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AN INEQUALITY CHARACTERIZES THE TRACE

L. TERRELL GARDNER

1. Introduction. While analogues of the Schwarz inequality have been much studied in the context of positive linear maps of operator algebras ([**1**], [**2**], [**6**], [**7**], [**10**]) the simpler triangle inequality $|\phi(x)| \leqq \phi(|x|)$ has been neglected, outside of (possibly non-commutative) integration theory—perhaps partly because except for the important and familiar example of traces, scalar maps satisfying the triangle inequality are rarely encountered. In fact we here prove that they are never encountered: every such map is a trace.

For C^*-algebras (norm-closed self-adjoint algebras of bounded operators on a Hilbert space) this means, for instance, that if the linear functional ϕ on the C^*-algebra $\mathscr{A}$ satisfies

(†) $|\phi(x)| \leqq \phi(|x|)$ for all x in $\mathscr{A}$,

then ϕ satisfies also the equivalent conditions

(i) $\phi(xy) = \phi(yx)$ for all x, y in $\mathscr{A}$;
(ii) $\phi(x^*x) = \phi(xx^*)$ for all x in $\mathscr{A}$;
(iii) $\phi(x) = \phi(uxu^*)$ for all x in $\mathscr{A}$ and all unitary u in A_e, the C^*-algebra formed from $\mathscr{A}$ by adjunction of a unit element.

The equivalence of these three conditions is well known; they together with linearity and positivity—$\phi(x^*x) \geqq 0$ for all x in $\mathscr{A}$—are the defining conditions of a (finite) trace ϕ on $\mathscr{A}$.

It is well-known and easy to prove that every finite trace on $\mathscr{A}$ does satisfy (†). The converse, proved here (Theorem 1) appears to have escaped notice.

If ϕ is a trace on the W^*-algebra $\mathscr{A}$ (for definitions, see § 2), it satisfies the triangle inequality in the following sense:

(T) If $x \in \mathscr{A}$ and $\phi(|x|) < +\infty$, x lies in the linear span L of $\{a \in \mathscr{A}^+ : \phi(a) < +\infty\}$; on L, ϕ has a unique extension as a linear functional, so $\phi(x)$ is defined and finite. Moreover, $|\phi(x)| \leqq \phi(|x|)$.

An apparently weaker condition, easier to prove necessary for a trace ϕ on $\mathscr{A}$ is:

(WT) For every projection e in $\mathscr{A}$ with $\phi(e) < +\infty$, $|\phi(x)| \leqq \phi(|x|)$ for all x in $e\mathscr{A}e$.

Here $\phi(x)$ makes sense as in (T), since $e\mathscr{A}e \subset L$. Conversely, we prove

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(Theorem 2) that for a normal, strongly semifinite weight ϕ on a W^*-algebra $\mathscr{A}$, (WT) is a sufficient condition that ϕ be a trace.

The proofs depend essentially on the special case of Theorem 1: $\mathscr{A} = M_2(\mathbf{C})$, the *-algebra of 2×2 complex matrices (Main Lemma). Using this, we prove Theorem 2 by showing first that ϕ must preserve equivalence of orthogonal projections, then reducing the problem in effect to the case in which ϕ is faithful and finite, and $\mathscr{A}$ is finite. The comparison theory of projections is our chief tool here.

Finally, to extract Theorem 1 from Theorem 2, we observe that the hypothesis persists for ϕ^{dd} on $\mathscr{A}^{dd}$, the second adjoint space of $\mathscr{A}$, which by Sherman's theorem [**9**] is a W^*-algebra. This argument requires an adaptation of Kaplansky's density theorem [**8**] to the *-strong topology, for which see [**5**].

In § 2 we set forth the basic definitions and conventions used in the rest of the paper; in § 3, the main results are stated; in § 4, the proofs and some subsidiary results appear.

