MAXIMIZING THE EXPECTED DURATION OF OWNING A RELATIVELY BEST OBJECT IN A POISSON PROCESS WITH RANKABLE OBSERVATIONS

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Abstract

Suppose that an unknown number of objects arrive sequentially according to a Poisson process with random intensity λ on some fixed time interval $[0, T]$. We assume a gamma prior density $G_\lambda(r, 1/a)$ for λ. Furthermore, we suppose that all arriving objects can be ranked uniquely among all preceding arrivals. Exactly one object can be selected. Our aim is to find a stopping time (selection time) which maximizes the time during which the selected object will stay relatively best. Our main result is the following. It is optimal to select the ith object that is relatively best and arrives at some time $s_i^{(r)}$ onwards. The value of $s_i^{(r)}$ can be obtained for each r and i as the unique root of a deterministic equation.

Keywords: Optimal stopping problem; secretary problem; Poisson arrival; duration problem

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1. Introduction

Our problem is as follows. We must select one object from an unknown number of rankable objects by some fixed finite time T. The objects are observed sequentially and appear according to a Poisson process with unknown intensity λ. Let X_i be the relative rank of the ith object, and set $X_i = 1, 2, \ldots$ when the ith object is the relatively best, second best, third best, etc., respectively. Each ordering of the objects is supposed to be equally likely. Then the distribution of X_i is $P(X_i = j) = 1/i$ for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, i$. We assume that the intensity λ of the Poisson arrival process $\{N(t)\}_{t \geq 0}$ has a gamma prior density $G_\lambda(r, 1/a)$, where $r \in \mathbb{N}$ and $a > 0$. We want to find a stopping time that maximizes the time during which the selected object will stay relatively best.

Our secretary problem is a combination of the duration problem with the Poisson arrival setting. For further information about the secretary problem, see [6]. The duration of owning a relatively best object is studied by Ferguson *et al.* [8]. They found optimal stopping times for many interesting versions of the duration problem in the 'no-information' and 'full-information' cases, for the problems treated in [10] and [11] with a random number of objects , and for the Poisson arrival models studied in [2] and [5]. Here we say no information if we can observe only the relative rank of the objects and if each ordering of the objects occurs equally likely. We

speak of full information if we can observe the value of each object, assumed to be independent and identically distributed from a known distribution. The optimal stopping time for Ferguson *et al.*'s problem with exponential prior intensity $G_\lambda(1, 1/a)$, $a > 0$, is given by

$$\tau_1^* = \min\{t \in [s^{(1)}, T]: X_{N(t)} = 1\},$$

where $s^{(1)} = [(T + a)/\mathrm{e}^2 - a]^+$ and $[x]^+ = \max\{x, 0\}$. It is then optimal to select the relatively best object which appears from time $s^{(1)}$ onwards. The authors have also shown that, for a close to 0 (large arrival rate), the optimal expected duration time is close to $2\mathrm{e}^{-2}T$.

Cowan and Zabczyk [5] assumed that the intensity of the process was known. The aim was to maximize the probability of selecting the best object overall. Bruss [2] succeeded in extending the problem to unknown intensities having exponential prior density $G_\lambda(1, 1/a)$, $a > 0$, and in finding the optimal stopping time. It is interesting to see that it has the same form as the optimal stopping time of Ferguson *et al.*'s duration problem (1) with the different value of $s^{(1)} = [(T + a)/\mathrm{e} - a]^+$. Ano [1] extended this problem to the problem of the unknown intensity having a gamma prior intensity $G_\lambda(r, 1/a)$, $r \in \mathbb{N}$, and found the optimal stopping time for $r = 2$. Kurushima and Ano [9] found the optimal stopping time for the cases in which $r = 3, 4, \ldots$. Furthermore, the game version of Bruss' problem has been studied in [13].

As we have seen above, our problem can be considered as an extension of both [2] and [8]. The main result of this paper is the proof that the optimal stopping time τ_r^* is a threshold stopping time:

$$\tau_r^* = \min\{t \in [s_i^{(r)}, T]: X_i = 1 \text{ for each } i = 1, 2, 3, \ldots\},$$

where $s_i^{(r)}$ is the unique root of a deterministic equation for each r and i. It is optimal to select the ith object to arrive provided it is relatively best and appears from time $s_i^{(r)}$ onwards. For each r, the threshold sequence $\{s_i^{(r)}\}_{i\geq 1}$ is nonincreasing. Furthermore, we see that the asymptotic value $s_i^{(r)}$ converges to $[(T + a)/\mathrm{e}^2 - a]^+$ as $i \to \infty$, which means that $N(T) \to \infty$ almost surely. Roughly speaking, when a large number of objects arrive in $[0, T]$, it is almost optimal to select the first relatively best object arriving after $[(T + a)/\mathrm{e}^2 - a]^+$.

The paper is organized as follows. In Section 2 we formulate the reward process. Our formulation is similar to that of [1]. We study the cases in which $r = 1, 2$, and 3. In Section 3 we prove the main result.

2. Formulation

2.1. Stopping reward

Let $S_1, S_2, \ldots$ denote the arrival times of the Poisson process $\{N(t)\}_{t\geq 0}$. We assume that it has an unknown intensity λ with gamma prior density

$$g(\lambda) = \frac{a^r \mathrm{e}^{-a\lambda} \lambda^{r-1}}{\Gamma(r)} \mathbf{1}(\lambda \geq 0), \tag{1}$$

where $r \in \mathbb{N}$ and $a > 0$. By Bayes' theorem, the posterior density of λ given $S_1 = s_1, \ldots, S_i = s$ can be computed and also turns out to be gamma, $G_\lambda(r + 1, 1/(a + s))$. Bruss [2] showed that after observing $S_1 = s_1, \ldots, S_i = s$, $0 < s < T$, the posterior distribution of $N(T)$ depends only on the values of i and S_i, and is a Pascal distribution with parameters $(r + 1, (s + a)/(T + a))$. Indeed, it was later shown that this is a special case of a Pascal process (see [3]).

