \leqslant cE_3; \tag{2.51}$$

and then the uniqueness of ρ_ε and u_ε. The uniqueness of μ_ε is deduced from (2.8). ∎

Theorem 2.1. *Let the assumptions of Lemma 2.1 hold. Then, there exists a trio of functions (ρ, μ, u), solution of (1.3)–(1.5) such that $\rho \in L^\infty(0,T; H^1_{\text{per}}(\Omega)) \cap L^2(0,T; H^2_{\text{per}}(\Omega)) \cap \mathcal{C}([0,T]; L^2(\Omega))$, $\mu \in L^2(0,T; H^1_{\text{per}}(\Omega))$, $|\rho| \leqslant 1$ almost everywhere in Ω_T, $\frac{\partial\rho}{\partial t} \in L^2(\Omega_T)$, $\phi(\rho) \in L^2(\Omega_T)$; and $u \in L^\infty(0,T; \dot{H}^1_{\text{per}}(\Omega)^n) \cap \mathcal{C}([0,T]; \dot{L}^2(\Omega)^n)$, with $\frac{\partial u}{\partial t} \in L^\infty(0,T;\ \dot{L}^2(\Omega)^n)$. Furthermore, we have the uniqueness of solution when $d = 0$.*

Proof: It follows from Lemma 2.1 that there exists a trio of functions (ρ, μ, u) and a subsequence $(\rho_\varepsilon, \mu_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ (which we still denote by $(\rho_\varepsilon, \mu_\varepsilon, u_\varepsilon)_{\varepsilon>0}$) such that

$$\begin{array}{ll}
\rho_\varepsilon,\ \nabla\rho_\varepsilon \to \rho,\ \nabla\rho & \text{strongly in } L^2(\Omega_T) \text{ and almost everywhere in } \Omega_T,\\
\dfrac{\partial\rho_\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial\rho}{\partial t} & \text{weakly in } L^2(\Omega_T),\\
\mu_\varepsilon,\ \nabla\mu_\varepsilon \rightharpoonup \mu,\ \nabla\mu & \text{weakly in } L^2(\Omega_T),\\
u_\varepsilon,\ \nabla u_\varepsilon \rightharpoonup u,\ \nabla u & \text{weakly-star in } L^\infty(0,T; \dot{L}^2(\Omega)),\\
\dfrac{\partial u_\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly-star in } L^\infty(0,T; \dot{L}^2(\Omega)),\\
\phi_\varepsilon(\rho_\varepsilon) \rightharpoonup \phi(\rho) & \text{weakly in } L^2(\Omega_T).
\end{array}$$

Passing to the limit in the regularised problem, we find that (ρ, μ, u) is a solution of (1.3)–(1.5). The other points of the theorem also come from passing to the limit in the uniform estimates (2.27) and (2.37), as ε goes to zero.

We don't succeed in obtaining the uniqueness of solution for (1.3)–(1.5) when $a \neq 0$ or/and $b \neq 0$, The difficulty appears in getting an estimate of the term $\left(\phi(\rho_1) - \phi(\rho_2), \frac{\partial\rho}{\partial t}\right)$ as we wanted. But, when we consider the case $d = 0$, it is a simple matter to obtain the uniqueness of solution (ρ, μ, u). Indeed, we take $q = \rho$, $\eta = u$, $\eta = \frac{\partial u}{\partial t}$ in the corresponding non-regularised versions of (2.39) and (2.40) respectively. Combining estimates obtained as above, we get an estimate of the form (2.51), therefore the result. ∎

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