2. Definitions and conventions. For the basic theory of C^*-algebra and W^*-algebra (von-Neumann algebras), we refer the reader to the books of J. Dixmier ([**4**], [**3**]).

If $\mathscr{A}$ is a C^*-algebra, $\mathscr{A}^+ = \{x^*x; x \in \mathscr{A}\}$ is a proper closed convex cone, linearly spanning $\mathscr{A}$. Every element a of A^+ has a unique square root $a^{1/2}$ in $\mathscr{A}^+$. If $x \in \mathscr{A}$, $|x| = (x^*x)^{1/2}$, and $\operatorname{Re} x = (x + x^*)/2$.

A *weight* on a W^*-algebra $\mathscr{A}$ is an additive, positively homogeneous map ϕ defined on the positive cone $\mathscr{A}^+$ of $\mathscr{A}$ with values in $[0, +\infty]$. ϕ is *normal* if for increasing, bounded nets $(a_\nu)_\nu$ in $\mathscr{A}^+$, $\sup [\phi(a_\nu)] = \phi(\sup [a_\nu])$, and *strongly semifinite* if for each a in A^+ there exists a non-zero a' in A^+ dominated by a, with $\phi(a') < +\infty$. ϕ is *faithful* if $0 \neq a \in \mathscr{A}^+$ implies $\phi(a) > 0$. A weight ϕ is a *trace* if it is unitarily invariant; that is, if $\phi(a) = \phi(uau^*)$ for all a in $\mathscr{A}^+$ and all unitary u in $\mathscr{A}$. An equivalent condition is that $\phi(x^*x) = \phi(xx^*)$ for all x in $\mathscr{A}$.

If ϕ is a weight on $\mathscr{A}$, and e a projection in $\mathscr{A}$ with $\phi(e) < +\infty$, call e *ϕ-finite*. Then ϕ is finite on the cone $(e\mathscr{A}e)^+ = e\mathscr{A}^+e$, and so extends uniquely to a positive linear functional on $e\mathscr{A}e$. Since it will not cause confusion, this functional will also be called ϕ.

3. Statement of main results. We state our results in the order of their decreasing intelligibility.

MAIN LEMMA. *Let ϕ be a linear functional on $\mathscr{A} = M_2(\mathbf{C})$, the *-algebra of all 2×2 complex matrices, satisfying $|\phi(x)| \leqq \phi(|x|)$ for all x of rank 1 in $\mathscr{A}$. Then ϕ is a non-negative scalar multiple of the trace.*

THEOREM 1. *The finite traces on a C^*-algebra $\mathscr{A}$ are precisely those (positive) linear functionals ϕ on $\mathscr{A}$ which satisfy*

$$|\phi(x)| \leqq \phi(|x|) \text{ for all } x \text{ in } \mathscr{A}.$$

THEOREM 2. *Let $\mathscr{A}$ be a W^*-algebra, and ϕ a normal, strongly semifinite weight on $\mathscr{A}$ satisfying the condition*

(WT) *For every ϕ-finite projection e in $\mathscr{A}$,*

$|\phi(x)| \leqq \phi(|x|)$ *for all x in $e\mathscr{A}e$.*

Then ϕ is a trace on $\mathscr{A}$.

4. Proofs. We prove first the Main Lemma, then Theorem 2, by way of some preliminaries and, finally, Theorem 1.

Proof of the Main Lemma. Well known, implicit in each of the Main Lemma and Theorem 2, and explicit in Theorem 1 is that traces satisfy the triangle inequality. As a gesture towards completeness, we prove this here for a finite trace ϕ on a W^*-algebra $\mathscr{A}$; the other cases can be inferred from this. In fact, for ϕ finite, the triangle inequality is equivalent to

$$\operatorname{Re} \phi(x) \leqq \phi(|x|),\ x \in \mathscr{A}.$$

Let $x = v|x|$ be the polar decomposition of x in $\mathscr{A}$. Then $x^* = |x|v^*$, and since ϕ is positive, so self-adjoint