Let (i, s) denote the state of the process when the ith object arrives at time s and is relatively best. Note that we can confine our interest to such states of the decision process. Recall that

X_i is the relative rank of the ith object. Let $p_{(i,s)}^{(k,u)}$ denote the transition probability from state (i, s) to $(i + k, s + u)$, which, as Ano [1] (or Kurushima and Ano [9]) showed, is given by

$$
\begin{aligned}
p_{(i,s)}^{(k,u)} &= \int_0^\infty P(S_{i+k} = s + u \mid S_i = s, \lambda)\, P(X_{i+k} = 1 \mid X_i = 1,\ S_i = s,\ S_{i+k} = s + u, \lambda) \\
&\qquad \times g(\lambda \mid S_i = s)\, d\lambda \\
&= \frac{\Gamma(i + k + r)}{\Gamma(k)\Gamma(i + r)(i + k)(i + k - 1)} \frac{i}{(i + k - 1)(i + k)} \frac{s + a}{(s + a + u)^2} \\
&\qquad \times \left(\frac{s + a}{s + a + u}\right)^{i + r - 1} \left(\frac{u}{s + a + u}\right)^{k - 1}.
\end{aligned}
\tag{2}
$$

The stopping reward is the expected duration time possessing the relatively best object when we stop at state (i, s). Denote the expected duration time when we stop at state (i, s) by $y_i^{(r)}(s)$. Using Bruss' result and (2), we have

$$
\begin{aligned}
y_i^{(r)}(s) &= \mathrm{E}(u \mid (i, s)) + (T - s)\, P(\text{no relatively best appears in } (s, T] \mid (i, s)) \\
&= \int_0^\infty u \sum_{k \geq 1} p_{(i,s)}^{(k,u)}\, du + (T - s) \sum_{n \geq i} \left(\frac{i}{n}\right) P(N(T) = n \mid S_i = s) \\
&= \int_0^{T-s} \sum_{k \geq 1} \frac{(i + r + k - 1)!}{(k - 1)!\, (i + r - 1)!} \frac{i}{(i + k)(i + k - 1)} x^{i+r}(1 - x)^k\, du \\
&\quad + (T - s) \sum_{n \leq i} \left(\frac{i}{n}\right) \frac{(n + r - 1)!}{(r + i - 1)!\, (n - i - 1)!} \theta^{r+i}(1 - \theta)^{n-i},
\end{aligned}
$$

where $x = (s + a)/(s + a + u)$ and $\theta = (s + a)/(T + a)$.

Using (2) and the identities

$$
\frac{(n + r - 1)!}{n} = (r - 1)! \sum_{i=0}^{r-1} \frac{(n + i - 1)!}{i!}, \qquad r = 1, 2, \ldots,
\tag{3}
$$

$$
1 = \sum_{k \geq 1} \frac{(n + k - 1)!}{(k - 1)!\, n!} x^{n+1}(1 - x)^{k-1}, \qquad 0 \leq x \leq 1,
\tag{4}
$$

we can calculate the expected duration time $y_i^{(r)}(s)$ as follows:

$$
\begin{aligned}
y_i^{(r)}(s) &= \binom{i + r - 1}{r - 1}^{-1} \int_0^{T-s} \sum_{k \geq 1} \sum_{j=0}^{r-1} \frac{(i + k + j - 1)!\, x^{i+r}(1 - x)^k}{(i - 1)!\, (k - 1)!\, j!\, (i + k - 1)}\, du \\
&\quad + \frac{i(r - 1)!\, (T - s)}{(i + r - 1)!} \sum_{j=0}^{r-1} \sum_{k \geq 1} \frac{(n + j - 1)!}{(n - i)!} \theta^{r+i}(1 - \theta)^{n-i} \\
&= \binom{i + r - 1}{r - 1}^{-1} \int_0^{T-s} \sum_{j=0}^{r-1} \sum_{l=0}^{j} \binom{i + l - 1}{l} \sum_{k \geq 1} \binom{i + k + l - 2}{k - 1} x^{r+i}(1 - x)^{n-i}\, du \\
&\quad + (T - s) \binom{i + r - 1}{r - 1}^{-1} \sum_{j=0}^{r-1} \binom{i + j - 1}{j} \theta^{r-j}.
\end{aligned}
\tag{5}
$$

