2

Differential geometry

The language of general relativity is differential geometry. The present chapter provides a brief review of the ideas and notions of differential geometry that will be used in this book. It also serves the purpose of setting the notation and conventions. The chapter assumes a prior knowledge of the subject at the level, say, of the first chapter of Choquet-Bruhat (2008) or Stewart (1991), or chapters 2 and 3 of Wald (1984). In view of the applications in later parts of this book, some topics which may not be regarded as belonging to the standard baggage of a relativist are discussed in some detail – for example, general (i.e. non-Levi-Civita) connections, the so-called $1+3$ split of tensors – that is, a split based on a congruence of timelike curves, rather than on a foliation, as in the usual $3+1$ – and the analysis of the geometry of submanifolds using a frame formalism.

2.1 Manifolds

The basic objects of study in differential geometry are differentiable manifolds. Intuitively, a manifold is a space that, locally, looks like $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Despite this simplicity at a small scale, the global structure of a manifold can be much more complicated and leads to considerations of differential topology.

2.1.1 On the definition of a manifold

A differentiable function f between open sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^n$, $f : \mathcal{U} \to \mathcal{V}$, is called a ***diffeomorphism*** if it is bijective and if its inverse $f^{-1} : \mathcal{V} \to \mathcal{U}$ is differentiable. If f and f^{-1} are C^k functions, then one has a ***C^k-diffeomorphism***. Furthermore, if f and f^{-1} are C^∞ functions, one speaks of a ***smooth diffeomorphism*** and one writes $\mathcal{U} \approx \mathcal{V}$. Throughout this book, the word ***smooth*** will be used as a synonym for C^∞. The words *function*, *map* and *mapping* will be used as synonyms of each other.

A ***topological space*** is a set with a well-defined notion of *open* and *closed* sets. Given some topological space $\mathcal{M}$, a ***chart*** on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$, with

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$\mathcal{U} \subset \mathcal{M}$ and φ a bijection from $\mathcal{U}$ to an open set $\varphi(\mathcal{U}) \subset \mathbb{R}^n$ such that given $p \in \mathcal{U}$

$$\varphi(p) \equiv (x^1, \ldots, x^n).$$

The entries $x^1, \ldots, x^n$ are called ***local coordinates*** of the point $p \in \mathcal{U}$. The set $\mathcal{U}$ is called the ***domain*** of the chart. Two charts $(\mathcal{U}_1, \varphi_1)$ and $(\mathcal{U}_2, \varphi_2)$ are said to be C^k***-related*** if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$$

and its inverse are C^k. The map $\varphi_1 \circ \varphi_2^{-1}$ defines changes of local coordinates $(x^\mu) = (x^1, \ldots, x^n) \mapsto (y^\mu) = (y^1, \ldots, y^n)$ in the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$; see Figure 2.1. Thus, one can regard the coordinates (y^μ) as functions of the coordinates (x^μ). *All throughout this book the Greek letters μ, $\nu, \ldots$ will be used to denote coordinate indices.* The functions $y^\mu(x^1, \ldots, x^n)$ are C^k and, moreover, the ***Jacobian*** $\det(\partial y^\mu / \partial x^\nu)$ is different from zero.

A C^k***-atlas*** on $\mathcal{M}$ is a collection of charts whose domains cover the set $\mathcal{M}$. The collection of all C^k-related charts is called a ***maximal atlas***. The pair consisting of the space $\mathcal{M}$ together with its maximal C^k-atlas is called a C^k***-differentiable manifold***. If the charts are C^∞-related, one speaks of a ***smooth differentiable manifold***. If for each φ in the atlas, the map $\varphi : \mathcal{U} \to \mathbb{R}^n$ has the same n, then the manifold is said to have ***dimension*** n. *In what follows, the discussion will be restricted to manifolds of dimension* 3 *and* 4.

Remark. In introductory discussions of differential geometry one generally considers smooth structures. However, as will be seen in later chapters, when one looks at general relativity from the perspective of conformal geometry, the

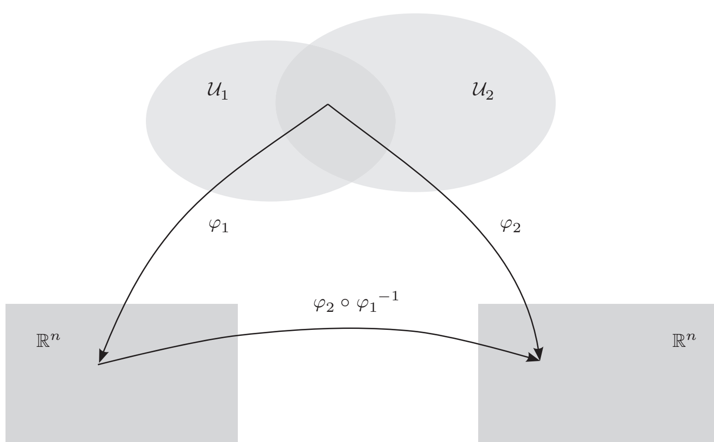

Figure 2.1 Schematic representation of the change of coordinates between charts – see the main text for further details. The figure is adapted from Stewart (1991).

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smoothness (or lack thereof) encodes important physical content. Accordingly, one is led to consider the more general class of C^k-differentiable manifolds.

The differentiable manifolds used in general relativity are generally assumed to be Hausdorff and paracompact. A differentiable manifold is ***Hausdorff*** if every two points in it admit non-intersecting open neighbourhoods. The reason for requiring the Hausdorff condition is to ensure that a convergent sequence of points cannot have more than one limit point. If $\mathcal{M}$ is ***paracompact***, then there exists a countable basis of open sets. Paracompactness is used in several basic constructions in differential geometry. In particular, it is required to show that every Riemannian manifold admits a metric. *In what follows, all differentiable manifolds to be considered will be assumed to be Hausdorff and paracompact.* Accordingly, in the rest of the book Hausdorff, paracompact differentiable manifolds will be simply called ***manifolds***.

Orientability

An open set of $\mathbb{R}^n$ is naturally oriented by the order of the coordinates $(x^\mu) = (x^1, \ldots, x^n)$. Hence, a chart $(\mathcal{U}, \varphi)$ inherits an orientation from its image in $\mathbb{R}^n$. In an orientable manifold the orientation of these charts matches together properly. More precisely, a manifold is said to be ***orientable*** if its maximal atlas is such that the Jacobian of the coordinate transformation for each pair of overlapping charts is positive.

An alternative description of the notion of orientability in terms of orthonormal frames will be given in Section 2.5.3. Orientability is a necessary and sufficient condition for the existence of a spinorial structure on $\mathcal{M}$; see, for example, Chapter 3.

2.1.2 Manifolds with boundary

Manifolds with boundary arise naturally when discussing general relativity from the perspective of conformal geometry. In order to introduce this concept one requires the following subsets of $\mathbb{R}^n$:

$$
\begin{aligned}
\mathbb{H}^n &\equiv \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}, \\
\partial\mathbb{H}^n &\equiv \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}.
\end{aligned}
$$

One says that $\mathcal{M}$ is a ***manifold with boundary*** if it can be covered with charts mapping open subsets of $\mathcal{M}$ either to open sets of $\mathbb{R}^n$ or to open subsets of $\mathbb{H}^n$. The ***boundary of*** $\mathcal{M}$, $\partial\mathcal{M}$, is the set of points $p \in \mathcal{M}$ for which there is a chart $(\mathcal{U}, \varphi)$ with $p \in \mathcal{U}$ such that $\varphi(\mathcal{U}) \subset \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$. The boundary $\partial\mathcal{M}$ is an $(n-1)$-dimensional differentiable manifold in its own right. Hence, it is a *submanifold* of $\mathcal{M}$ – see Section 2.7.1.

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2.2 Vectors and tensors on a manifold

In order to probe the geometric properties of a manifold one needs vectors and, more generally, tensors. This section provides a brief discussion of these fundamental notions.

2.2.1 Some ancillary notions

Derivations

Denote by $\mathfrak{X}(\mathcal{M})$ the set of ***scalar fields (i.e. functions) over*** $\mathcal{M}$; that is, smooth functions $f : \mathcal{M} \to \mathbb{R}$.

Definition 2.1 (***derivations***) *A* ***derivation*** *is a map* $\mathcal{D} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ *such that:*

(i) ***Action on constants.*** *For all constant fields* c, $\mathcal{D}(c) = 0$.
(ii) ***Linearity.*** *For all* f, $g \in \mathfrak{X}(\mathcal{M})$, $\mathcal{D}(f + g) = \mathcal{D}(f) + \mathcal{D}(g)$.
(iii) ***Leibnitz rule.*** *For all* f, $g \in \mathfrak{X}(\mathcal{M})$, $\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$.

The connection between derivations and covariant derivatives is discussed in Section 2.4.1.

Curves

The notion of a vector is intimately related to that of a curve. Given an open interval $I = (a, b) \subset \mathbb{R}$ where either or both of a, b can be infinite, a smooth ***curve*** on $\mathcal{M}$ is a map $\gamma : I \to \mathcal{M}$ such that for any chart $(\mathcal{U}, \varphi)$, the composition $\varphi \circ \gamma : I \to \mathbb{R}^n$ is a smooth map. One often speaks of the curve $\gamma(\mathrm{s})$ with $\mathrm{s} \in (a, b)$; s is called the ***parameter of the curve***. If the domain (a, b) of a curve can be extended to, say, $[a, b]$ while keeping $\gamma(\mathrm{s})$ smooth, one has an ***extendible curve***. A curve which is not extendible is called ***inextendible***.

A ***tangent vector to a curve*** $\gamma(\mathrm{s})$ at a point $p \in \mathcal{M}$, to be denoted as $\dot{\boldsymbol{\gamma}}(p)$, is the map defined by

$$\dot{\boldsymbol{\gamma}}(p) : f \mapsto \frac{\mathrm{d}}{\mathrm{ds}}(f \circ \gamma)\big|_p = \dot{\boldsymbol{\gamma}}(f)\big|_p, \qquad f \in \mathfrak{X}(\mathcal{M}).$$

Given a chart $(\mathcal{U}, \varphi)$ with local coordinates (x^μ), the components of $\dot{\boldsymbol{\gamma}}(p)$ with respect to the chart are given by

$$\dot{x}^\mu(p) \equiv \frac{\mathrm{d}}{\mathrm{ds}} x^\mu(\gamma(\mathrm{s}))\big|_p.$$

In a slight abuse of notation the points of the curve γ will often be denoted by $x(\mathrm{s}) \in \mathcal{M}$ and its tangent vector by $\dot{\boldsymbol{x}}(\mathrm{s})$.

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2.2.2 Tangent vectors and covectors

To each point $p \in \mathcal{M}$, one can associate a vector space $T|_p(\mathcal{M})$, the ***tangent space at*** p consisting of all the tangent vectors at p. In what follows, the elements of this space will be simply known as ***vectors***. All throughout, vectors will mostly be denoted with lowercase bold Latin letters: $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{w}, \ldots$ Abstract index notation will also be used to denote vectors; see Section 2.2.6. The tangent space $T|_p(\mathcal{M})$ can be characterised either as *the set of derivations at p of smooth functions on* $\mathcal{M}$ or as *the set of equivalence classes of curves through p under a suitable equivalence relation*. With the first characterisation one considers the vectors as *directional derivatives*, while with the second one they are considered as *velocities*. If the dimension of the manifold $\mathcal{M}$ is n, then $T|_p(\mathcal{M})$ is a vector space of dimension n. Local coordinates (x^μ) in a neighbourhood of the point p give a basis of $T|_p(\mathcal{M})$ consisting of the partial derivative operators $\{\boldsymbol{\partial}/\boldsymbol{\partial} x^\mu\}$; where no confusion arises about which coordinates are meant, one simply writes $\{\boldsymbol{\partial}_\mu\}$. In particular, for the vector tangent to a curve one has that $\dot{\boldsymbol{x}}(\mathrm{s}) = \dot{x}^\mu(\mathrm{s})\boldsymbol{\partial}_\mu$. In this last expression and in what follows, ***Einstein's summation convention*** has been adopted – that is, repeated up and down coordinate indices indicate summation for all values of the range of the index. That is,

$$\dot{x}^\mu(\mathrm{s})\boldsymbol{\partial}_\mu \equiv \sum_{\mu=1}^{n} \dot{x}^\mu(\mathrm{s})\boldsymbol{\partial}_\mu.$$

Covectors

The ***dual space*** $T^*|_p(\mathcal{M})$, the ***cotangent space at*** p, is the vector space of linear maps $\boldsymbol{\omega} : T|_p(\mathcal{M}) \to \mathbb{R}$. Generic elements of $T^*|_p(\mathcal{M})$ will be denoted by lowercase bold Greek letters: $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\omega}$, $\ldots$. Being dual to $T|_p(\mathcal{M})$, the space $T^*|_p(\mathcal{M})$ has also dimension n, and its elements are called ***covectors***. If $\boldsymbol{\omega}$ acts on $\boldsymbol{v} \in T|_p(\mathcal{M})$, then one writes $\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \in \mathbb{R}$.

Given $f \in \mathfrak{X}(\mathcal{M})$, for each $\boldsymbol{v} \in T|_p(\mathcal{M})$, one has that $\boldsymbol{v}(f)$ is a scalar. Hence, f defines a map, the ***differential*** of f, $\mathbf{d}f : T|_p(\mathcal{M}) \to \mathbb{R}$ via

$$\mathbf{d}f(\boldsymbol{v}) = \boldsymbol{v}(f).$$

As a consequence of the linearity of $\boldsymbol{v}$ one has that $\mathbf{d}f$ is linear, and thus $\mathbf{d}f \in T^*|_p(\mathcal{M})$. Given a chart $(\mathcal{U}, \varphi)$ with coordinates (x^μ), the coordinate differentials $\mathbf{d}x^\mu$ form a basis for $T^*|_p(\mathcal{M})$, the so-called ***dual basis***. The dual basis satisfies $\langle \mathbf{d}x^\mu, \boldsymbol{\partial}_\nu \rangle = \delta_\nu{}^\mu$, where $\delta_\nu{}^\mu$ is the so-called ***Kronecker's delta***. It follows that every covector $\boldsymbol{\omega}$ at $p \in \mathcal{M}$ can be written as $\boldsymbol{\omega} = \langle \boldsymbol{\omega}, \boldsymbol{\partial}_\mu \rangle \mathbf{d}x^\mu$.