$$\operatorname{Re} \phi(x) = \phi(\operatorname{Re} x) = \phi\left(\frac{|x|v^* + v|x|}{2}\right)$$
$$= \phi\left(\frac{v^* + v}{2}|x|\right) = \phi(|x|^{\frac{1}{2}}(\operatorname{Re} v)|x|^{\frac{1}{2}}) \leqq \phi(|x|),$$

since $\|v\| \leqq 1$ implies $\|\operatorname{Re} v\| \leqq 1$, so $\operatorname{Re} v \leqq I$, and $a \to |x|^{1/2}a|x|^{1/2}$ is a positive map.

Now suppose $0 \neq \phi$ is a (necessarily positive) linear functional on $\mathscr{A} = M_2(\mathbf{C})$ satisfying

(‡) $|\phi(x)| \leqq \phi(|x|)$ for all $x \in \mathscr{A}$ of rank 1.

Then if u is unitary in A and if we define

$$\phi_u(x) = \phi(uxu^*)\ (x \in \mathscr{A}),$$

it is easy to check that ϕ_u also satisfies (‡), and of course if ϕ_u is a trace, so is ϕ. Replacing ϕ by an appropriate ϕ_u we may therefore suppose that $\phi(x) = \operatorname{tr}(ax)$ where a is a diagonal positive matrix with $a_{22} \neq 0$; and then, replacing ϕ by $a_{22}^{-1}\phi$, that

$$\phi(x) = \tau x_{11} + x_{22} \text{ for } \tau = a_{11}a_{22}^{-1} \geqq 0.$$

If we show on this assumption that $\tau \geqq 1$, then in particular $\tau > 0$, so by symmetry $\tau \leqq 1$, so $\tau = 1$ and $\phi = \operatorname{tr}$, as desired. But consider the rank 1 matrices

$$x = \begin{bmatrix} p & 1 \\ p^2 & p \end{bmatrix},\quad p > 1.$$

For such x,

$$|x| = \begin{bmatrix} p^2 & p \\ p & 1 \end{bmatrix},$$

so by (‡),

$$\tau p + p \leqq \tau p^2 + 1 \text{ for } p > 1.$$

Then

$$\tau(p - p^2) \leqq 1 - p, \text{ or}$$
$$\tau \geqq 1/p \text{ for all } p > 1.$$

Thus $\tau \geqq 1$, and the proof is complete.

A Geometric Corollary. Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$, and $[\ ,\ ]$ a second inner product on $\mathscr{H}$ with the property that for every pair of unit vectors x, y in $(H, \langle\ ,\ \rangle)$, $|[x, y]| \leqq [y, y]$. Then $[\ ,\ ]$ is proportional to $\langle\ ,\ \rangle$.

Proof. It is routine to reduce the problem to the case

$$H = \mathbf{C}^2,\ \langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2.$$

If we use the canonical identifications of $\mathscr{H} \otimes \overline{\mathscr{H}}$ with $M_2(\mathbf{C}) \simeq L\ (\mathscr{H})$, whereby $x \otimes \bar{y}(z) = \langle z, y\rangle x\ (x, y, z \in H)$, then $\langle x, y\rangle = \operatorname{tr}\,(x \otimes \bar{y})$. If we define

$$\phi(x \otimes \bar{y}) = [x, y],$$

ϕ extends to a linear functional on $M_2(\mathbf{C})$, and since if $y \neq 0$,

$$|x \otimes \bar{y}| = [(y \otimes \bar{x})(x \otimes \bar{y})]^{1/2} = y \otimes \bar{y}\ \|x\|/\|y\|,$$

the hypothesis reads

$$|\phi(x \otimes \bar{y})| \leqq \phi(|x \otimes \bar{y}|)$$

for all rank 1 elements $x \otimes \bar{y}$ of $M_2(\mathbf{C})$. Thus by the Main Lemma, $\phi = k \operatorname{tr}$ for some $k > 0$, and $[\ ,\ ] = k\ \langle\ ,\ \rangle$, as desired.