A straightforward calculation using identity (4) and setting $x = (s+a)/(s+a+u)$ shows that the first terms on the right-hand side of (5) become

$$(s+a)\binom{i+r-1}{r-1}^{-1}\int_{(s+a)/(T+a)}^{1}\sum_{j=0}^{r-1}\sum_{l=0}^{j}\binom{i+l-1}{l}x^{r-l}(1-x)\,dx$$

$$= (s+a)\binom{i+r-1}{r-1}^{-1}\int_{\theta}^{1}\left\{\binom{i+r-2}{r-1}x^{-1}+\sum_{l=0}^{r-1}\binom{i+l-1}{l}x^{r-l-2}\right\}dx$$

$$+ (s+a)\binom{i+r-1}{r-1}^{-1}\int_{\theta}^{1}\left\{\sum_{j=0}^{r-2}\sum_{l=0}^{j}\binom{i+l-1}{l}x^{r-l-2}\right.$$

$$\left. -\sum_{j=0}^{r-1}\sum_{l=0}^{j}\binom{i+l-1}{l}x^{r-l-1}\right\}dx$$

$$= (s+a)\binom{i+r-1}{r-1}^{-1}\left\{\binom{i+r-2}{r-1}(-\ln\theta)+\sum_{l=0}^{r-2}\binom{i+l-1}{l}\frac{1-\theta^{r-l-1}}{r-l-1}\right\}$$

$$+ (s+a)\binom{i+r-1}{r-1}^{-1}\left\{\sum_{l=0}^{r-2}\binom{i+l-1}{l}(1-\theta^{r-l-1})\right.$$

$$\left. +\sum_{l=0}^{r-1}\binom{i+l-1}{l}(1-\theta^{r-l})\right\}. \tag{6}$$

Substituting (6) into (5), completes the proof of the following lemma.

Lemma 1. *Let $y_i^{(r)}(s)$ be the stopping reward in state (i,s), that is, the expected duration time when we stop at state (i,s). Then,*

$$y_i^{(r)}(s) = (s+a)\binom{i+r-1}{r-1}^{-1}\left\{\binom{i+r-2}{r-1}(-\ln\theta)+\sum_{j=0}^{r-2}\binom{i+j-1}{j}\frac{1-\theta^{r-j-1}}{r-j-1}\right\},$$

where $\theta = (s+a)/(T+a)$.

2.2. Monotone property of the stopping problem

First we shortly summarize the monotone stopping problem based on Chow *et al.* [4, pp. 54–55]. Let $\{Y_i\}_{i\geq 1}$ be the stopping reward process. If the process is supermartingale then the stopping problem is said to be monotone. Let *the one-stage look-ahead (OLA) stopping time* be denoted by $\tau = \min\{i \geq 1: Y_i \geq E(Y_{i+1} \mid \mathcal{F}_i)\}$. Let *the OLA stopping region* be denoted by $B = \{i: Y_i \geq E(Y_{i+1} \mid \mathcal{F}_i)\}$. If B is 'closed', that is, $P(j \in B \mid i \in B) = 1$ for $j = i+1, i+2, \ldots$, then the problem is monotone and τ is optimal. Throughout the paper, we use the term 'closed' in this sense. This monotone property also holds for the reward process related to a stationary Markov chain (see Chapter 5 of [4], [7], or Chapter 6 of [12]).

We shall obtain the OLA stopping region for our problem and prove that it is closed. For each $r = 1, 2, \ldots$, let

$$H_i^{(r)}(s) = \binom{i+r-1}{r-1}(s+a)^{-1}\left\{y_i^{(r)}(s) - \int_0^\infty \sum_{k\geq 1} p_{(i,s)}^{(k,u)} y_{i+k}^{(r)}(s+u)\,du\right\}. \tag{7}$$

Then we have the OLA stopping time τ_r and the OLA stopping region B_r,

$$\tau_r = \min\{i \geq 1 \colon H_i^{(r)}(s) \geq 0\} \quad \text{and} \quad B_r = \{(i, s) \colon H_i^{(r)}(s) \geq 0\}.$$

Now we calculate $H_i^{(r)}(s)$. From Lemma 1 we have

$$\int_0^\infty \sum_{k \geq 1} p_{(i,s)}^{(k,u)} y_{i+k}^{(r)}(s + u) \, du$$

$$= \int_0^\infty \sum_{k \geq 1} \frac{(i + r + k - 2)!}{(k - 1)! \, (i + r - 1)!} \frac{i x^{i+r} (1 - x)^{k-1} (-\ln \tilde{\theta})}{i + k - 1} \, du$$

$$+ \int_0^\infty \sum_{k \geq 1} \frac{i (r - 1)! \, x^{i+r} (1 - x)^{k-1}}{(k - 1)! \, (i + r - 1)!} \sum_{j=0}^{r-2} \frac{(i + k + j - 1)! \, (1 - \tilde{\theta}^{r-j-1})}{j! \, (i + k - 1)(r - j - 1)} \, du,$$

where $\tilde{\theta} = (s + a + u)/(T + a)$. Using identities (3) and (4) again and calculating in a similar way to $y_i^{(r)}(s)$, we have

$$\int_0^\infty \sum_{k \geq 1} p_{(i,s)}^{(k,u)} y_{i+k}^{(r)}(s + u) \, du$$

$$= -\binom{i + r - 2}{r - 1} \frac{1}{2} (\ln \theta)^2 - \sum_{j=0}^{r-2} \binom{i + j - 1}{j} \frac{1 - \theta^{r-j-1}}{r - j - 1} \ln \theta$$

$$+ \sum_{j=1}^{r-2} \sum_{l=0}^{j-1} \binom{i + l - 1}{l} \frac{(j - l) + (r - j - 1)\theta^{r-l-1} - (r - l - 1)\theta^{r-j-1}}{(j - l)(r - j - 1)(r - l - 1)}.$$