Bases

The previous discussion is extended in a natural way to more general bases. Given any basis $\{\boldsymbol{e_a}\}$ of $T|_p(\mathcal{M})$, its ***dual basis*** $\{\boldsymbol{\omega^b}\}$ of $T^*|_p(\mathcal{M})$ is defined by the

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condition $\langle \boldsymbol{\omega}^{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{a}} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. In the rest of the book, lowercase bold indices such as $\boldsymbol{a}$, $\boldsymbol{b}, \ldots$ denote ***spacetime frame indices*** ranging $\mathbf{0}, \ldots, \mathbf{3}$. These will be used when working with four-dimensional manifolds. The lowercase bold Latin letters $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}, \ldots$ will range, depending on the context, over either $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$. *For simplicity of presentation, and unless explicitly stated, a four-dimensional manifold will be assumed in the subsequent discussion.*

Given another pair of bases $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ and $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ of $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$, respectively, these are related to the bases $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ and $\{\boldsymbol{\omega}^{\boldsymbol{b}}\}$ by non-singular matrices $(A_{\boldsymbol{a}}{}^{\boldsymbol{b}})$ and $(A^{\boldsymbol{a}}{}_{\boldsymbol{b}})$ such that

$$\tilde{\boldsymbol{e}}_{\boldsymbol{a}} = A_{\boldsymbol{a}}{}^{\boldsymbol{b}} \boldsymbol{e}_{\boldsymbol{b}}, \qquad \tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} = A^{\boldsymbol{a}}{}_{\boldsymbol{b}} \boldsymbol{\omega}^{\boldsymbol{b}}, \tag{2.1}$$

satisfying $A^{\boldsymbol{a}}{}_{\boldsymbol{b}} A^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \delta_{\boldsymbol{c}}{}^{\boldsymbol{a}}$ so that $(A_{\boldsymbol{a}}{}^{\boldsymbol{b}})$ and $(A^{\boldsymbol{a}}{}_{\boldsymbol{b}})$ are inverses of each other. In these last expressions and in what follows, ***Einstein's summation convention*** for repeated contravariant and covariant *frame* indices has been adopted so that a sum from $\boldsymbol{b} = \mathbf{0}$ to $\boldsymbol{b} = \mathbf{3}$ is implied.

Condition (2.1) ensures that the new bases $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ and $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ are dual to each other; that is, $\langle \tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}, \tilde{\boldsymbol{e}}_{\boldsymbol{a}} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. Given $\boldsymbol{v} \in T|_p(\mathcal{M})$, $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$, the above transformation rules for the bases imply

$$\begin{aligned} \boldsymbol{v} &= v^{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{a}} = \tilde{v}^{\boldsymbol{a}} \tilde{\boldsymbol{e}}_{\boldsymbol{a}} = (\tilde{v}^{\boldsymbol{a}} A_{\boldsymbol{a}}{}^{\boldsymbol{b}}) \boldsymbol{e}_{\boldsymbol{b}}, \\ \boldsymbol{\alpha} &= \alpha_{\boldsymbol{a}} \boldsymbol{\omega}^{\boldsymbol{a}} = \tilde{\alpha}_{\boldsymbol{a}} \tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} = (\tilde{\alpha}_{\boldsymbol{a}} A^{\boldsymbol{a}}{}_{\boldsymbol{b}}) \boldsymbol{\omega}^{\boldsymbol{b}}. \end{aligned}$$

The two bases are said to have the ***same orientation*** if $\det(A_{\boldsymbol{a}}{}^{\boldsymbol{b}}) > 0$.

2.2.3 Higher rank tensors

Higher rank tensors can be constructed using elements of $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$ as basic building blocks. A ***contravariant tensor of rank*** k at the point p is a multilinear map

$$\boldsymbol{M} : \underbrace{T^*|_p(\mathcal{M}) \times \cdots \times T^*|_p(\mathcal{M})}_{k \text{ terms}} \longrightarrow \mathbb{R},$$

that is, a function taking k covectors as arguments. Similarly, a ***covariant tensor of rank*** l at the point p is a multilinear map

$$\boldsymbol{N} : \underbrace{T|_p(\mathcal{M}) \times \cdots \times T|_p(\mathcal{M})}_{l \text{ terms}} \longrightarrow \mathbb{R},$$

that is, a function taking l vectors as arguments. More generally, one can also have ***tensors of mixed type***: a (k, l) tensor at p is a multilinear map

$$\boldsymbol{T} : \underbrace{T^*|_p(\mathcal{M}) \times \cdots \times T^*|_p(\mathcal{M})}_{k \text{ terms}} \times \underbrace{T|_p(\mathcal{M}) \times \cdots \times T|_p(\mathcal{M})}_{l \text{ terms}} \longrightarrow \mathbb{R},$$

so that $\boldsymbol{T}$ takes k covectors and l vectors as arguments. In particular, a $(k, 0)$-tensor corresponds to a contravariant tensor of rank k, while a $(0, l)$-tensor is

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a covariant tensor of rank l. The ***space of*** (k,l)***-tensors at the point*** p will be denoted by $T^k_l|_p(\mathcal{M})$. In particular, one has the identifications $T^1|_p(\mathcal{M}) = T|_p(\mathcal{M})$ and $T_1|_p(\mathcal{M}) = T^*|_p(\mathcal{M})$. Formally, the space $T^k_l|_p(\mathcal{M})$ is obtained as the *tensor product* of k copies of $T^*|_p(\mathcal{M})$ and l copies of $T|_p(\mathcal{M})$. That is, one has that

$$T^k_l|_p(\mathcal{M}) = \underbrace{T|_p(\mathcal{M}) \otimes \cdots \otimes T|_p(\mathcal{M})}_{k \text{ terms}} \otimes \underbrace{T^*|_p(\mathcal{M}) \otimes \cdots \otimes T^*|_p(\mathcal{M})}_{l \text{ terms}}.$$

The ordering given in the previous expression is known as the ***standard order***. Notice, however, that an arbitrary tensor does not need not to have its arguments in standard order.

As an example of the previous discussion consider $\boldsymbol{v} \in T|_p(\mathcal{M})$ and $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$. Their ***tensor product*** $\boldsymbol{v} \otimes \boldsymbol{\alpha}$ is then defined by

$$(\boldsymbol{v} \otimes \boldsymbol{\alpha})(\boldsymbol{u}, \boldsymbol{\beta}) = \langle \boldsymbol{\beta}, \boldsymbol{v} \rangle \langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle, \qquad \boldsymbol{u} \in T|_p(\mathcal{M}), \qquad \boldsymbol{\beta} \in T^*|_p(\mathcal{M}). \tag{2.2}$$

One readily sees that $\boldsymbol{v} \otimes \boldsymbol{\alpha}$ is a bilinear map and thus a $(1,1)$-tensor at $p \in \mathcal{M}$. The action of the tensor product given in Equation (2.2) can be extended directly to an arbitrary (finite) number of tensors and covectors. If $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ and $\{\boldsymbol{\omega}^{\boldsymbol{b}}\}$ denote, respectively, bases of $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$, then a basis of $T^k_l|_p(\mathcal{M})$ is given by

$$\{\boldsymbol{e}_{\boldsymbol{b}_1} \otimes \cdots \otimes \boldsymbol{e}_{\boldsymbol{b}_k} \otimes \boldsymbol{\omega}^{\boldsymbol{a}_1} \otimes \cdots \otimes \boldsymbol{\omega}^{\boldsymbol{a}_l}\}.$$

The collection of all the tensor spaces of the form $T^k_l|_p(\mathcal{M})$ is called the ***tensor algebra*** at p and will be denoted by $T^\bullet|_p(\mathcal{M})$. The tensor algebra is defined by means of a *direct sum*.

Symmetries of tensors

A covariant tensor of rank l, say, $\boldsymbol{S}$, is said to be ***symmetric*** with respect to its ith and jth arguments if

$$\boldsymbol{S}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_l) = \boldsymbol{S}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_l). \tag{2.3}$$

Similarly, $\boldsymbol{A}$ it is said to be ***antisymmetric*** if

$$\boldsymbol{A}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_l) = -\boldsymbol{A}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_l). \tag{2.4}$$

If the properties (2.3) and (2.4) hold under interchange of any arbitrary pair of indices, one says that $\boldsymbol{S}$ is ***totally symmetric*** and $\boldsymbol{A}$ is ***totally antisymmetric***, respectively. The above definitions can be extended to contravariant tensors of arbitrary rank. A totally antisymmetric covariant tensor of rank l is also called an l***-form***. Symmetry properties of tensors are best expressed in terms of abstract index notation.

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2.2.4 Tensor fields

The discussion in the previous subsections concerned the notion of a tensor at a point $p \in \mathcal{M}$. The ***tensor bundle over*** $\mathcal{M}$, $\mathfrak{T}^\bullet(\mathcal{M})$, is the *disjoint union* of the tensor algebras $T^\bullet|_p(\mathcal{M})$ for all $p \in \mathcal{M}$:

$$\mathfrak{T}^\bullet(\mathcal{M}) \equiv \coprod_{p\in\mathcal{M}} T^\bullet|_p(\mathcal{M}).$$

The disjoint union emphasises that although for p, $q \in \mathcal{M}$, $p \neq q$, the spaces $T^\bullet|_p(\mathcal{M})$ and $T^\bullet|_q(\mathcal{M})$ are *isomorphic*; they are regarded as different sets. Important subsets of the tensor bundles are the ***tangent bundle*** and the ***cotangent bundle*** given, respectively, by

$$T(\mathcal{M}) \equiv \coprod_{p\in\mathcal{M}} T|_p(\mathcal{M}), \qquad T^*(\mathcal{M}) \equiv \coprod_{p\in\mathcal{M}} T^*|_p(\mathcal{M}).$$

A ***smooth tensor field*** over $\mathcal{M}$ is a prescription of a tensor $\boldsymbol{T} \in T^\bullet|_p(\mathcal{M})$ at each $p \in \mathcal{M}$ such that when $\boldsymbol{T}$ is represented locally in a system of coordinates around p, the corresponding components are smooth functions on the local chart and, more generally, across the atlas. This idea can be naturally extended to consider tensor fields which are not smooth but just C^k for some positive integer k. An important property of tensor fields is that they are multilinear over $\mathfrak{X}(\mathcal{M})$. This property is often referred to as ***$\mathfrak{X}$-linearity***. It can be used to characterise tensors. More precisely, one has the following lemma which will be used repeatedly (see Penrose and Rindler (1984) for a proof):

Lemma 2.1 (***characterisation of tensors***) *A map*

$$\boldsymbol{T} : T^*(\mathcal{M}) \times \cdots \times T^*(\mathcal{M}) \times T(\mathcal{M}) \times \cdots \times T(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

is induced by a (k, l)-tensor field if and only if it is multilinear over $\mathfrak{X}(\mathcal{M})$.

The discussion of tensor fields and the tensor bundle is naturally carried out using the language of *fibre bundles*; see, for example, Kobayashi and Nomizu (2009). This point of view will, however, not be used in this book.

2.2.5 The commutator of vector fields

Given $\boldsymbol{u}, \boldsymbol{v} \in T(\mathcal{M})$, their ***commutator*** $[\boldsymbol{u}, \boldsymbol{v}] \in T(\mathcal{M})$ is the vector field defined by

$$[\boldsymbol{u}, \boldsymbol{v}]f \equiv \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f)),$$

for $f \in \mathfrak{X}(\mathcal{M})$. Given a basis $\{\boldsymbol{e_a}\}$ one has that the components of the commutator with respect to this basis are given by

$$[\boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{a}} = \boldsymbol{u}(v^{\boldsymbol{a}}) - \boldsymbol{v}(u^{\boldsymbol{a}}), \qquad u^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{a}}, \boldsymbol{u} \rangle, \quad v^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{a}}, \boldsymbol{v} \rangle.$$

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One can readily verify that

$$
\begin{aligned}
&[\boldsymbol{u}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}], \\
&[\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{u}, \boldsymbol{w}] + [\boldsymbol{v}, \boldsymbol{w}], \\
&[[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w}] + [[\boldsymbol{v}, \boldsymbol{w}], \boldsymbol{u}] + [[\boldsymbol{w}, \boldsymbol{u}], \boldsymbol{v}] = 0.
\end{aligned}
$$

The last identity is known as the ***Jacobi identity*** – not to be confused with the Jacobi identity for spinors, to be discussed in Chapter 3.

2.2.6 Abstract index notation for tensors

The presentation of tensors in this section has so far used an ***index-free notation***. In the sequel, the so-called ***abstract index notation*** will also be used where convenient; see Penrose and Rindler (1984). To this end, lowercase Latin indices will be employed. Accordingly, a vector field $\boldsymbol{v} \in T(\mathcal{M})$ will also be written as v^a. Similarly, for $\boldsymbol{\alpha} \in T^*(\mathcal{M})$ one writes α_a. More generally, a (k, l)-tensor $\boldsymbol{T}$ will be denoted by $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$. *It is important to stress that the indices in these expressions do not represent components with respect to some coordinates or frame.* These components are denoted, respectively, by Greek indices and bold lowercase Latin indices such as in v^μ and $v^{\boldsymbol{a}}$. The role of the abstract indices is to specify in a simple way the nature of the object under consideration and to describe in a convenient fashion operations between tensors. In particular, the action $\langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle$ of a 1-form on a vector is denoted in abstract index notation by $\alpha_a v^a$, while its tensor product $\boldsymbol{\alpha} \otimes \boldsymbol{v}$ is written as $\alpha_a v^b$. Similarly, the operation defined in Equation (2.2) is expressed as $\alpha_a u^a \beta_b v^b$.