Alternate Proof of the Corollary. Instead of forcing the proof in series with the Main Lemma, we could give a simpler, parallel proof thus: As before, reduce to the case of a two dimensional Hilbert space, for which choose an orthonormal basis diagonalizing the positive, non-singular operator a defined by $[x, y] = \langle ax, y\rangle$. We may assume a has eigenvalues τ and 1, so that

$$[x, y] = \tau x_1\bar{y}_1 + x_2\bar{y}_2.$$

Then the choice $x = (1, p)$ and $y = (p, 1)$, $(p > 1)$ yields in the limit $\tau \geqq 1$, while $x = (p, 1)$, $y = (1, p)$ yields $\tau \leqq 1$.

LEMMA 1. *If $\mathscr{A}$ and ϕ are as in the hypothesis of Theorem 2 and e and f are equivalent, mutually orthogonal projections in $\mathscr{A}$, then $\phi(e) = \phi(f)$.*

Proof. Let v be a partial isometry in $\mathscr{A}$ such that $v^*v = e$, $vv^* = f$. Suppose $\phi(e) < +\infty$. By the strong semifiniteness of ϕ and spectral theory there exists a projection f', $0 \neq f' \leqq f$, with $\phi(f') < +\infty$. Put $e' = v^*f'v \leqq e$. Then since $\phi(e' + f') < +\infty$, our hypothesis together with the Main Lemma applied to the copy of $M_2(\mathbf{C})$ generated in $(e' + f')\mathscr{A}(e' + f')$ by $f'v$ yields $\phi(e') = \phi(f')$. It follows that for every pair

$$e = v^*v \perp vv^* = f,$$

there exist non-zero sub-projections $e' \leqq e, f' \leqq f$ such that

$$f' = ve'v^* \text{ and } \phi(e') = \phi(f').$$

Let $(e_\delta)_{\delta \in D}$ be a maximal pairwise orthogonal family of non-zero sub-projections of e such that

$$\phi(e_\delta) = \phi(ve_\delta v^*) \text{ for each } \delta \in D,$$

and put

$$e_1 = \sum e_\delta, f_1 = \sum ve_\delta v^* = ve_1v^*.$$

Then $e_1 = e$ and $f_1 = f$ by maximality, and since ϕ is normal,

$$\phi(e) = \sum \phi(e_\delta) = \sum \phi(ve_\delta v^*) = \phi(f).$$

This proves the Lemma.

COROLLARY 1. *Suppose $\mathscr{A}$, ϕ as in the hypothesis of Theorem 2. Then there exists a unique central projection p of $\mathscr{A}$ such that ϕ vanishes on $p\mathscr{A}^+$ and is faithful on $(I - p)\mathscr{A}^+$.*

Proof. Let $(e_\delta)_{\delta \in D}$ be a maximal, pairwise orthogonal family of projections in $\mathscr{A}$ such that $\phi(e_\delta) = 0$ for each $\delta \in D$, and put $p = \sum e_\delta$. Then by normalcy, $\phi(p) = 0$, so ϕ vanishes on $p\mathscr{A}^+$, while if $0 \leqq a \in (I - p)\mathscr{A}^+$, there exists by spectral theory a projection $0 \neq e \leqq ta < t\|a\|(I - p)$ for some real $t > 0$. Then $0 \neq e \leqq (I - p)$, and $\phi(e) > 0$ by maximality of $(e_\delta)_{\delta \in D}$, and so $\phi(ta) = t\phi(a) > 0$: ϕ is faithful on $(I - p)\mathscr{A}$. Finally to the existence proof, if p were not central, there would be a non-zero sub-projection e of p equivalent in $\mathscr{A}$ to some sub-projection e' of $(I - p)$, so that by Lemma 1 and the above, we would have

$$0 = \phi(e) = \phi(e') > 0.$$

So p is central. Its uniqueness follows from the fact that any p with the properties claimed is easily seen to be the largest projection in $\mathscr{A}$ on which ϕ vanishes.