Substituting the above equation into (7), we obtain

$$H_i^{(r)}(s) \equiv h_i^{(r)}(\theta)$$

$$= -\binom{i + r - 2}{r - 1} \ln \theta \left(1 + \frac{1}{2} \ln \theta\right) + \sum_{j=0}^{r-2} \binom{i + j - 1}{j} \frac{1 - \theta^{r-j-1}}{r - j - 1} (1 + \ln \theta)$$

$$- \sum_{j=1}^{r-2} \sum_{l=0}^{j-1} \binom{i + l - 1}{l} \frac{(j - l) + (r - j - 1)\theta^{r-l-1} - (r - l - 1)\theta^{r-j-1}}{(j - l)(r - j - 1)(r - l - 1)}, \quad (8)$$

where $\theta = (s + a)/(T + a) \in [0, 1]$. Thus, for each r, τ_r and B_r can be rewritten as

$$\tau_r = \min\{i \geq 1 \colon h_i^{(r)}(\theta) \geq 0\} \quad \text{and} \quad B_r = \{(i, \theta) \colon h_i^{(r)}(\theta) \geq 0\}.$$

2.3. The cases in which $r = 1, 2,$ and 3

From (8), the functions $h_i^{(r)}$ for $r = 1, 2,$ and 3 are respectively given by

$$h_i^{(1)}(\theta) = -\ln \theta \left(1 + \frac{\ln \theta}{2}\right),$$

$$h_i^{(2)}(\theta) = -i \ln \theta \left(1 + \frac{\ln \theta}{2}\right) + (1 - \theta)(1 + \ln \theta),$$

$$h_i^{(3)}(\theta) = -\frac{i(i + 1)}{2} \ln \theta \left(1 + \frac{\ln \theta}{2}\right) + \frac{1 - \theta^2}{2} (1 + \ln \theta) + i(1 - \theta)(1 + \ln \theta) - \frac{(1 - \theta)^2}{2}.$$

Case 1: $r = 1$. (See [8].) This is the same problem as that given in Section 4.1 of [8]. The function $h_i^{(1)}(\theta)$ does not depend on i and so we may write it as $h^{(1)}(\theta)$. We can see that $h^{(1)}(\theta)$ is a unimodal function in $\theta \in (0, 1]$ with $h^{(1)}(0+) = -\infty$ and $h^{(1)}(1) = 0$, since $h^{(1)'}(\theta)$ is increasing in $\theta \in (0, e^{-1}]$, $h^{(1)'}(\theta)$ is decreasing in $\theta \in [e^{-1}, 1]$, and $h^{(1)''}(\theta) \leq 0$ for $\theta \in (0, 1]$. Therefore, B_1 can be written as

$$B_1 = \{\theta : h^{(1)}(\theta) \geq 0\} = \{\theta : \theta \geq e^{-2}\} = \left\{ s : s \geq \left[\frac{T + a}{e^2} - a \right]^+ \right\}.$$

It is closed because once $h^{(1)}(\theta)$ becomes nonnegative then it stays nonnegative in $(e^{-2}, 1]$. Thus, the optimal stopping time is given by (1).

Case 2: $r = 2$. We first study the function $h_i^{(2)}(\theta)$ for each $i = 1, 2, \ldots$.

Lemma 2. *For $r = 2$, the following statements hold.*

(i) *There exists a unique root $\theta_i^{(2)} \in (e^{-2}, e^{-1})$ of the equation $h_i^{(2)}(\theta) = 0$ for each $i = 1, 2, \ldots$. Also, $h_i^{(2)}(\theta) \geq 0$ implies that $h_i^{(2)}(\theta + \eta) \geq 0$ for $0 \leq \eta \leq 1 - \theta$.*

(ii) *For each $i = 1, 2, \ldots$, $h_i^{(2)}(\theta) \geq 0$ implies that $h_{i+1}^{(2)}(\theta) \geq 0$.*

Proof. (i) For $\theta \in (0, 1]$, we have $h_i^{(2)''}(\theta) \leq 0$, implying that $h_i^{(2)}(\theta)$ is a unimodal function with $h_i^{(2)}(0+) = \infty$ and $h_i^{(2)}(1) = 0$. We also have $h_i^{(2)}(e^{-2}) < 0$ and $h_i^{(2)}(e^{-1}) > 0$. Therefore, (i) holds.

(ii) From (i), $h_i^{(2)}(\theta)$ changes sign once from negative to nonnegative in $[e^{-2}, 1]$. On the other hand, for $\theta \in [e^{-2}, 1]$,

$$h_{i+1}^{(2)}(\theta) - h_i^{(2)}(\theta) = -\ln \theta \left(1 + \frac{\ln \theta}{2} \right) \geq 0.$$

Thus, if $h_i^{(2)}(\theta) \geq 0$ then $h_{i+1}^{(2)}(\theta) \geq h_i^{(2)}(\theta) \geq 0$.

Theorem 1. *If $r = 2$, the optimal stopping time τ_2^* is given by $\tau_2^* = \min\{t \in [s_i^{(2)}, T] : X_i = 1, i = 1, 2, \ldots\}$, where $s_i^{(2)} \in (0, T]$ is the unique root of the equation $H_i^{(2)}(s) = 0$ for each $i = 1, 2, \ldots$.*

Proof. Lemma 2 implies the following.

(i) There exists a unique root $s_i^{(2)} \in (0, T]$ of the equation $H_i^{(2)}(\theta) = 0$, and $H_i^{(2)}(s) \geq 0$ implies that $H_i^{(2)}(s + u) \geq 0$ for $0 \leq u \leq 1 - s$.

(ii) For each $i = 1, 2, \ldots$, $H_i^{(2)}(s) \geq 0$ implies that $H_{i+1}^{(2)}(s) \geq 0$, or, equivalently, $H_i^{(2)}(s) \geq 0$ implies that $H_{i+k}^{(2)}(s) \geq 0$ for $k = 1, 2, \ldots$.