The idea behind the use of abstract indices is to have a notation for tensorial expressions that mirrors the expressions for their basis components (had a basis been introduced). Using the index notation one can write only tensorial expressions since no basis has been specified; see, for example, Wald (1984) for a further discussion on this subject.

Each type of notation has its own advantages. In particular, the index-free notation is better to describe conceptual and structural aspects, while the abstract index notation is useful in explicit computations. In particular, the abstract index notation allows the expression, in a convenient way, of tensors whose arguments are not given in standard order as in $F_{ab}{}^c{}_d$.

An operation which has a particularly convenient description in terms of abstract indices is the ***contraction*** between a contravariant and a covariant index. For example, given $F_{ab}{}^c{}_d$, the contraction between the contravariant index ^{c} and, say, the covariant index ${}_d$ is denoted by $F_{ab}{}^c{}_c$. Following the convention that repeated indices are *dummy* indices one has, for example, that $F_{ab}{}^c{}_c = F_{ab}{}^d{}_d$. Given a basis $\{\boldsymbol{e_a}\}$ and a cobasis $\{\boldsymbol{\omega^a}\}$, their elements are denoted, using abstract index notation, as $e_{\boldsymbol{a}}{}^a$ and $\omega^{\boldsymbol{a}}{}_a$, respectively. If $F_{\boldsymbol{ab}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \equiv F_{ab}{}^c{}_d e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c e_{\boldsymbol{d}}{}^d$ denotes the components of $F_{ab}{}^c{}_d$ with respect to a basis $\{\boldsymbol{e_a}\}$ and its associated cobasis $\{\boldsymbol{\omega^a}\}$, then the components of the contraction $F_{ab}{}^c{}_c$

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are given by $F_{\boldsymbol{ab}}{}^{\boldsymbol{c}}{}_{\boldsymbol{c}}$. *Following Einstein's summation convention, a sum on the index* $\boldsymbol{c}$ *is understood.* Although this definition is given in terms of components with respect to a basis, the contraction is a *geometric* (i.e. coordinate- and base-independent) operation transforming a tensor of rank (k, l) into a tensor of rank $(k-1, l-1)$.

Symmetries of tensors are expressed in a convenient fashion using abstract index notation. For example, if S_{ab} and A_{ab} denote, respectively, symmetric and antisymmetric covariant tensors of rank 2, then $S_{ab} = S_{ba}$ and $A_{ab} = -A_{ba}$. More generally, given M_{ab}, its ***symmetric*** and ***antisymmetric parts*** are defined, respectively, by the expressions

$$M_{(ab)} \equiv \frac{1}{2}(M_{ab} + M_{ba}), \qquad M_{[ab]} \equiv \frac{1}{2}(M_{ab} - M_{ba}).$$

The operations of symmetrisation and antisymmetrisation can be extended to higher rank tensors. In particular, it is noticed that for a rank 3 covariant tensor T_{abc} one has

$$T_{[abc]} \equiv \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{cba} - T_{bac}).$$

If a tensor $S_{a_1 \cdots a_l}$ is symmetric with respect to the indices $a_1, \ldots, a_l$, then one writes $S_{a_1 \cdots a_l} = S_{(a_1 \cdots a_l)}$. Similarly, if $A_{a_1 \cdots a_l}$ is antisymmetric with respect to $a_1, \ldots, a_l$, one writes $A_{a_1 \cdots a_l} = A_{[a_1 \cdots a_l]}$ and $A_{a_1 \cdots a_l}$ is said to be an ***l-form***.

Consistent with the abstract index notation for tensors, it is convenient to introduce a similar convention to denote the various tensor spaces. Accordingly the bundle $\mathfrak{T}^k_l(\mathcal{M})$ will, in the following, be denoted by $\mathfrak{T}^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}(\mathcal{M})$. In particular, in this notation the tangent bundle $T(\mathcal{M})$ is denoted by $\mathfrak{T}^a(\mathcal{M})$, while the cotangent bundle $T^*(\mathcal{M})$ is given by $\mathfrak{T}_a(\mathcal{M})$.

A further discussion of the abstract index notation with specific remarks in the treatment of spinors is given in Section 3.1.4.

2.3 Maps between manifolds

This section discusses maps between manifolds. In what follows let $\mathcal{M}$ and $\mathcal{N}$ denote two manifolds. These manifolds could be the same.

2.3.1 Push-forwards and pull-backs

A map $\varphi : \mathcal{N} \to \mathcal{M}$ is said to be smooth (C^∞) if for every smooth function $f \in \mathfrak{X}(\mathcal{M})$, the composition $\varphi^* f \equiv f \circ \varphi : \mathcal{N} \to \mathbb{R}$ is also smooth. Given $p \in \mathcal{N}$, let $T|_p(\mathcal{N})$, $T|_{\varphi(p)}(\mathcal{M})$ denote, respectively, the tangent spaces at $p \in \mathcal{N}$ and $\varphi(p) \in \mathcal{M}$. The map $\varphi : \mathcal{N} \to \mathcal{M}$ induces a map $\varphi_* : T|_p(\mathcal{N}) \to T|_{\varphi(p)}(\mathcal{M})$, the ***push-forward***, through the formula

$$(\varphi_* \boldsymbol{v}) f(p) \equiv \boldsymbol{v}(f \circ \varphi)(p), \qquad \boldsymbol{v} \in T|_p(\mathcal{N}).$$

It can be readily verified that φ_* so defined is a $\mathfrak{X}$-linear map; that is, given $\boldsymbol{v},\ \boldsymbol{u} \in T|_p(\mathcal{N})$ and a function $f \in \mathfrak{X}(\mathcal{M})$ one has $\varphi_*(f\boldsymbol{v} + \boldsymbol{u}) = f\varphi_*\boldsymbol{v} + \varphi_*\boldsymbol{u}$.

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Note that the above definition is made in a point-wise manner. *Smooth vector fields do not, in general, push forward to smooth vector fields, except in the case of diffeomorphisms.* For example, if φ is not surjective, then there is no way of deciding which vector to assign to a point not on the image of φ. If φ is not injective, then for some points of $\mathcal{M}$, there may be several different vectors obtained as push-forwards of a vector on $\mathcal{N}$. However, given $\varphi : \mathcal{N} \to \mathcal{M}$ a diffeomorphism, for every $\boldsymbol{v} \in T(\mathcal{N})$ there exists a unique vector field on $T(\mathcal{M})$ obtained as the pull-back of $\boldsymbol{v}$; see Lee (2002).

The push-forward $\varphi_* : T(\mathcal{N}) \to T(\mathcal{M})$ can be used, in turn, to define a map $\varphi^* : T^*(\mathcal{M}) \to T^*(\mathcal{N})$, the ***pull-back***, as

$$\langle \varphi^* \boldsymbol{\omega}, \boldsymbol{v} \rangle \equiv \langle \boldsymbol{\omega}, \varphi_* \boldsymbol{v} \rangle, \qquad \boldsymbol{\omega} \in T^*(\mathcal{M}), \qquad \boldsymbol{v} \in T(\mathcal{N}).$$

Again, it can be readily verified that φ^* so defined is $\mathfrak{X}$-linear: $\varphi^*(f\boldsymbol{\omega} + \boldsymbol{\zeta}) = f^* \varphi^* \boldsymbol{\omega} + \varphi^* \boldsymbol{\zeta}$ for $\boldsymbol{\omega}$, $\boldsymbol{\zeta} \in T^*(\mathcal{M})$. The pull-back commutes with the differential $\mathbf{d}$; that is, $\varphi^*(\mathbf{d}f) = \mathbf{d}(\varphi^* f)$. *Contrary to the case of push-forwards, pull-backs of smooth covector fields always lead to smooth covector fields. There is no ambiguity in the construction.* In the case that $\varphi : \mathcal{N} \to \mathcal{M}$ is a diffeomorphism, then the inverse pull-back $(\varphi^*)^{-1}$ is well defined so that covectors can be pulled back from $T^*(\mathcal{N})$ to $T^*(\mathcal{M})$.

The operations of push-forward and pull-back can be extended in a natural way, respectively, to arbitrary contravariant and covariant tensors. The case of most relevance for the subsequent discussion is that of a covariant tensor of rank 2, $\boldsymbol{g} \in \mathfrak{T}_2(\mathcal{M})$. Its pull-back $\varphi^* \boldsymbol{g} \in \mathfrak{T}_2(\mathcal{N})$ satisfies

$$(\varphi^* \boldsymbol{g})(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\varphi_* \boldsymbol{u}, \varphi_* \boldsymbol{v}), \qquad \boldsymbol{u}, \boldsymbol{v} \in T(\mathcal{N}).$$

2.3.2 Lie derivatives

Smooth maps of the manifold into itself, $\varphi : \mathcal{M} \to \mathcal{M}$, lead to the notion of the Lie derivative. Given a vector $\boldsymbol{v}$, the ***Lie derivative*** $\pounds_{\boldsymbol{v}}$ measures the change of a tensor field along the integral curves of $\boldsymbol{v}$.

In what follows, let $f \in \mathfrak{X}(\mathcal{M})$ denote a smooth function and $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{M})$, $\boldsymbol{\alpha} \in T^*(\mathcal{M})$. The action of $\pounds_{\boldsymbol{v}}$ on functions and vectors is given by

$$\pounds_{\boldsymbol{v}} f \equiv \boldsymbol{v}(f), \qquad \pounds_{\boldsymbol{v}} \boldsymbol{u} \equiv [\boldsymbol{v}, \boldsymbol{u}].$$

The Lie derivative can be extended to act on covectors by requiring the *Leibnitz rule*

$$\pounds_{\boldsymbol{v}} \langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle = \langle \pounds_{\boldsymbol{v}} \boldsymbol{\alpha}, \boldsymbol{u} \rangle + \langle \boldsymbol{\alpha}, \pounds_{\boldsymbol{v}} \boldsymbol{u} \rangle.$$

A coordinate expression can be obtained from the latter. The action of $\pounds_{\boldsymbol{v}}$ can be extended to arbitrary tensor fields by means of the Leibnitz rule

$$\pounds_{\boldsymbol{v}} (\boldsymbol{S} \otimes \boldsymbol{T}) = \pounds_{\boldsymbol{v}} \boldsymbol{S} \otimes \boldsymbol{T} + \boldsymbol{S} \otimes \pounds_{\boldsymbol{v}} \boldsymbol{T}.$$

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The reader interested in the derivation of the above expressions and their precise relation to the notions of push-forward and pull-back of tensor fields is referred to, for example, Stewart (1991) where a list of coordinate expressions for the computation of the derivatives is also provided.

2.4 Connections, torsion and curvature

This section discusses the further structure required on a manifold to describe the geometric notion of curvature – a key ingredient of the equations of general relativity.

2.4.1 Covariant derivatives and connections

The notion of linear connection allows one to relate tensors at different points of the manifold $\mathcal{M}$.

Definition 2.2 (***linear connection***) *A* ***linear connection*** *(****connection*** *for short) is a map* $\boldsymbol{\nabla} : \mathfrak{T}^1(\mathcal{M}) \times \mathfrak{T}^1(\mathcal{M}) \to \mathfrak{T}^1(\mathcal{M})$ *sending the pair of vector fields* $(\boldsymbol{u}, \boldsymbol{v})$ *to a vector field* $\nabla_{\boldsymbol{v}}\boldsymbol{u}$ *satisfying:*

(i) $\nabla_{\boldsymbol{u}+\boldsymbol{v}}\boldsymbol{w} = \nabla_{\boldsymbol{u}}\boldsymbol{w} + \nabla_{\boldsymbol{v}}\boldsymbol{w}$
(ii) $\nabla_{\boldsymbol{u}}(\boldsymbol{v} + \boldsymbol{w}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \nabla_{\boldsymbol{u}}\boldsymbol{w}$
(iii) $\nabla_{f\boldsymbol{u}}\boldsymbol{v} = f\nabla_{\boldsymbol{u}}\boldsymbol{v}$
(iv) $\nabla_{\boldsymbol{u}}(f\boldsymbol{v}) = \boldsymbol{u}(f)\boldsymbol{v} + f\nabla_{\boldsymbol{u}}\boldsymbol{v}$

for $f \in \mathfrak{X}(\mathcal{M})$*. The vector* $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ *is called the* ***covariant derivative of*** $\boldsymbol{v}$ ***with respect to*** $\boldsymbol{u}$*.*

Any manifold admits a connection. In four dimensions this can be shown through the specification of 4^3 functions on the spacetime manifold $\mathcal{M}$; see, for example, Willmore (1993). The reason behind this result becomes more transparent once the so-called *connection coefficients* have been introduced; see Section 2.6.

As a consequence of the requirement (iv) $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ is not $\mathfrak{X}$-linear in $\boldsymbol{v}$; however, it is $\mathfrak{X}$-linear in $\boldsymbol{u}$. Thus, using Lemma 2.1 for a fixed second argument it defines a mixed $(1,1)$-tensor. Using abstract index notation the latter is denoted by $\nabla_a v^b$, so that $\nabla_a v^b \in \mathfrak{T}_a{}^b(\mathcal{M})$.