Corollary 1 permits us to assume, in proving Theorem 2, that ϕ is faithful. The usefulness of this property is revealed by the next corollary.

COROLLARY 2. *Suppose that $\mathscr{A}$, ϕ satisfy the hypothesis of Theorem 2, and that ϕ is faithful. Then every ϕ-finite projection in $\mathscr{A}$ is finite.*

Proof. If e is infinite, $e\mathscr{A}e$ contains a sequence $(e_j)_{j\in N}$ of equivalent, mutually orthogonal non-zero projections. Since ϕ is faithful, $\phi(e_j)$ is positive, independent of j by Lemma 1, and

$$\phi(e) \geqq \sum \phi(e_j) = +\infty,$$

so e cannot be ϕ-finite.

LEMMA 2. *If e and f are equivalent projections in $\mathscr{A}$, $\phi(e) = \phi(f)$.*

Proof. We may suppose ϕ faithful (Corollary 1), and e ϕ-finite. Then e, f and so $e \vee f$ are finite. Now

$$e_1 = e \vee f - e \sim f - e \wedge f \sim e - e \wedge f = e_2,$$

and since $e_1 \perp e_2$, Lemma 1 yields $\phi(e_1) = \phi(e_2)$, or

$$(*) \quad \phi(e \vee f - e) = \phi(e - e \wedge f) \leqq \phi(e) < +\infty.$$

Then $\phi(e \vee f) = \phi(e \vee f - e) + \phi(e) < +\infty$, so $\phi(f)$ is finite, too. This symmetrizes the roles of e, f, while expanding the equation in (*) above and solving for $\phi(e)$, we have

$$\phi(e) = \tfrac{1}{2}[\phi(e \vee f) + \phi(e \wedge f)].$$

By symmetry, $\phi(f)$ is given by the same expression and $\phi(e) = \phi(f)$, as claimed.

Proof of Theorem 2. Lemma 2 tells us that the restriction of ϕ to the convex cone $\mathscr{C}$ of elements of the form

$$\sum_{j=1}^{n} \alpha_j e_j, \ \alpha_j \text{ non-negative scalars and } e_j \text{ projections in } \mathscr{A},$$

is unitarily invariant. Spectral theory tells us that every a in $\mathscr{A}^+$ is the supremum of a family of such operators. The normalcy of ϕ then tells us that the unitary invariance of ϕ on $\mathscr{C}$ persists to all of $\mathscr{A}^+$: ϕ is a trace on $\mathscr{A}$.

Proof of Theorem 1. Let ϕ be a positive linear functional on the C^*-algebra $\mathscr{A}$, satisfying $|\phi(x)| \leqq \phi(|x|)$ for all $x \in \mathscr{A}$. Then the second transpose map ϕ^{dd} is *-strongly continuous on $\mathscr{A}^{\mathrm{dd}}$, and since $x \rightarrow |x|$ is *-strongly continuous on bounded subsets of A^{dd} [**8**], while the unit ball in $\mathscr{A}$ is *-strongly dense in that of $\mathscr{A}^{\mathrm{dd}}$ [**8**], [**5**], the inequality persists:

$$|\phi^{\mathrm{dd}}(x)| \leqq \phi^{\mathrm{dd}}(|x|) \text{ for all } x \text{ in } \mathscr{A}^{\mathrm{dd}}.$$

ϕ^{dd} is also normal and finite, so Theorem 2 applies: ϕ^{dd} is a trace on $\mathscr{A}^{\mathrm{dd}}$, so its restriction ϕ is a trace on $\mathscr{A}$.

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University of Toronto,
Toronto, Ontario

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