Therefore, B_2 is given by $B_2 = \{(i, \theta) : h_i^{(2)}(\theta) \geq 0\} = \{(i, s) : H_i^{(2)}(s) \geq 0\}$ and it is closed. So the first hitting time τ_2^* for B_2 is the optimal stopping time given by $\tau_2^* = \min\{s \geq s_i^{(2)} : (i, s) \in B_2\} = \min\{t \in [s_i^{(2)}, T] : X_i = 1, i = 1, 2, \ldots\}$.

Case 3: $r = 3$.

Lemma 3. *For $r = 3$, the following statements hold.*

(i) *There exists a unique root $\theta_i^{(3)} \in (e^{-2}, e^{-1})$ of the equation $h_i^{(3)}(\theta) = 0$. Moreover, $h_i^{(3)}(\theta) \geq 0$ implies that $h_i^{(3)}(\theta + \eta) \geq 0$ for $0 \leq \eta \leq 1 - \theta$.*

(ii) *For each $i = 1, 2, \ldots$, $h_i^{(3)}(\theta) \geq 0$ implies that $h_{i+1}^{(3)}(\theta) \geq 0$.*

Proof. (i) We have, for $\theta \in (0, e^{-1}]$,

$$\theta h_i^{(3)'}(\theta) = -\left(\frac{i(i+1)}{2} + i\theta + \theta^2\right)(1 + \ln\theta) + \frac{(1-\theta^2)}{2} + (i+\theta)(1-\theta) \geq 0,$$

so that $h_i^{(3)}(\theta)$ is nondecreasing in $\theta \in (0, e^{-1}]$. For $\theta \in [e^{-1}, 1]$, we have

$$\theta^2 h_i^{(3)''}(\theta) = -\left(\theta^2 - \frac{i(i+1)}{2}\right)\ln\theta - \frac{5}{2}\theta^2 - i(1+\theta) - \frac{3}{2} < 0,$$

so that $h_i^{(3)}(\theta)$ is concave in $\theta \in (e^{-1}, 1]$. Since $h_i^{(3)}(0+) = -\infty$, $h_i^{(3)}(1) = 0$, $h_i^{(3)}(e^{-2}) < 0$, and $h_i^{(3)}(e^{-1}) > 0$, (i) holds.

(ii) By virtue of (i), if $h_{i+1}^{(3)}(\theta_i^{(3)}) > 0$ then (ii) holds. There exists a unique root $\theta_i^{(3)}$ in (e^{-2}, e^{-1}), and so $1 + \ln\theta_i^{(3)} < 0$. From $h_i^{(3)}(\theta_i^{(3)}) = 0$, the difference $h_{i+1}^{(3)}(\theta_i^{(3)}) - h_i^{(3)}(\theta_i^{(3)})$ is equal to $h_{i+1}^{(3)}(\theta_i^{(3)})$, which is given by

$$h_{i+1}^{(3)}(\theta_i^{(3)}) = -(i+1)\ln\theta_i^{(3)}\left(1 + \frac{\ln\theta_i^{(3)}}{2}\right) + (1-\theta_i^{(3)})(1 + \ln\theta_i^{(3)}).$$

On the other hand, $h_i^{(3)}(\theta_i^{(3)}) = 0$ is equivalent to

$$-(i+1)\ln\theta_i^{(3)}\left(1 + \frac{\ln\theta_i^{(3)}}{2}\right) = -\frac{1-\theta_i^{(3)2}}{i}(1 + \ln\theta_i^{(3)}) - 2(1-\theta_i^{(3)})(1 + \ln\theta_i^{(3)})$$

$$+ \frac{(1-\theta_i^{(3)})^2}{i}.$$

Substituting the above equation into $h_{i+1}^{(3)}(\theta_i^{(3)})$, we have

$$h_{i+1}^{(3)}(\theta_i^{(3)}) = -\frac{1-\theta_i^{(3)2}}{i}(1 + \ln\theta_i^{(3)}) - (1-\theta_i^{(3)})(1 + \ln\theta_i^{(3)}) + \frac{(1-\theta_i^{(3)})^2}{i} \geq 0,$$

where the last inequality follows from $1 + \ln\theta_i^{(3)} < 0$. This completes the proof.

Theorem 2. *For $r = 3$, the optimal stopping time τ_3^* is given by $\tau_3^* = \min\{t \in [s_i^{(3)}, T]\colon X_i = 1, i = 1, 2, \ldots\}$, where $s_i^{(3)} \in (0, T]$ is the unique root of the equation $H_i^{(3)}(s) = 0$.*

Proof. The proof is similar to that of Theorem 2 and is therefore omitted.

3. Main results

First we study the function $h_i^{(r)}(\theta)$ for each $i = 1, 2, \ldots$.

Proposition 1. *For each $r = 1, 2, \ldots$ and $i = 1, 2, \ldots$,*

(i) $h_i^{(r)}(\theta)$ *is nondecreasing in $\theta \in (0, e^{-1}]$,*

(ii) $h_i^{(r)}(0+) = -\infty$, $h_i^{(r)}(1) = 0$, *and* $h_i^{(r)}(e^{-1}) > 0$,

(iii) *there exists a $\hat{\theta}$ such that $h_i^{(r)}(\theta)$ is nondecreasing in $\theta \in [e^{-1}, \hat{\theta}]$ and nonincreasing in $\theta \in [\hat{\theta}, 1]$.*

Proof. It is hard to show these statements directly. We look at the behavior of $\theta h_i^{(r)'}(\theta)$ instead of $h_i^{(r)'}(\theta)$.