From the discussion in the previous paragraph it follows that one can regard the connection $\boldsymbol{\nabla}$ as a map $\nabla_a : \mathfrak{T}^b(\mathcal{M}) \to \mathfrak{T}_a{}^b(\mathcal{M})$. Moreover, a connection $\boldsymbol{\nabla}$ induces a map $\nabla_a : \mathfrak{T}_b(\mathcal{M}) \to \mathfrak{T}_{ab}(\mathcal{M})$ via

$$(\nabla_a \omega_b) v^b = \nabla_a(\omega_b v^b) - \omega_b(\nabla_a v^b).$$

This map is fixed if one requires the *Leibnitz rule* to hold between the product of a vector and a covector. To extend the covariant derivative to arbitrary tensors one uses again the Leibnitz rule. For example, from

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$$\begin{aligned}\nabla_e(\omega_a T^a{}_{bcd}u^b v^c w^d) = (\nabla_e\omega_a)T^a{}_{bcd}u^b v^c w^d + \omega_a(\nabla_e T^a{}_{bcd})u^b v^c w^d \\ + \omega_a T^a{}_{bcd}(\nabla_e u^b)v^c w^d + \omega_a T^a{}_{bcd}u^b(\nabla_e v^c)w^d \\ + \omega_a T^a{}_{bcd}u^b v^c(\nabla_e w^d),\end{aligned}$$

it follows that

$$\begin{aligned}(\nabla_e T^a{}_{bcd})\omega_a u^b v^c w^d = \nabla_e(\omega_a T^a{}_{bcd}u^b v^c w^d) - (\nabla_e\omega_a)T^a{}_{bcd}u^b v^c w^d \\ - \omega_a T^a{}_{bcd}(\nabla_e u^b)v^c w^d - \omega_a T^a{}_{bcd}u^b(\nabla_e v^c)w^d \\ - \omega_a T^a{}_{bcd}u^b v^c(\nabla_e w^d),\end{aligned}$$

so that one obtains a $\mathfrak{X}$-linear map $\mathfrak{T}^a{}_{bcd}(\mathcal{M}) \to \mathfrak{T}_e{}^a{}_{bcd}(\mathcal{M})$.

The subsequent discussion will make use of the ***commutator of covariant derivatives***. This is defined as

$$[\nabla_a, \nabla_b] \equiv 2\nabla_{[a}\nabla_{b]}.$$

One has that

$$\begin{aligned}[\nabla_a, \nabla_b](T_{\mathcal{A}} + S_{\mathcal{A}}) = [\nabla_a, \nabla_b]T_{\mathcal{A}} + [\nabla_a, \nabla_b]S_{\mathcal{A}}, \\ [\nabla_a, \nabla_b](T_{\mathcal{A}}R_{\mathcal{B}}) = ([\nabla_a, \nabla_b]T_{\mathcal{A}})R_{\mathcal{B}} + T_{\mathcal{A}}([\nabla_a, \nabla_b]R_{\mathcal{B}}),\end{aligned}$$

where ${}_{\mathcal{A}}$ and ${}_{\mathcal{B}}$ denote an arbitrary string of (covariant and contravariant) indices.

Covariant derivatives and derivations on a manifold are related in a natural way: given a derivation $\mathcal{D}$ and a connection $\boldsymbol{\nabla}$ on $\mathcal{M}$ there exists a unique $\boldsymbol{v} \in T(\mathcal{M})$ such that $\mathcal{D}f = v^a\nabla_a f$ for any $f \in \mathfrak{X}(\mathcal{M})$; see, for example, O'Neill (1983).

2.4.2 Torsion of a connection

The notion of ***torsion*** arises from the analysis of the action of the commutator of covariant derivatives on scalar fields. For convenience the abstract index notation is used. Consider $x^{ab} \in \mathfrak{T}^{ab}(\mathcal{M})$ and $f,\ g \in \mathfrak{X}(\mathcal{M})$. One readily has that

$$\begin{aligned}x^{ab}[\nabla_a, \nabla_b](f + g) = x^{ab}[\nabla_a, \nabla_b]f + x^{ab}[\nabla_a, \nabla_b]g, \\ x^{ab}[\nabla_a, \nabla_b](fg) = (x^{ab}[\nabla_a, \nabla_b]f)g + f(x^{ab}[\nabla_a, \nabla_b]g).\end{aligned}$$

It follows from the latter that the operator $x^{ab}[\nabla_a, \nabla_b]$ *must be a derivation*; see Definition 2.1. Thus, there exists $u^a \in \mathfrak{T}^a(\mathcal{M})$ such that

$$x^{ab}[\nabla_a, \nabla_b] = u^a\nabla_a. \tag{2.5}$$

The map $x^{ab} \mapsto u^a\nabla_a$ defined by Equation (2.5) is $\mathfrak{X}$-linear. It defines a tensor $\boldsymbol{\Sigma}$, the ***torsion tensor*** of the connection $\boldsymbol{\nabla}$, via $u^c = x^{ab}\Sigma_a{}^c{}_b$. Hence,

$$\nabla_a\nabla_b f - \nabla_b\nabla_a f = \Sigma_a{}^c{}_b\nabla_c f, \qquad f \in \mathfrak{X}(\mathcal{M}). \tag{2.6}$$

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One readily sees that

$$\Sigma_a{}^c{}_b = -\Sigma_b{}^c{}_a.$$

That is, the torsion is an antisymmetric tensor. If a connection $\boldsymbol{\nabla}$ is such that $\Sigma_a{}^c{}_b = 0$, then it is said to be ***torsion-free***.

Remark. Alternatively, one could have defined the torsion via the relation

$$\boldsymbol{\Sigma}(\boldsymbol{u}, \boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}], \qquad \boldsymbol{u},\, \boldsymbol{v} \in T(\mathcal{M}). \tag{2.7}$$

2.4.3 Curvature of a connection

In order to discuss the notion of curvature of a connection it is convenient to define the ***modified commutator*** of covariant derivatives

$$[\![\nabla_a, \nabla_b]\!] \equiv [\nabla_a, \nabla_b] - \Sigma_a{}^c{}_b\nabla_c.$$

Clearly, one has that $[\![\nabla_a, \nabla_b]\!]f = 0$ for $f \in \mathfrak{X}(\mathcal{M})$ so that

$$[\![\nabla_a, \nabla_b]\!](fT_{\mathcal{A}}) = f[\![\nabla_a, \nabla_b]\!]T_{\mathcal{A}},$$

for ${}_{\mathcal{A}}$ denoting an arbitrary string of covariant or contravariant indices. In particular, one has that

$$[\![\nabla_a, \nabla_b]\!](fu^c) = f[\![\nabla_a, \nabla_b]\!]u^c,$$
$$[\![\nabla_a, \nabla_b]\!](u^c + v^c) = [\![\nabla_a, \nabla_b]\!]u^c + [\![\nabla_a, \nabla_b]\!]v^c.$$

From the previous expressions one concludes that the map $u^d \mapsto [\![\nabla_a, \nabla_b]\!]u^d$ is $\mathfrak{X}$-linear. Thus, using Lemma 2.1 it defines a tensor field $R^d{}_{cab}$, the ***Riemann curvature tensor*** of the connection $\boldsymbol{\nabla}$. One writes

$$[\![\nabla_a, \nabla_b]\!]u^d = \left([\nabla_a, \nabla_b] - \Sigma_a{}^c{}_b\nabla_c\right)u^d = R^d{}_{cab}u^c. \tag{2.8}$$

Alternatively, one has that

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)u^d = R^d{}_{cab}u^c + \Sigma_a{}^c{}_b\nabla_c u^d.$$

The antisymmetry of $[\![\nabla_a, \nabla_b]\!]$ on the indices ${}_a$ and ${}_b$ is inherited by the Riemann curvature tensor, so that

$$R^d{}_{cab} = -R^d{}_{cba}.$$

The action of the commutator of covariant derivatives can be extended to other tensors using the Leibnitz rule. For example, from

$$[\![\nabla_a, \nabla_b]\!](\omega_d v^d) = \left([\![\nabla_a, \nabla_b]\!]\omega_d\right)v^d + \omega_d[\![\nabla_a, \nabla_b]\!]v^d,$$

one can conclude that

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)\omega_d = -R^c{}_{dab}\omega_c + \Sigma_a{}^c{}_b\nabla_c\omega_d.$$

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Similarly, evaluating $[\![\nabla_a, \nabla_b]\!](S^d{}_{ef}\omega_d u^e v^f)$, one concludes that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\, S^d{}_{ef} = R^d{}_{cab}S^c{}_{ef} - R^c{}_{eab}S^d{}_{cf} - R^c{}_{fab}S^d{}_{ec} + \Sigma_a{}^c{}_b\nabla_c S^d{}_{ef}.$$

Remark. The curvature can be defined in an alternative way via the relation

$$\boldsymbol{Riem}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \qquad \boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in T(\mathcal{M}), \quad (2.9)$$

where the expression $\boldsymbol{Riem}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}$ corresponds to $R^d{}_{cab}w^c u^a v^b$ in abstract index notation.

Bianchi identities

In order to investigate further symmetries of the curvature tensor, consider the triple derivative $\nabla_{[a}\nabla_b\nabla_{c]}f$ of $f \in \mathfrak{X}(\mathcal{M})$. A computation shows, on the one hand, that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}f &= 2\nabla_{[[a}\nabla_{b]}\nabla_{c]}f = [\nabla_{[a},\nabla_b]\nabla_{c]}f \\ &= \Sigma_{[a}{}^d{}_b\nabla_{|d|}\nabla_{c]}f - R^d{}_{[cab]}\nabla_d f, \end{aligned}$$

and on the other hand that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}f &= 2\nabla_{[a}\nabla_{[b}\nabla_{c]]}f \\ &= \nabla_{[a}[\nabla_b,\nabla_{c]}]f = \nabla_{[a}\left(\Sigma_b{}^d{}_{c]}\nabla_d f\right) \\ &= \nabla_{[a}\Sigma_b{}^d{}_{c]}\nabla_d f + \Sigma_{[b}{}^d{}_c\nabla_{a]}\nabla_d f. \end{aligned}$$

Putting these two computations together and using the definition of the torsion tensor, Equation (2.6), one concludes that

$$\nabla_{[a}\Sigma_b{}^d{}_{c]}\nabla_d f + R^d{}_{[cab]}\nabla_d f + \Sigma_{[a}{}^d{}_b\Sigma_{c]}{}^e{}_d\nabla_e f = 0.$$

As the scalar field f is arbitrary, one concludes that

$$R^d{}_{[cab]} + \nabla_{[a}\Sigma_b{}^d{}_{c]} + \Sigma_{[a}{}^e{}_b\Sigma_{c]}{}^d{}_e = 0. \quad (2.10)$$

This is the so-called ***first Bianchi identity***. In the case of a *torsion-free connection* it takes the familiar form

$$R^d{}_{[cab]} = 0.$$

As a consequence of the antisymmetry in the last two indices, the latter can be written as

$$R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} = 0.$$

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Next, consider the action of $\nabla_{[a}\nabla_b\nabla_{c]}$ on a vector field v^d. As in the case of the first Bianchi identity, one can compute this object in two different ways. On the one hand, one has that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}v^d &= 2\nabla_{[[a}\nabla_{b]}\nabla_{c]}v^d \\ &= [\nabla_{[a},\nabla_b]\nabla_{c]}v^d \\ &= [\![\nabla_{[a},\nabla_b]\!]\nabla_{c]}v^d + \Sigma_{[a}{}^e{}_b\nabla_{|e|}\nabla_{c]}v^d \\ &= -R^e{}_{[cab]}\nabla_e v^d + R^d{}_{e[ab}\nabla_{c]}v^e + \Sigma_{[a}{}^e{}_b\nabla_{|e|}\nabla_{c]}v^d, \end{aligned}$$

and on the other hand that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}v^d &= 2\nabla_{[a}\nabla_{[b}\nabla_{c]]}v^d \\ &= 2\nabla_{[a}[\![\nabla_b,\nabla_{c]}]\!]v^d + \nabla_{[a}\left(\Sigma_b{}^e{}_{c]}\nabla_e v^d\right) \\ &= \nabla_{[a}R^d{}_{|e|bc]}v^e + R^d{}_{e[bc}\nabla_{a]}v^e + \nabla_{[a}\Sigma_b{}^e{}_{c]}\nabla_e v^d + \Sigma_{[b}{}^e{}_c\nabla_{a]}\nabla_e v^d. \end{aligned}$$

Equating the two expressions for $2\nabla_{[a}\nabla_b\nabla_{c]}v^d$ and using the first Bianchi identity, Equation (2.10), to eliminate covariant derivatives of the torsion tensor one concludes that

$$\nabla_{[a}R^d{}_{|e|bc]} + \Sigma_{[a}{}^f{}_b R^d{}_{|e|c]f} = 0. \tag{2.11}$$

This is the so-called ***second Bianchi identity***. For a *torsion-free connection* one obtains the well-known expression

$$\nabla_{[a}R^d{}_{|e|bc]} = 0. \tag{2.12}$$

2.4.4 Change of connection

Consider two connections $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$ on the manifold $\mathcal{M}$. A natural question to be asked is whether there is any relation between these connections and their associated torsion and curvature tensors. By definition one has that

$$(\bar{\nabla}_a - \nabla_a)f = 0, \qquad f \in \mathfrak{X}(\mathcal{M}).$$

Moreover, one also has that

$$(\bar{\nabla}_a - \nabla_a)(fv^a) = f(\bar{\nabla}_a - \nabla_a)v^a.$$

It follows that the map $v^b \mapsto (\bar{\nabla}_a - \nabla_a)v^a$ is $\mathfrak{X}$-linear, so that, invoking Lemma 2.1, there exists a tensor field, the ***transition tensor*** $Q_a{}^b{}_c$, such that

$$(\bar{\nabla}_a - \nabla_a)v^b = Q_a{}^b{}_c v^c. \tag{2.13}$$

Now, from

$$(\bar{\nabla}_a - \nabla_a)(\omega_b v^b) = 0,$$

one readily concludes that

$$(\bar{\nabla}_a - \nabla_a)\omega_b = -Q_a{}^c{}_b\omega_c. \tag{2.14}$$

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A different choice of covariant derivatives gives rise to a different choice of transition tensor. The set of connections over a manifold $\mathcal{M}$ is an *affine space*: given a connection $\boldsymbol{\nabla}$ on the manifold, any other connection can be obtained by a suitable choice of transition tensor. If $\boldsymbol{Q}$ denotes the index-free version of the tensor $Q_a{}^b{}_c$, then the relation between the connection $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$ will be denoted, in a schematic way, as

$$\bar{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{Q}.$$

In Chapter 5 specific forms for the transition tensor will be investigated.

Transformation of the torsion and the curvature

A direct computation using Equations (2.6) and (2.13) renders the following relation between the torsion tensors of the connections $\bar{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$:

$$\bar{\Sigma}_a{}^c{}_b - \Sigma_a{}^c{}_b = -2Q_{[a}{}^c{}_{b]}. \tag{2.15}$$

In particular, it follows that if $Q_a{}^c{}_b = \frac{1}{2}\Sigma_a{}^c{}_b$, then $\bar{\Sigma}_a{}^c{}_b = 0$. That is, *it is always possible to construct a connection which is torsion-free.*

An analogous, albeit lengthier computation using Equations (2.6) and (2.8) renders the following relation between the respective curvature tensors:

$$\bar{R}^c{}_{dab} - R^c{}_{dab} = 2\nabla_{[a}Q_{b]}{}^c{}_d - \Sigma_a{}^e{}_b Q_e{}^c{}_d + 2Q_{[a}{}^c{}_{|e|}Q_{b]}{}^e{}_d. \tag{2.16}$$

2.4.5 The geodesic and geodesic deviation equations

Given a covariant derivative $\boldsymbol{\nabla}$, one can introduce the notion of ***parallel propagation***. Given $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{M})$, then $\boldsymbol{u}$ is said to be parallely propagated in the direction of $\boldsymbol{v}$ if it satisfies the equation $\nabla_{\boldsymbol{v}}\boldsymbol{u} = 0$.

A ***geodesic*** $\gamma \subset \mathcal{M}$ is a curve whose tangent vector is parallely propagated along itself. Following the convention of Section 2.2.1, let $\dot{\boldsymbol{x}}$ denote the tangent vector to γ. One has that

$$\nabla_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0. \tag{2.17}$$

A ***congruence of geodesics*** is the set of integral curves of a vector field $\dot{\boldsymbol{x}}$ satisfying Equation (2.17). Any vector $\boldsymbol{z}$ such that $[\dot{\boldsymbol{x}}, \boldsymbol{z}] = 0$ is called a ***deviation vector*** of the congruence of geodesics. *Assuming that the connection* $\boldsymbol{\nabla}$ *is torsion-free* so that $\nabla_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \nabla_{\boldsymbol{z}}\dot{\boldsymbol{x}}$, a computation shows that $\boldsymbol{z}$ satisfies the ***geodesic deviation equation***

$$\nabla_{\dot{\boldsymbol{x}}}\nabla_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \boldsymbol{Riem}(\dot{\boldsymbol{x}}, \boldsymbol{z})\dot{\boldsymbol{x}}.$$

Remark. The set of geodesics emanating from a point $p \in \mathcal{M}$ allows one to define a diffeomorphism between a neighbourhood of the origin of $T|_p(\mathcal{M})$ and a suitably small neighbourhood $\mathcal{U}$ of p, the so-called ***exponential map***. A precise

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definition of the exponential map is given in Section 11.6.2. Further properties and applications are given in Sections 14.2 and 18.4.1.

2.5 Metric tensors

A ***metric*** on the manifold $\mathcal{M}$ is a symmetric rank 2 covariant tensor field $\boldsymbol{g}$ – to be denoted by g_{ab} in abstract index notation. The metric tensor $\boldsymbol{g}$ is said to be ***non-degenerate*** if $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = 0$ for all $\boldsymbol{u}$ if and only if $\boldsymbol{v} = 0$. In the sequel, and unless otherwise explicitly stated, it is assumed that all the metrics under consideration are non-degenerate. If $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = 0$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be ***orthogonal***. Pointwise, the components $g_{\boldsymbol{ab}} \equiv \boldsymbol{g}(\boldsymbol{e_a},\boldsymbol{e_b})$ with respect to a basis $\{\boldsymbol{e_a}\}$ define a symmetric $(n\times n)$-matrix $(g_{\boldsymbol{ab}})$. As this matrix is symmetric, it has n real eigenvalues. The ***signature*** of $\boldsymbol{g}$ is the difference between the number of positive and negative eigenvalues. If the signature is n or $-n$, then $\boldsymbol{g}$ is said to be a ***Riemannian metric***. If the signature is $\pm(n-2)$, then $\boldsymbol{g}$ is a ***Lorentzian metric***.

From the non-degeneracy of $\boldsymbol{g}$ it follows that there exists a unique contravariant rank 2 tensor to be denoted by either $\boldsymbol{g}^\sharp$ or g^{ab} such that

$$g_{ab}g^{bc} = \delta_a{}^c.$$

In terms of components with respect to a basis this means that the matrices $(g_{\boldsymbol{ab}})$ and $(g^{\boldsymbol{ab}})$ are inverses of each other. Accordingly, $\boldsymbol{g}^\sharp$ is also non-degenerate and one obtains an isomorphism between the vector spaces $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$. More precisely, given $\boldsymbol{v} \in T|_p(\mathcal{M})$, then $\boldsymbol{v}^\flat \equiv \boldsymbol{g}(\boldsymbol{v},\cdot) \in T^*|_p(\mathcal{M})$ as $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{R}$ for any $\boldsymbol{u} \in T|_p(\mathcal{M})$. Similarly, given $\boldsymbol{\omega} \in T^*|_p(\mathcal{M})$, one has that $\boldsymbol{\omega}^\sharp \equiv \boldsymbol{g}^\sharp(\boldsymbol{\omega},\cdot) \in T|_p(\mathcal{M})$. In terms of abstract indices, the operations $^\flat$ (*flat*) and $^\sharp$ (*sharp*) correspond to the operations of ***lowering and raising of indices*** by means of g_{ab} and g^{ab}:

$$v_a \equiv g_{ab}v^b, \qquad \omega^a \equiv g^{ab}\omega_b.$$

The operations $^\flat$ and $^\sharp$ are inverses of each other. They can be extended in a natural way to tensors of arbitrary rank.

Given two manifolds $\mathcal{M}$ and $\bar{\mathcal{M}}$ with metrics $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$, respectively, a diffeomorphism $\varphi : \mathcal{M} \to \bar{\mathcal{M}}$ is called an ***isometry*** if $\varphi^*\bar{\boldsymbol{g}} = \boldsymbol{g}$. If an isometry exists, then the pairs $(\mathcal{M},\boldsymbol{g})$ and $(\bar{\mathcal{M}},\bar{\boldsymbol{g}})$ are said to be ***isometric***. If $\mathcal{M} = \bar{\mathcal{M}}$ and $\boldsymbol{g} = \bar{\boldsymbol{g}}$, one speaks of an isometry of $\mathcal{M}$.

Remark. Most of the Lorentzian metrics to be considered in this book will be associated to four-dimensional manifolds. These Lorentzian metrics will be assumed to have signature -2. This convention leads one to consider three-dimensional ***negative-definite*** Riemannian metrics, that is, metrics with signature -3. In this book, only three-dimensional Riemannian manifolds will be considered. In the sequel, the symbol $\boldsymbol{g}$ will be used to denote a generic Lorentzian metric, while $\boldsymbol{h}$ will be used for a generic negative-definite Riemannian metric.

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Specifics for Lorentzian metrics

Following the standard terminology of general relativity, a pair $(\mathcal{M}, \boldsymbol{g})$ consisting of a four-dimensional manifold and a Lorentzian metric will be called a ***spacetime***. The metric $\boldsymbol{g}$ can be used to classify vectors in a pointwise manner as ***timelike***, ***null*** or ***spacelike*** depending on whether $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$ or $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$, respectively. A basis $\{\boldsymbol{e_a}\}$ is said to be ***orthonormal*** if

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}, \qquad \eta_{\boldsymbol{ab}} \equiv \mathrm{diag}(1, -1, -1, -1).$$

It follows that $\boldsymbol{g}$ can be written as

$$\boldsymbol{g} = \eta_{\boldsymbol{ab}} \boldsymbol{\omega^a} \otimes \boldsymbol{\omega^b}, \tag{2.18}$$

where $\{\boldsymbol{\omega^a}\}$ denotes the coframe dual to $\{\boldsymbol{e_a}\}$. A change of basis, as given by Equation (2.1), preserving Equation (2.18), is called a ***Lorentz transformation***. A calculation readily shows that for a Lorentz transformation one has that

$$\eta_{\boldsymbol{ab}} A^{\boldsymbol{a}}{}_{\boldsymbol{c}} A^{\boldsymbol{b}}{}_{\boldsymbol{d}} = \eta_{\boldsymbol{cd}}.$$

Further aspects of Lorentz transformations are discussed in Sections 3.1.9, 3.1.12 and 5.1.1.

The set of null vectors at a point $p \in \mathcal{M}$ is called the ***null cone at*** p and will be denoted by $\mathcal{C}_p$. By definition timelike vectors lie inside the null cone, while spacelike ones lie outside it. The null cone is made of two half cones. If one of these half cones can be singled out and called the ***future half cone*** $\mathcal{C}_p^+$ and the other the ***past half cone*** $\mathcal{C}_p^-$, then $T|_p(\mathcal{M})$ is said to be ***time oriented***. A timelike vector inside $\mathcal{C}_p^+$ is said to be ***future directed***; similarly a timelike vector inside $\mathcal{C}_p^-$ is called ***past directed***. If $T(\mathcal{M})$ can be time oriented in a continuous manner for all $p \in \mathcal{M}$, then $(\mathcal{M}, \boldsymbol{g})$ is said to be a ***time-oriented spacetime***. A curve $\gamma \subset \mathcal{M}$ with a *timelike, future-oriented* tangent vector $\dot{\boldsymbol{x}}$ is said to be ***parametrised by its proper time*** if $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$.

Specifics for Riemannian metrics

A Riemannian metric $\boldsymbol{h}$ endows the tangent spaces of the manifold with an inner product. Because of the signature conventions, this inner product is negative definite. A basic result of Riemannian geometry is that every differential manifold admits a Riemannian metric. The proof of this argument relies heavily on the paracompactness of the manifold; see, for example, Choquet-Bruhat et al. (1982).

In the case of a Riemannian metric $\boldsymbol{h}$, a basis $\{\boldsymbol{e_i}\}$ is said to be ***orthonormal*** if

$$\boldsymbol{h}(\boldsymbol{e_i}, \boldsymbol{e_j}) = -\delta_{\boldsymbol{ij}}, \qquad \delta_{\boldsymbol{ij}} \equiv \mathrm{diag}(1, 1, 1).$$

Thus, using the associated coframe basis $\{\boldsymbol{\omega^i}\}$ one can write

$$\boldsymbol{h} = -\delta_{\boldsymbol{ij}} \boldsymbol{\omega^i} \otimes \boldsymbol{\omega^j}.$$

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2.5.1 Metric connections and Levi-Civita connections

Two further conditions which are usually required from a connection are metric compatibility and torsion-freeness. In this section the consequences of these assumptions are briefly reviewed.

Metric connections

A connection $\boldsymbol{\nabla}$ on $\mathcal{M}$ is said to be ***metric with respect to $\boldsymbol{g}$*** if $\boldsymbol{\nabla g} = 0$ (i.e. $\nabla_a g_{bc} = 0$). The Riemann curvature tensor of the connection $\boldsymbol{\nabla}$ acquires, by virtue of the metricity condition, a further symmetry. This can be better seen by applying the modified commutator $[\![\nabla_a, \nabla_b]\!]$ to the metric g_{ab}. On the one hand, by the assumption of metricity one has $[\![\nabla_a, \nabla_b]\!] g_{cd} = 0$, while on the other hand

$$[\![\nabla_a, \nabla_b]\!] g_{cd} = -R^e{}_{cab} g_{ed} - R^e{}_{dab} g_{ce} = -R_{dcab} - R_{cdab},$$

where $R_{dcab} \equiv g_{de} R^e{}_{cab}$. Hence, one concludes that

$$R_{cdab} = -R_{dcab}. \tag{2.19}$$

The Levi-Civita connection

A connection $\boldsymbol{\nabla}$ is said to be the ***Levi-Civita connection of the metric $\boldsymbol{g}$*** if $\boldsymbol{\nabla}$ is *torsion-free* and *metric* with respect to $\boldsymbol{g}$. The *Fundamental Theorem of Riemannian Geometry* (also valid in the Lorentzian case) ensures that the Levi-Civita connection of a metric $\boldsymbol{g}$ is unique. The proof of this result is well known and readily available in most books on Riemannian geometry; see, for example, Choquet-Bruhat et al. (1982). The Levi-Civita connection $\boldsymbol{\nabla}$ of the metric $\boldsymbol{g}$ is characterised by the so-called ***Koszul formula***

$$\begin{aligned} 2\boldsymbol{g}(\nabla_{\boldsymbol{v}}\boldsymbol{u}, \boldsymbol{w}) &= \boldsymbol{v}(\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{w})) + \boldsymbol{u}(\boldsymbol{g}(\boldsymbol{w}, \boldsymbol{v})) - \boldsymbol{w}(\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})) \\ &\quad - \boldsymbol{g}(\boldsymbol{v}, [\boldsymbol{u}, \boldsymbol{w}]) + \boldsymbol{g}(\boldsymbol{u}, [\boldsymbol{w}, \boldsymbol{v}]) + \boldsymbol{g}(\boldsymbol{w}, [\boldsymbol{v}, \boldsymbol{u}]). \end{aligned} \tag{2.20}$$