(i) We have

$$\theta h_i^{(r)'}(\theta) = -\binom{i+r-2}{r-1}(1+\ln\theta) - \sum_{j=0}^{r-2}\binom{i+j-1}{j}\theta^{r-j-1}(1+\ln\theta)$$
$$+ \sum_{j=0}^{r-2}\binom{i+j-1}{j}\frac{1-\theta^{r-j-1}}{r-j-1} - \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{\theta^{r-l-1}-\theta^{r-j-1}}{j-l}. \quad (9)$$

As is easily shown, $(1+\ln\theta) \le 0$ for $\theta \in (0, e^{-1}]$ and $\theta^{r-l-1} \le \theta^{r-j-1}$ for $\theta \in [0, 1]$, from which it follows that $h_i^{(r)'} \ge 0$ for $\theta \in (0, e^{-1}]$.

(ii) It is easy to show that the first two statements hold. From (9) we have

$$h_i^{(r)}(e^{-1}) = \frac{1}{2}\binom{i+r-1}{r-2} - f_1^{(r)}(e^{-1}),$$

where

$$f_1^{(r)}(\theta) = \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{(j-l)+(r-j-1)\theta^{r-l-1}-(r-l-1)\theta^{r-j-1}}{(j-l)(r-j-1)(r-l-1)}.$$

The function $f_1^{(r)}(\theta)$ is a decreasing, nonnegative function on $[0, 1]$, since $f_1^{(r)}(0) > 0$, $f_1^{(r)}(1) = 0$, and $f_1^{(r)'}(\theta) < 0$. Therefore, if we could show that

$$\frac{1}{2}\binom{i+r-1}{r-2} - f_1^{(r)}(0) > 0$$

then $h_i^{(r)}(e^{-1}) > 0$. We shall show this now.

Using identity (4) repeatedly, we have

$$\frac{1}{2}\binom{i+r-1}{r-2} = \frac{1}{2}\sum_{j=0}^{r-2}\binom{i+j}{j} = \frac{1}{2}\sum_{j=0}^{r-2}\sum_{l=0}^{j}\binom{i+l-1}{l}.$$

Then, for each $r = 3, 4, \ldots,$

$$\frac{1}{2}\binom{i+r-1}{r-2} - f_1^{(r)}(0)$$
$$= \binom{i-1}{0}\left[\frac{1}{2} - \frac{1}{(r-2)(r-1)}\right]$$
$$+ \binom{i-1}{0}\left[\frac{1}{2} - \frac{1}{(r-3)(r-1)}\right] + \binom{i-2}{1}\left[\frac{1}{2} - \frac{1}{(r-3)(r-2)}\right] + \cdots$$
$$+ \binom{i-1}{0}\left[\frac{1}{2} - \frac{1}{1(r-1)}\right] + \cdots + \binom{i+r-4}{r-3}\left[\frac{1}{2} - \frac{1}{2}\right]$$
$$+ \frac{1}{2}\left\{\binom{i-1}{0} + \binom{i-2}{1} + \cdots + \binom{i+r-3}{r-2}\right\}$$
$$> 0.$$

The last inequality follows since the differences in all the square brackets are nonnegative for each $r = 3, 4, \ldots$. For $r = 1, 2$, it is easy to show that $h^{(1)}(e^{-1}) > 0$ and $h_i^{(2)}(e^{-1}) > 0$, and, thus, the proof of (ii) is complete.

(iii) We want to prove that the function $h_i^{(r)'}(\theta)$ changes sign once from negative to nonnegative in $(0, 1)$. To show this, let $\theta h_i^{(r)'}(\theta) = -f_2^{(r)}(\theta) + f_3^{(r)}(\theta)$, where $f_2^{(r)}(\theta)$ and $f_3^{(r)}(\theta)$ are given by

$$f_2^{(r)}(\theta) = \binom{i+r-2}{r-1}(1+\ln\theta) - \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{\theta^{r-j-1}}{j-l},$$

$$f_3^{(r)}(\theta) = -\sum_{j=0}^{r-2}\binom{i+j-1}{j}\left\{\theta^{r-j-1}(1+\ln\theta) - \frac{1-\theta^{r-j-1}}{r-j-1}\right\}$$
$$- \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{\theta^{r-l-1}}{j-l}.$$

If

(a) $f_3^{(r)}(e^{-1}) > f_2^{(r)}(e^{-1})$,

(b) $f_2^{(r)}(1) > f_3^{(r)}(1)$,

(c) $f_2^{(r)}(\theta)$ is a concave function for $\theta \in [e^{-1}, 1]$, and

(d) $f_3^{(r)}(\theta)$ is a decreasing and concave function for $\theta \in [e^{-1}, 1]$

hold, then there exists a unique root $\hat{\theta}$ in $[e^{-1}, 1]$ of the equation $f_2^{(r)}(\theta) = f_3^{(r)}(\theta)$, or, equivalently, $h_i^{(r)'}(\theta) = 0$. This shows that the function $h_i^{(r)'}(\theta)$ changes sign once from negative to nonnegative in $(0, 1)$. For an example of $r = 3$, see Figure 1. We now prove conditions (a)–(d). Since $e^{-(r-l-1)} < e^{-(r-j-1)}$, we have

$$f_3^{(r)}(e^{-1}) - f_2^{(r)}(e^{-1}) = \sum_{j=0}^{r-2}\frac{1-e^{-(r-j-1)}}{r-j-1} - \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\frac{e^{-(r-l-1)}-e^{-(r-j-1)}}{j-l} > 0.$$
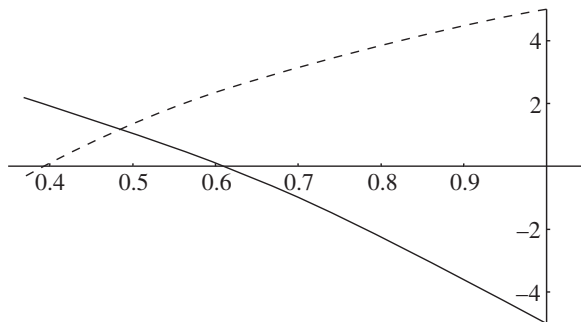