Of particular interest are the further symmetries that the Riemann tensor of a Levi-Civita connection possesses. First of all, because of the metricity, the curvature tensor has the symmetry given in Equation (2.19). Furthermore, as the connection is torsion-free, the first Bianchi identity implies $R_{c[dab]} = 0$. From the latter one readily has that

$$\begin{aligned} 2R_{cdab} &= R_{cdab} + R_{dcba} \\ &= -R_{cabd} - R_{cbda} - R_{dbac} - R_{dacb} \\ &= -R_{acdb} - R_{bcad} - R_{bdca} - R_{adbc} \\ &= R_{abcd} + R_{badc}. \end{aligned}$$

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Hence, one recovers the well-known symmetry of interchange of pairs

$$R_{cdab} = R_{abcd}.$$

Characterisation of flatness

An open subset $\mathcal{U} \subset \mathcal{M}$ of a spacetime $(\mathcal{M}, \boldsymbol{g})$ is said to be flat if the metric $\boldsymbol{g}$ on $\mathcal{U}$ is isometric to the ***Minkowski metric***

$$\boldsymbol{\eta} \equiv \eta_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu, \qquad (\eta_{\mu\nu}) \equiv \mathrm{diag}(1, -1, -1, -1).$$

In the case of a three-dimensional Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$, flatness implies a local isometry with the three-dimensional ***Euclidean metric***

$$\boldsymbol{\delta} \equiv -\delta_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta, \qquad (\delta_{\alpha\beta}) \equiv \mathrm{diag}(1, 1, 1).$$

The Riemann tensor of a Levi-Civita connection provides a local characterisation of the flatness of a manifold. More precisely, a metric is flat on $\mathcal{U}$ if and only if its Riemann tensor vanishes on $\mathcal{U}$. The *if* part of the result follows by direct evaluation of the Riemann tensor. The *only if* part is more complicated; see, for example, Choquet-Bruhat et al. (1982), page 310 for a proof.

Traces

A metric $\boldsymbol{g}$ on a manifold $\mathcal{M}$ allows one to introduce a further operation on tensors which reduces their rank by 2 – the ***trace*** with respect to $\boldsymbol{g}$. Given $\boldsymbol{T} \in \mathfrak{T}_2(\mathcal{M})$, its trace, $\mathbf{tr}_{\boldsymbol{g}}\boldsymbol{T}$, is the scalar described in abstract index notation by $g^{ab}T_{ab}$. Observing that $g^{ab}T_{ab} = T^a{}_a$, one sees that taking the trace of a tensor is a generalisation of the operation of contraction. The operation of taking the trace can be generalised to any pair of indices of the same type in an arbitrary tensor – for example, $g^{ac}M_{abcd}$ and $g^{bc}M_{abcd}$ denote the traces of M_{abcd} with respect to the first and third arguments and the second and third ones, respectively.

Given a symmetric tensor on a four-dimensional manifold $\mathcal{M}$, $T_{ab} = T_{(ab)} \in \mathfrak{T}_{ab}(\mathcal{M})$, its ***trace-free part*** $T_{\{ab\}}$ is given by

$$T_{\{ab\}} \equiv T_{ab} - \frac{1}{4}g_{ab}g^{cd}T_{cd}.$$

In the case of a three-dimensional manifold $\mathcal{S}$ with metric $\boldsymbol{h}$, the above definition has to be modified to

$$T_{\{ij\}} \equiv T_{ij} - \frac{1}{3}h_{ij}h^{kl}T_{kl},$$

for a symmetric tensor $T_{ij} \in \mathfrak{T}_{ij}(\mathcal{S})$. The operation of taking the trace-free part of a tensor can be extended to tensors of arbitrary rank. Unfortunately, the expressions to compute them become increasingly cumbersome. A more efficient approach to describe this operation is in terms of spinors; see Chapters 3 and 4. A tensor $M_{a_1\cdots a_k}$ is said to be ***trace-free*** if $M_{a_1\cdots a_k} = M_{\{a_1\cdots a_k\}}$.

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2.5.2 Decomposition of the Riemann tensor

In what follows, consider a spacetime $(\mathcal{M}, \boldsymbol{g})$ and a connection $\bar{\boldsymbol{\nabla}}$ on $\mathcal{M}$ – not necessarily the Levi-Civita connection of the metric $\boldsymbol{g}$. Let $\bar{R}^a{}_{bcd}$ denote the Riemann curvature tensor of the connection $\bar{\boldsymbol{\nabla}}$. A ***concomitant*** of $\bar{R}^a{}_{bcd}$ is any tensorial object which can be constructed from the curvature tensor by means of the operations of covariant differentiation and contraction with g_{ab} and g^{ab}. The basic concomitant of $\bar{R}^a{}_{bcd}$ is the ***Ricci tensor*** $\bar{R}_{cd}$ defined by the contraction

$$\bar{R}_{bd} \equiv \bar{R}^a{}_{bad}.$$

When working in index-free notation the Ricci tensor will be denoted by $\boldsymbol{Ric}$. Using the contravariant metric g^{ab} one can define a further concomitant, the ***Ricci scalar*** relative to the metric $\boldsymbol{g}$, $\bar{R}$, as

$$\bar{R} \equiv g^{bd}\bar{R}_{bd}.$$

A concomitant of $\bar{R}^a{}_{bcd}$ which will appear recurrently in this book is the ***Schouten tensor*** relative to $\boldsymbol{g}$, $\bar{L}_{ab}$. In four dimensions it is defined as

$$\bar{L}_{ab} = \frac{1}{2}\bar{R}_{ab} - \frac{1}{12}\bar{R}g_{ab}.$$

The definition of the Schouten tensor is dimension dependent. The definition for three dimensions will be discussed in Section 2.7. When working in index-free notation the Schouten tensor will be denoted by $\boldsymbol{Schouten}$. In the discussion of spinors in Chapter 3 a further concomitant arises in a natural way: the ***trace-free Ricci tensor*** $\bar{\Phi}_{ab}$. In four dimensions one has that

$$\bar{\Phi}_{ab} \equiv \frac{1}{2}\bar{R}_{\{ab\}} = \frac{1}{2}\left(\bar{R}_{(ab)} - \frac{1}{4}\bar{R}g_{ab}\right),$$

where the overall factor of $\frac{1}{2}$ is conventional. *It is important to observe that the tensors $\bar{R}_{ab}$ and $\bar{L}_{ab}$ are not symmetric unless $\bar{\nabla}$ is a Levi-Civita connection.*

Finally, one can define the ***Weyl tensor*** of $\bar{\boldsymbol{\nabla}}$ relative to $\boldsymbol{g}$, $\bar{C}^a{}_{bcd}$, as the fully trace-free part of $\bar{R}^a{}_{bcd}$. When working in index-free notation the Weyl tensor will be denoted by $\boldsymbol{Weyl}$.

The case of a Levi-Civita connection

If $\bar{\boldsymbol{\nabla}}$ is the Levi-Civita connection of the metric $\boldsymbol{g}$, so that $\bar{\boldsymbol{\nabla}} = \boldsymbol{\nabla}$, it can be shown that

$$\begin{aligned} R^c{}_{dab} &= C^c{}_{dab} + 2(\delta^c{}_{[a}L_{b]d} - g_{d[a}L_{b]}{}^c), && \text{(2.21a)} \\ &= C^c{}_{dab} + 2S_{d[a}{}^{ce}L_{b]e}, && \text{(2.21b)} \end{aligned}$$

where

$$S_{ab}{}^{cd} = \delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd}.$$

This tensor will play a special role in the context of conformal geometry; see Chapter 5. A spinorial derivation of this decomposition is provided in Chapter 3.

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Remark. The decomposition given by Equations (2.21a) and (2.21b) is unique; that is, the Rieman tensor cannot be reconstructed from any other combination of the Schouten and Weyl tensors. Moreover, if $C^c{}_{dab} = 0$ and $L_{ab} = 0$, then necessarily $R^c{}_{dab} = 0$. These remarks also hold for the generalisations of the decomposition to Weyl connections; see Section 5.3 and, in particular, Equation (5.28a).

The Einstein tensor

An important concomitant of the Riemann tensor of a Levi-Civita connection $\boldsymbol{\nabla}$ is the ***Einstein tensor*** $\boldsymbol{G}$ defined in four dimensions by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab}.$$

Starting from the second Bianchi identity, Equation (2.12), contracting the indices ^{d} and ${}_b$ and then contracting the resulting expression with g^{ae} yields

$$\nabla^a R_{ab} = \frac{1}{2}\nabla_b R, \qquad \text{that is,} \qquad \nabla^a G_{ab} = 0.$$

That is, the Einstein tensor is divergence-free.

2.5.3 Volume forms and Hodge duals

The spacetime ***volume form*** of the metric $\boldsymbol{g}$, ϵ_{abcd}, is defined by the conditions

$$\epsilon_{abcd} = \epsilon_{[abcd]}, \qquad \epsilon_{abcd}\epsilon^{abcd} = -24,$$

and

$$\epsilon_{abcd} e_{\boldsymbol{0}}{}^a e_{\boldsymbol{1}}{}^b e_{\boldsymbol{2}}{}^c e_{\boldsymbol{3}}{}^d = 1,$$

where $\{\boldsymbol{e_a}\}$ is a $\boldsymbol{g}$-orthonormal frame. A spacetime $(\mathcal{M}, \boldsymbol{g})$ has a non-vanishing volume element if and only if $\mathcal{M}$ is orientable; see, for example, O'Neill (1983); Willmore (1993). The following properties can be directly verified:

$$\epsilon_{abcd}\epsilon^{pqrs} = -24\delta_a{}^{[p}\delta_b{}^q\delta_c{}^r\delta_d{}^{s]}, \tag{2.22a}$$
$$\epsilon_{abcd}\epsilon^{pqrd} = -6\delta_a{}^{[p}\delta_b{}^q\delta_c{}^{r]}, \tag{2.22b}$$
$$\epsilon_{abcd}\epsilon^{pqcd} = -4\delta_a{}^{[p}\delta_b{}^{q]}, \tag{2.22c}$$
$$\epsilon_{abcd}\epsilon^{pbcd} = -6\delta_a{}^p; \tag{2.22d}$$

see, for example, Penrose and Rindler (1984). If $\boldsymbol{\nabla}$ denotes the Levi-Civita covariant derivative of the metric $\boldsymbol{g}$, one can then readily verify that $\nabla_a\epsilon_{bcde} = 0$. That is, the volume form is compatible with the Levi-Civita connection of the metric $\boldsymbol{g}$.

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The Hodge duals

Given an antisymmetric tensor $F_{ab} = F_{[ab]}$, one can use the volume form to define its ***Hodge dual*** ${}^*F_{ab}$ as

$$
{}^*F_{ab} \equiv -\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}.
$$

This definition can be naturally extended to any tensor with a pair of antisymmetric indices. Using the identity (2.22c) one readily finds that

$$
{}^{**}F_{ab} = -F_{ab}.
$$

Of special relevance are the Hodge duals of the Riemann and Weyl tensors. If R_{abcd} denotes the Riemann curvature of the Levi-Civita connection $\boldsymbol{\nabla}$, then one can define a ***left dual*** and a ***right dual***, respectively, by

$$
{}^*R_{abcd} \equiv -\frac{1}{2}\epsilon_{ab}{}^{pq}R_{pqcd}, \qquad R^*_{abcd} \equiv -\frac{1}{2}\epsilon_{cd}{}^{pq}R_{abpq}.
$$

The Hodge dual can be used to recast the Bianchi identities in an alternative way. More precisely, one has that

$$
R_{a[bcd]} = \delta_{[b}{}^{p}\delta_{c}{}^{q}\delta_{d]}{}^{r}R_{apqr} = -\frac{1}{6}\epsilon_{sbcd}\left(\epsilon^{spqr}R_{apqr}\right) = \frac{1}{3}\epsilon_{sbcd}R^*{}_{ap}{}^{sp}.
$$

Thus, the first Bianchi identity $R_{a[bcd]} = 0$ is equivalent to

$$
R^*_{ab}{}^{cb} = 0. \tag{2.23}
$$

Furthermore,

$$
\frac{1}{2}\epsilon_f{}^{abc}\nabla_{[a}R^d{}_{|e|bc]} = \nabla_a\left(\frac{1}{2}\epsilon_f{}^{abc}R^d{}_{ebc}\right) = -\nabla_a R^{*d}{}_{ef}{}^{a}.
$$

Thus, one has that

$$
\nabla^a R^*{}_{abcd} = 0.
$$

Finally, it is noticed that the duals of the Weyl tensor satisfy

$$
{}^*C_{abcd} = C^*_{abcd}.
$$

Sometimes it is convenient to make use of operations of dualisation on one or three indices. Given an arbitrary tensor J_a and another tensor K_{abc} antisymmetric in ${}_{abc}$ one defines

$$
{}^\dagger J_{abc} \equiv \epsilon_{abc}{}^{d}J_d, \qquad {}^\ddagger K_a \equiv \frac{1}{6}\epsilon_a{}^{bcd}K_{bcd}. \tag{2.24}
$$

Using the properties of contractions of the volume form, it can be shown that

$$
{}^{\ddagger\dagger}J_a = J_a, \qquad {}^{\dagger\ddagger}K_{abc} = K_{abc}.
$$

Further details on the calculations required to obtain all of the properties discussed in this section can be found in Penrose and Rindler (1984).