FIGURE 1: Graphs of $f_2^{(3)}(\theta)$ (*dashed line*) and $f_3^{(3)}(\theta)$ (*solid line*) in $[e^{-1}, 1]$.
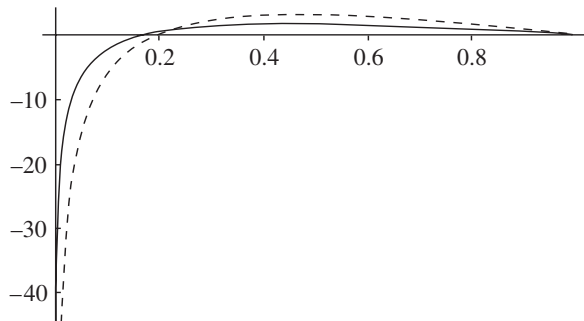
FIGURE 2: Graphs of $h_2^{(3)}(\theta)$ (*dashed line*) and $h_3^{(3)}(\theta)$ (*solid line*) in $(0, 1]$.

Thus, (a) holds. Since

$$-f_2^{(r)}(1) + f_3^{(r)}(1) = h_i^{(r)'}(1) = -\binom{i+r-2}{r-1} - \sum_{j=0}^{r-2}\binom{i+j-1}{j} < 0,$$

(b) holds. For $\theta \in [e^{-1}, 1]$, a straightforward calculation yields $f_2^{(r)''}(\theta) < 0$, $f_3^{(r)'}(\theta) < 0$, and $f_3^{(r)''}(\theta) < 0$. Thus, (c) and (d) hold, and the proof is complete.

Now we have the following lemma.

Lemma 4. *For each $r = 1, 2, \ldots$ and $i = 1, 2, \ldots$,*

(i) *there exists a unique root $\theta_i^{(r)}$ on $(0, e^{-1})$ of the equation $h_i^{(r)}(\theta) = 0$,*

(ii) *$h_i^{(r)}(\theta) \geq 0$ implies that $h_i^{(r)}(\theta + \eta) \geq 0$, $0 \leq \eta \leq 1 - \theta$,*

(iii) *$h_i^{(r)}(\theta) \geq 0$ implies that $h_{i+1}^{(r)}(\theta) \geq 0$.*

Proof. Parts (i) and (ii) follow immediately from Proposition 1. Part (ii) states that once the function $h_i^{(r)}(\theta)$ becomes nonnegative, it remains nonnegative (for an example of $r = 3$, see Figure 2). From (i) and (ii), for the proof of (iii), it suffices to show that $h_{i+1}^{(r)}(\theta_i^{(r)})$ is nonnegative.

First, we have $h_i^{(r)}(\theta_i^{(r)}) = 0$ implies that

$$\binom{i+r-2}{r-1}\ln\theta_i^{(r)}\left(1 + \frac{\ln^2\theta_i^{(r)}}{2}\right)$$
$$= \sum_{j=0}^{r-2}\binom{i+j-1}{j}\frac{(1-\theta_i^{(r)r-j-1})(1+\ln\theta_i^{(r)})}{r-j-1}$$
$$\quad - \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{f_4^{(r)}(\theta_i^{(r)})}{(j-l)(r-j-1)(r-l-1)}, \tag{10}$$

where we set $f_4^{(r)}(\theta) = (j - l) + (r - j - 1)\theta^{r-l-1} - (r - l - 1)\theta^{r-j-1}$. Now note that

$$i(h_{i+1}^{(r)}(\theta) - h_i^{(r)}(\theta)) = -(r-1)\binom{i+r-2}{r-1}\ln\theta\left(1 + \frac{\ln\theta}{2}\right)$$
$$+ \sum_{j=0}^{r-2} j\binom{i+j-1}{j}\frac{1-\theta^{r-j-1}}{r-j-1}(1+\ln\theta)$$
$$- \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{lf_4^{(r)}(\theta)}{(j-l)(r-j-1)(r-l-1)}.$$

Since $h_i^{(r)}(\theta_i^{(r)}) = 0$, we have

$$ih_{i+1}^{(r)}(\theta_i^{(r)}) = -(r-1)\binom{i+r-2}{r-1}\ln\theta_i^{(r)}\left(1 + \frac{\ln\theta_i^{(r)}}{2}\right)$$
$$+ \sum_{j=0}^{r-2} j\binom{i+j-1}{j}\frac{1-\theta_i^{(r)r-j-1}}{r-j-1}(1+\ln\theta_i^{(r)})$$
$$- \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{lf_4^{(r)}(\theta_i^{(r)})}{(j-l)(r-j-1)(r-l-1)}. \qquad (11)$$

Substituting (10) into (11) we obtain

$$ih_{i+1}^{(r)}(\theta_i^{(r)}) = -\sum_{j=0}^{r-2}\binom{i+j-1}{j}(1-\theta_i^{(r)r-j-1})(1+\ln\theta_i^{(r)})$$
$$+ \sum_{j=1}^{r-2}\sum_{l=0}^{j-1}\binom{i+l-1}{l}\frac{(r-l-1)f_4^{(r)}(\theta_i^{(r)})}{(j-l)(r-j-1)(r-l-1)}. \qquad (12)$$

From Proposition 1 we already know that $\theta_i^{(r)} \le e^{-1}$ and, hence, $1 + \ln\theta_i^{(r)} < 0$. Recall that $f_4^{(r)}(\theta) \ge 0$ for all $\theta \in [0, 1]$. Then the right-hand side of (12) is positive, which shows that $h_{i+1}^{(r)}(\theta_i^{(r)}) > 0$. This completes the proof.

We now state and prove the main result.

Theorem 3. *For a Poisson process with gamma prior intensity $G_\lambda(r, 1/a)$, $a > 0$ and $r = 1, 2, \ldots$, the optimal stopping time τ_r^* maximizing the expected duration time owning a relatively best object is given by $\tau_r^* = \min\{t \in [s_i^{(r)}, T]: X_i = 1, i = 1, 2, \ldots\}$, where $s_i^{(r)}$ is the unique root of the equation $H_i^{(r)}(s) = 0$ in $[0, T]$ for each r and i, and $H_i^{(r)}(s)$ is given by (8).*

Proof. We have $B_r = \{(i, s): H_i^{(r)}(s) \ge 0\} = \{(i, \theta): h_i^{(r)}(\theta) \ge 0\}$. Lemma 4 directly shows that B_r is closed and the problem becomes monotone. Therefore, B_r is the optimal stopping region and the OLA stopping time τ_r^* is optimal.

Now we shall see the characteristic of the threshold values precisely.

Proposition 2. *For each $r = 1, 2, \ldots$ and $i = 1, 2, \ldots$, there exists a unique root $\theta_i^{(r)} \in (e^{-2}, e^{-1})$ of the equation $h_i^{(r)}(\theta) = 0$.*

Proof. We have

$$h_i^{(r)}(e^{-2}) = -\sum_{j=0}^{r-2} \binom{i+j-1}{j} \frac{1 - e^{-2(r-j-1)}}{(r-j-1)}$$

$$- \sum_{j=1}^{r-2} \sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{f_4^{(r)}(e^{-2})}{(j-l)(r-j-1)(r-l-1)},$$

where we set $f_4^{(r)}(\theta) = (j-l) + (r-j-1)\theta^{r-l-1} - (r-l-1)\theta^{r-j-1}$. Since $f_4^{(r)}(0) > 0$, $f_4^{(r)}(1) = 0$, and $f_4^{(r)'}(\theta) \leq 0$, we have $f_4^{(r)}(\theta) > 0$ for $\theta \in [0, 1)$. Therefore, $f_4^{(r)}(e^{-2}) > 0$, which yields $h_i^{(r)}(e^{-2}) < 0$.

Theorem 4. *The threshold sequence has the following properties.*

(i) *For each $r = 1, 2, \ldots$, the threshold sequence $\{s_i^{(r)}\}_{i \geq 1}$ is nonincreasing.*

(ii) *The limiting value of the threshold $s_i^{(r)}$ for all $r = 1, 2, \ldots$ is given by $\lim_{i \to \infty} s_i^{(r)} = [(T+a)/e^2 - a]^+$.*

Proof. (i) The proof follows immediately from Lemma 4.

(ii) Since $s_i^{(r)} = \theta_i^{(r)}(T+a) - a$, it suffices to prove that $\lim_{i \to \infty} \theta_i^{(r)} = e^{-2}$. When $h_i^{(r)}(\theta) = 0$, we have

$$-\ln\theta\left(1 + \frac{\ln\theta}{2}\right) = \binom{i+r-2}{r-1}^{-1} \sum_{j=0}^{r-2} \binom{i+j-1}{j} \frac{(1-\theta^{r-j-1})(1+\ln\theta)}{r-j-1}$$

$$+ \binom{i+r-2}{r-1}^{-1} \sum_{j=1}^{r-2} \sum_{l=0}^{j-1} \binom{i+l-1}{l} \frac{f_4^{(r)}(\theta)}{(j-l)(r-j-1)(r-l-1)}. \tag{13}$$

It is easy to check that, as i tends to ∞,

$$\binom{i+r-2}{r-1}^{-1} \sum_{j=1}^{r-2} \binom{i+j-1}{j} = \frac{(r-1)!}{(i+r-2)\cdots i} + \cdots + \frac{r-1}{i+r-2} \to 0$$

and

$$\binom{i+r-2}{r-1}^{-1} \sum_{j=1}^{r-2} \sum_{l=0}^{j-1} \binom{i+l-1}{l} = \left\{ \frac{(r-1)!}{(i+r-2)\cdots i} + \cdots + \frac{(r-1)!}{(i+r-2)\cdots i} \right\}$$

$$+ \cdots + \left\{ \frac{(r-1)!}{(i+r-2)\cdots i} + \cdots + \frac{(r-1)(r-2)}{(i+1)!} \right\}$$

$$\to 0.$$

Therefore, the right-hand side of (13) goes to 0 and $-\ln\theta(1 + (\ln\theta/2)) \to 0$ as $i \to \infty$. Hence, $\lim_{i \to \infty} \theta_i^{(r)} = e^{-2}$, which completes the proof.

Acknowledgements

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