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2.6 Frame formalisms

Frame formalisms have been used in many areas of relativity to analyse the properties of the Einstein field equations and their solutions; see, for example, Ellis and van Elst (1998); Ellis et al. (2012); Wald (1984). One of the advantages of frame formalisms is that they lead to consider scalar objects and equations, which are, in general, simpler to manipulate than their tensorial counterparts. A further advantage of frames is that they lead to a straight forward transcription of tensorial expressions into spinors; see Chapter 3.

The purpose of this section is to develop and fix the conventions of a frame formalism used in Friedrich (2004).

2.6.1 Basic definitions and conventions

Given a spacetime $(\mathcal{M}, \boldsymbol{g})$, let $\{\boldsymbol{e_a}\}$ denote a frame and let $\{\boldsymbol{\omega^b}\}$ denote its dual coframe basis. *For the time being, this frame is not assumed to be $\boldsymbol{g}$-orthogonal.* By definition one has that

$$\langle \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}. \tag{2.25}$$

In what follows, it will be assumed one has a connection $\boldsymbol{\nabla}$ *which, for the time being, is assumed to be general*; that is, it is not necessarily metric or torsion-free. The ***connection coefficients*** of $\boldsymbol{\nabla}$ with respect to the frame $\{\boldsymbol{e_a}\}$, to be denoted by $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$, are defined via

$$\nabla_{\boldsymbol{a}} \boldsymbol{e_b} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c}, \tag{2.26}$$

where $\nabla_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^{a} \nabla_a$ denotes the ***covariant directional derivative*** in the direction of $\boldsymbol{e_a}$. As $\nabla_{\boldsymbol{a}} \boldsymbol{e_b}$ is a vector, it follows that

$$\langle \boldsymbol{\omega^c}, \nabla_{\boldsymbol{a}} \boldsymbol{e_b} \rangle = \langle \boldsymbol{\omega^c}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \boldsymbol{e_d} \rangle = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \langle \boldsymbol{\omega^c}, \boldsymbol{e_d} \rangle = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}.$$

This expression could have been used, alternatively, as a definition of the connection coefficients. In order to carry out computations one also needs an expression for $\nabla_{\boldsymbol{a}} \boldsymbol{\omega^b}$. By analogy with Equation (2.26) one can write $\nabla_{\boldsymbol{a}} \boldsymbol{\omega^b} = \mho_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \boldsymbol{\omega^c}$. The coefficients $\mho_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ can be expressed in terms of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ by differentiating Equation (2.25) with respect to $\nabla_{\boldsymbol{d}}$. Noting that $\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ is a constant scalar one has, on the one hand, that

$$\nabla_{\boldsymbol{d}}(\langle \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle) = \boldsymbol{e_d}(\langle \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle) = \boldsymbol{e_d}(\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}) = 0,$$

while, on the other hand, one has

$$\nabla_{\boldsymbol{d}}(\langle \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle) = \langle \nabla_{\boldsymbol{d}} \boldsymbol{\omega^b}, \boldsymbol{e_a} \rangle + \langle \boldsymbol{\omega^b}, \nabla_{\boldsymbol{d}} \boldsymbol{e_a} \rangle = \left(\mho_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \Gamma_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \right) \langle \boldsymbol{\omega^c}, \boldsymbol{e_a} \rangle,$$

so that $\mho_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = -\Gamma_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. Consequently, one has

$$\nabla_{\boldsymbol{a}} \boldsymbol{\omega^b} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \boldsymbol{\omega^c}. \tag{2.27}$$

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It is observed that the specification of the 4^3 connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ fully determines the connection $\boldsymbol{\nabla}$; a generalisation of this argument shows that every manifold admits a connection; see, for example, Willmore (1993).

Consider now $\boldsymbol{v} \in T(\mathcal{M})$ and $\boldsymbol{\alpha} \in T^*(\mathcal{M})$. Writing the above in terms of the frame and coframe, respectively, one has

$$\begin{aligned} \boldsymbol{v} &= v^{\boldsymbol{a}}\boldsymbol{e}_{\boldsymbol{a}}, \qquad v^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{a}}, \boldsymbol{v} \rangle, \\ \boldsymbol{\alpha} &= \alpha_{\boldsymbol{a}}\boldsymbol{\omega}^{\boldsymbol{a}}, \qquad \alpha_{\boldsymbol{a}} \equiv \langle \boldsymbol{\alpha}, \boldsymbol{e}_{\boldsymbol{a}} \rangle. \end{aligned}$$

In order to further develop the frame formalism it will be convenient to define

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{b}}, \nabla_{\boldsymbol{a}} \boldsymbol{v} \rangle, \qquad \nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} \equiv \langle \nabla_{\boldsymbol{a}} \boldsymbol{\alpha}, \boldsymbol{e}_{\boldsymbol{b}} \rangle.$$

It follows from Equations (2.26) and (2.27) that

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} = \boldsymbol{e}_{\boldsymbol{a}}(v^{\boldsymbol{b}}) + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} v^{\boldsymbol{c}}, \qquad \nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} = \boldsymbol{e}_{\boldsymbol{a}}(\alpha_{\boldsymbol{b}}) - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \alpha_{\boldsymbol{c}}. \tag{2.28}$$

The above expressions extend in the obvious way to higher rank components. Notice, in particular, that

$$\nabla_{\boldsymbol{a}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{d}} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0.$$

Metric connections

Now assume that the connection $\boldsymbol{\nabla}$ is $\boldsymbol{g}$-compatible (i.e. $\boldsymbol{\nabla g} = 0$) and that the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ is $\boldsymbol{g}$-orthogonal; that is, $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}$. It follows then that

$$\nabla_{\boldsymbol{a}} \left(\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) \right) = \boldsymbol{e}_{\boldsymbol{a}}(\eta_{\boldsymbol{bc}}) = 0$$

and that

$$\nabla_{\boldsymbol{a}} \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) = \boldsymbol{g}(\nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) + \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{c}}).$$

Thus, using Equation (2.26) one concludes that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \eta_{\boldsymbol{dc}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} \eta_{\boldsymbol{bd}} = 0. \tag{2.29}$$

Finally, in the case of a Levi-Civita connection and with the choice of a coordinate basis $\{\boldsymbol{\partial}_\mu\}$, the Koszul formula, Equation (2.20), shows that the connection coefficients reduce to the classical expression for the Christoffel symbols:

$$\Gamma_\mu{}^\nu{}_\lambda = \frac{1}{2} g^{\nu\rho} (\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}).$$

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2.6.2 Frame description of the torsion and curvature

Following the spirit of the previous subsections, let

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \boldsymbol{e_a}{}^a \boldsymbol{e_b}{}^b \boldsymbol{\omega^c}{}_c \Sigma_a{}^c{}_b$$

denote the ***components of the torsion tensor*** $\Sigma_a{}^c{}_b$ with respect to $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^a}\}$. Given $f \in \mathfrak{X}(\mathcal{M})$, a short computation shows that

$$\begin{aligned}
\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c}(f) &= \nabla_{\boldsymbol{a}} \boldsymbol{e_b}(f) - \nabla_{\boldsymbol{b}} \boldsymbol{e_a}(f) \\
&= (\boldsymbol{e_a e_b}(f) - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c}(f)) - (\boldsymbol{e_b e_a}(f) - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}} \boldsymbol{e_c}(f)) \\
&= [\boldsymbol{e_a}, \boldsymbol{e_b}](f) - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\, \boldsymbol{e_c}(f),
\end{aligned}$$

where it has been used that $\nabla_{\boldsymbol{a}} f = \boldsymbol{e_a}(f)$. Thus, one obtains that

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c} = [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\, \boldsymbol{e_c}. \tag{2.30}$$

To obtain a frame description of the Riemann curvature tensor one makes use of Equation (2.8) with $u^c = e_{\boldsymbol{d}}{}^c$, and contracts with $e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_d$. One then has that

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b e_{\boldsymbol{d}}{}^d \omega^{\boldsymbol{c}}{}_c R^c{}_{dab}.$$

Furthermore, one can compute

$$\begin{aligned}
e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c \nabla_a \nabla_b e_{\boldsymbol{d}}{}^c &= \omega^{\boldsymbol{c}}{}_c \nabla_{\boldsymbol{a}}(\nabla_{\boldsymbol{b}} e_{\boldsymbol{d}}{}^c) - \omega^{\boldsymbol{c}}{}_c (\nabla_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^b)(\nabla_b e_{\boldsymbol{d}}{}^c), \\
&= \omega^{\boldsymbol{c}}{}_c \nabla_{\boldsymbol{a}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} e_{\boldsymbol{f}}{}^c) - \omega^{\boldsymbol{c}}{}_c \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \nabla_{\boldsymbol{f}} e_{\boldsymbol{d}}{}^c \\
&= \omega^{\boldsymbol{c}}{}_c \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}) e_{\boldsymbol{f}}{}^c + \omega^{\boldsymbol{c}}{}_c \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \nabla_{\boldsymbol{a}} e_{\boldsymbol{f}}{}^c - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \\
&= \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}.
\end{aligned}$$

A similar computation can be carried out for $e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c \nabla_b \nabla_a e_{\boldsymbol{d}}{}^c$ so that one obtains

$$\begin{aligned}
R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e_b}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) \\
+ \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}.
\end{aligned} \tag{2.31}$$

Remark. Equations (2.30) and (2.31) are sometimes known as the ***(Cartan) structure equations***. They can be conveniently expressed in the language of differential forms; see, for example, Frankel (2003); Wald (1984).

2.7 Congruences and submanifolds

The formulation of an initial value problem in general relativity requires the decomposition of tensorial objects in terms of *temporal* and *spatial components*. This decomposition requires, in turn, an understanding of the way geometric structures of the spacetime are inherited by suitable subsets thereof. For concreteness, in what follows a spacetime $(\mathcal{M}, \boldsymbol{g})$ is assumed. Hence $\mathcal{M}$ is a four-dimensional manifold and $\boldsymbol{g}$ denotes a Lorentzian metric.

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2.7.1 Basic notions

Submanifolds

Intuitively, a ***submanifold*** of $\mathcal{M}$ is a set $\mathcal{N} \subset \mathcal{M}$ which inherits a manifold structure from $\mathcal{M}$. A more precise definition of submanifolds requires the concept of embedding. Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, an ***embedding*** is a map $\varphi : \mathcal{N} \to \mathcal{M}$ such that:

(a) The push-forward $\varphi_* : T|_p(\mathcal{N}) \to T|_{\varphi(p)}(\mathcal{M})$ is injective for every point $p \in \mathcal{N}$.
(b) The manifold $\mathcal{N}$ is diffeomorphic to the image $\varphi(\mathcal{N})$.

In terms of the above, one defines a ***submanifold*** $\mathcal{N}$ of $\mathcal{M}$ as the image, $\varphi(\mathcal{S}) \subset \mathcal{M}$, of a k-dimensional manifold $\mathcal{S}$ ($k < 4$) by an embedding $\varphi : \mathcal{S} \to \mathcal{M}$. Often it is convenient to identify $\mathcal{N}$ with $\varphi(\mathcal{S})$ and denote, in an abuse of notation, both manifolds by $\mathcal{N}$. A three-dimensional submanifold of $\mathcal{M}$ is called a ***hypersurface***. In what follows, a generic hypersurface will be denoted by $\mathcal{S}$. As a consequence of its manifold structure, one can associate to $\mathcal{S}$ tangent and cotangent bundles, $T(\mathcal{S})$ and $T^*(\mathcal{S})$ and, more generally, a tensor bundle $\mathfrak{T}^\bullet(\mathcal{S})$.

A vector $\boldsymbol{u}$ (u^i) on $\mathcal{S}$ can be associated to a vector of $\mathcal{M}$ by the push-forward $\varphi_*\boldsymbol{u}$. A vector on $\boldsymbol{v} \in T(\mathcal{M})$ is said to be ***normal*** to $\mathcal{S}$ if $\boldsymbol{g}(\boldsymbol{v}, \varphi_*\boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in T(\mathcal{S})$. If $\epsilon \equiv \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = \pm 1$, one speaks of a ***unit normal vector*** – in this case the surface is said to be ***timelike*** if $\epsilon = -1$ and ***spacelike*** if $\epsilon = 1$. A hypersurface $\mathcal{S}$ of a Lorentzian manifold $\mathcal{M}$ is orientable if and only if there exists a unique smooth normal vector field on $\mathcal{S}$; see, for example, O'Neill (1983).

A natural way of specifying a hypersurface is as the level surface of some function $f \in \mathfrak{X}(\mathcal{M})$. In this case one has that the gradient $\mathbf{d}f \in T^*(\mathcal{M})$ gives rise to a normal vector $(\mathbf{d}f)^\sharp \in T(\mathcal{M})$. The ***unit normal*** of $\mathcal{S}$, $\boldsymbol{\nu}$ (ν_a), is then defined as a unit 1-form in the direction of $\mathbf{d}f$; that is, $\boldsymbol{g}^\sharp(\boldsymbol{\nu}, \boldsymbol{\nu}) = \epsilon$. The normal of $\mathcal{S}$ is defined in the restriction to $\mathcal{S}$ of the cotangent bundle $T^*(\mathcal{M})$. In the case of a spacelike hypersurface, the normal constructed in this way is taken, conventionally, to be future pointing.

Foliations

A ***foliation*** of a spacetime $(\mathcal{M}, \boldsymbol{g})$ is a family, $\{\mathcal{S}_t\}_{t\in\mathbb{R}}$, of *spacelike* hypersurfaces $\mathcal{S}_t$, such that

$$\bigcup_{t\in\mathbb{R}} \mathcal{S}_t = \mathcal{M}, \qquad \mathcal{S}_{t_1} \cap \mathcal{S}_{t_2} = \emptyset \quad \text{for} \quad t_1 \neq t_2.$$

The hypersurfaces $\mathcal{S}_t$ are called the ***leaves*** or ***slices*** of the foliation. The foliation $\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ can be defined in terms of a scalar field $f \in \mathfrak{X}(\mathcal{M})$ such that the leaves of the foliation are level surfaces of f. That is, given $p \in \mathcal{S}_t$, then $f(p) = t$. The scalar field f is said to be a ***time function***. In what follows, it will be convenient

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to identify f and t. The ***normal of a foliation*** is a normalised vector field $\boldsymbol{\nu}$ orthogonal to each leaf of a foliation. The gradient $\mathbf{d}t$ provides a further 1-form normal to the leaves. In general, one has that

$$\boldsymbol{\nu} = N\mathbf{d}t.$$

The proportionality factor N is called the ***lapse*** of the foliation.

Distributions

A ***distribution*** Π is an assignment at each $p \in \mathcal{M}$ of a k-dimensional subspace $\Pi|_p$ of the tangent space $T|_p(\mathcal{M})$. The vector spaces $\Pi|_p$ are called ***hyperplanes*** if their dimension is one less than that of $\mathcal{M}$. A submanifold $\mathcal{N}$ of $\mathcal{M}$ such that $\Pi|_p = T|_p(\mathcal{N})$ for all $p \in \mathcal{N}$ is said to be an ***integrable manifold*** of Π. If for every $p \in \mathcal{M}$ there is an integrable manifold, then Π is said to be ***integrable***. One has the following result (see e.g. Choquet-Bruhat et al. (1982) for details):

Theorem 2.1 (***Frobenius theorem***) *A distribution Π on $\mathcal{M}$ is integrable if and only if for $\boldsymbol{u}$, $\boldsymbol{v} \in \Pi$, one has $[\boldsymbol{u}, \boldsymbol{v}] \in \Pi$.*

The ***projector associated to the distribution*** Π is a tensor field ${h_a}^b$ satisfying ${h_a}^b {h_b}^c = {\delta_a}^c$ such that for $v^a \in \mathfrak{T}(\mathcal{M})$ one has that ${h_a}^b v^a \in \Pi$.

2.7.2 Geometry of congruences

Integral curves

A curve $\gamma : I \to \mathcal{M}$ is the ***integral curve*** of a vector $\boldsymbol{v}$ if the tangent vector of the curve γ coincides with $\boldsymbol{v}$. Standard theorems of the theory of ordinary differential equations – see, for example, Hartman (1987) – ensure that, given $\boldsymbol{v} \in T(\mathcal{M})$, for all $p \in \mathcal{M}$ there exists an interval $I \ni 0$ and a unique integral curve $\gamma : I \to \mathcal{M}$ of $\boldsymbol{v}$ such that $\gamma(0) = p$. If the domain of an integral curve is $\mathbb{R}$, then the integral curve is said to be ***complete***.

Congruences

The notion of a *congruence of geodesics* has been discussed in Section 2.4.5. More generally, a ***congruence of curves*** is the set of integral curves of a (nowhere vanishing) vector field $\boldsymbol{v}$ on $\mathcal{M}$. In the remaining part of this section *it will be assumed that the curves of a congruence are non-intersecting and timelike*. This will be the case of most relevance in this book. In what follows, $\boldsymbol{t}$ will denote the vector field generating a timelike congruence. Without loss of generality it is assumed that $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t}) = 1$.

As in previous sections let $\{\boldsymbol{e_a}\}$ denote a $\boldsymbol{g}$-orthonormal frame. The orthonormal frame can be adapted to the congruence defined by the vector field $\boldsymbol{t}$ by

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setting $\boldsymbol{e_0} = \boldsymbol{t}$. Given a point $p \in \mathcal{M}$, the tangent space $T|_p(\mathcal{M})$ is naturally split in a part tangential to $\boldsymbol{t}$, to be denoted by $\langle \boldsymbol{t} \rangle|_p$ (the one-dimensional subspace spanned by $\boldsymbol{t}$), and a part orthogonal to it which will be denoted by $\langle \boldsymbol{t} \rangle^\perp|_p = \langle \boldsymbol{e_i} \rangle|_p$ (the three-dimensional subspace generated by $\{\boldsymbol{e_i}\}$ with $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$). The space $\langle \boldsymbol{t} \rangle^\perp|_p$ is an example of a hyperplane. One writes then

$$T|_p(\mathcal{M}) = \langle \boldsymbol{t} \rangle|_p \oplus \langle \boldsymbol{t} \rangle^\perp|_p, \tag{2.32}$$

where $\oplus$ denotes the ***direct sum*** of vectorial spaces – that is, any vector in $T|_p(\mathcal{M})$ can be written in a unique way as the sum of an element in $\langle \boldsymbol{t} \rangle|_p$ and an element in $\langle \boldsymbol{t} \rangle^\perp|_p$. Hence, one sees that *the congruence generated by $\boldsymbol{t}$ gives rise to a three-dimensional distribution* Π. At every point $p \in \mathcal{M}$, the subspace $\Pi_p \subset T|_p(\mathcal{M})$ corresponds to $\langle \boldsymbol{e_i} \rangle|_p$; that is, $\{\boldsymbol{e_i}\}$ is a basis of Π_p. In the sequel, $\langle \boldsymbol{t} \rangle$ and $\langle \boldsymbol{t} \rangle^\perp$ will denote, respectively, the *disjoint union* of all the spaces $\langle \boldsymbol{t} \rangle|_p$ and $\langle \boldsymbol{t} \rangle|_p^\perp$, $p \in \mathcal{M}$, and one has that $\Pi = \langle \boldsymbol{t} \rangle^\perp$. The Frobenius theorem, Theorem 2.1, gives the necessary and sufficient conditions for the distribution defined by $\langle \boldsymbol{t} \rangle|_p^\perp$ to be integrable; that is, for the vector $\boldsymbol{t}$ to be the unit normal of a foliation $\{\mathcal{S}_t\}$ of the spacetime.

Making use of $\boldsymbol{g}^\sharp$ one obtains an analogous decomposition for the cotangent space. Namely, one has that

$$T^*|_p(\mathcal{M}) = \langle \boldsymbol{t}^\flat \rangle|_p \oplus \langle \boldsymbol{t}^\flat \rangle^\perp|_p, \tag{2.33}$$

with $\langle \boldsymbol{t}^\flat \rangle^\perp|_p = \langle \boldsymbol{\omega_i} \rangle|_p$. The decompositions (2.32) and (2.33) can be extended in a natural way to higher rank tensors by considering tensor products. Given a tensor T_{ab} with components with respect to the frame $\{\boldsymbol{e_a}\}$ given by $T_{\boldsymbol{ab}}$, one has that $T_{\boldsymbol{ij}} \equiv e_{\boldsymbol{i}}{}^a e_{\boldsymbol{j}}{}^b T_{ab}$ and $T_{\mathbf{00}} \equiv t^a t^b T_{ab}$ correspond, respectively, to the components of T_{ab} ***transversal*** and ***longitudinal*** to $\boldsymbol{t}$; finally, $T_{\mathbf{0}\boldsymbol{i}} \equiv t^a e_{\boldsymbol{i}}{}^b T_{ab}$ and $T_{\boldsymbol{i}\mathbf{0}} \equiv e_{\boldsymbol{i}}{}^a t^b T_{ab}$ are ***mixed transversal-longitudinal*** components.

The covariant derivative of $\boldsymbol{t}$

To further discuss the geometry of the congruence generated by the timelike vector $\boldsymbol{t}$ it is convenient to introduce the ***Weingarten map*** $\boldsymbol{\chi} : \langle \boldsymbol{t} \rangle^\perp \to \langle \boldsymbol{t} \rangle^\perp$ defined by

$$\boldsymbol{\chi}(\boldsymbol{u}) \equiv \nabla_{\boldsymbol{u}} \boldsymbol{t}, \qquad \boldsymbol{u} \in \langle \boldsymbol{t} \rangle^\perp.$$

One can readily verify that

$$\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{\chi}(\boldsymbol{u})) = \boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{u}} \boldsymbol{t}) = \frac{1}{2} \nabla_{\boldsymbol{u}} (\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t})) = 0, \tag{2.34}$$

so that indeed $\boldsymbol{\chi}(\boldsymbol{u}) \in \langle \boldsymbol{t} \rangle^\perp$. Hence, it is enough to consider the Weingarten map evaluated on a basis $\{\boldsymbol{e_i}\}$ of $\langle \boldsymbol{t} \rangle^\perp$. Accordingly, one defines

$$\boldsymbol{\chi_i} \equiv \boldsymbol{\chi}(\boldsymbol{e_i}) = \chi_{\boldsymbol{i}}{}^{\boldsymbol{j}} \boldsymbol{e_j}, \qquad \chi_{\boldsymbol{i}}{}^{\boldsymbol{j}} \equiv \langle \boldsymbol{\omega^j}, \boldsymbol{\chi_i} \rangle.$$

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In the following, it will be more convenient to work with $\chi_{\boldsymbol{ij}} \equiv \eta_{\boldsymbol{jk}}\chi_{\boldsymbol{i}}{}^{\boldsymbol{k}}$. The scalars $\chi_{\boldsymbol{ij}}$ can be considered as the components of a rank 2 covariant tensor on $\boldsymbol{\chi} \in \langle \boldsymbol{t}\rangle^{\perp} \otimes \langle \boldsymbol{t}\rangle^{\perp}$ – the ***Weingarten (or shape) tensor*** of the congruence. The symmetric part $\theta_{\boldsymbol{ij}} \equiv \chi_{(\boldsymbol{ij})}$ and the antisymmetric part $\omega_{\boldsymbol{ij}} \equiv \chi_{[\boldsymbol{ij}]}$ are called the ***expansion*** and the ***twist*** of the congruence, respectively. From $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{e_i}) = 0$ it follows that $\boldsymbol{g}(\nabla_{\boldsymbol{j}}\boldsymbol{t}, \boldsymbol{e_i}) = -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{j}}\boldsymbol{e_i})$. Hence, one can compute

$$\begin{aligned}
\chi_{\boldsymbol{ij}} &= \boldsymbol{g}(\boldsymbol{e_i}, \boldsymbol{\chi_j}) = \boldsymbol{g}(\boldsymbol{e_i}, \nabla_{\boldsymbol{j}}\boldsymbol{t}) = -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{j}}\boldsymbol{e_i}) \\
&= -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{i}}\boldsymbol{e_j} - [\boldsymbol{e_i}, \boldsymbol{e_j}]) = \boldsymbol{g}(\nabla_{\boldsymbol{i}}\boldsymbol{t}, \boldsymbol{e_j}) + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e_i}, \boldsymbol{e_j}]) \\
&= \boldsymbol{g}(\boldsymbol{\chi_i}, \boldsymbol{e_j}) + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e_i}, \boldsymbol{e_j}]) \\
&= \chi_{\boldsymbol{ji}} + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e_i}, \boldsymbol{e_j}]),
\end{aligned}$$

where in the third line it has been used that $\nabla_{\boldsymbol{i}}\boldsymbol{e_i} - \nabla_{\boldsymbol{j}}\boldsymbol{e_i} = [\boldsymbol{e_i}, \boldsymbol{e_j}]$ as $\boldsymbol{\nabla}$ is torsion-free. Hence, by the Frobenius theorem, Theorem 2.1, the symmetry relation $\chi_{\boldsymbol{ij}} = \chi_{\boldsymbol{ji}}$ holds if and only if the distribution $\langle \boldsymbol{t}\rangle^{\perp}$ is integrable. The components $\chi_{\boldsymbol{ij}}$ are related to the connection coefficients of $\boldsymbol{\nabla}$ as can be seen from

$$\chi_{\boldsymbol{i}}{}^{\boldsymbol{j}} = \langle \omega^{\boldsymbol{j}}, \chi_{\boldsymbol{i}}\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{j}}, \nabla_{\boldsymbol{i}}\boldsymbol{e}_0\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{j}}, \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{b}}{}_{\boldsymbol{0}}\boldsymbol{e_b}\rangle = \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}}.$$

Alternatively, one has that

$$\chi_{\boldsymbol{ij}} = \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}}\eta_{\boldsymbol{cj}} = -\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{j}}\eta_{\boldsymbol{c0}} = -\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}},$$

where the last two equalities follow from the metricity of the connection; see Equation (2.29). Now, from $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t}) = 1$, it readily follows that $\boldsymbol{g}(\nabla_{\boldsymbol{a}}\boldsymbol{t}, \boldsymbol{t}) = 0$. Consequently, one has that the ***acceleration*** of the congruence, $\boldsymbol{a} \equiv \nabla_{\boldsymbol{0}}\boldsymbol{t} = \nabla_{\boldsymbol{0}}\boldsymbol{e_0}$, if non-vanishing, must be spatial; that is, $\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{t}) = 0$ so that $\boldsymbol{a} \in \langle \boldsymbol{t}\rangle^{\perp}$. Using the definition of connection coefficients of the connection $\boldsymbol{\nabla}$ it follows that

$$\boldsymbol{a}^{\boldsymbol{i}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{i}}, \boldsymbol{a}\rangle = \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{i}}{}_{\boldsymbol{0}}.$$

2.7.3 Geometry of hypersurfaces

Given a spacetime $(\mathcal{M}, \boldsymbol{g})$ and a hypersurface thereof, $\mathcal{S}$, the embedding $\varphi : \mathcal{S} \to \mathcal{M}$ induces on $\mathcal{S}$ a rank 2 covariant tensor $\boldsymbol{h}$, the ***intrinsic metric*** or ***first fundamental form*** of $\mathcal{S}$ via the pull-back of $\boldsymbol{g}$ to $\mathcal{S}$:

$$\boldsymbol{h} \equiv \varphi^*\boldsymbol{g}.$$

As a consequence of the definition of an embedding, the intrinsic metric $\boldsymbol{h}$ will be non-degenerate if the hypersurface $\mathcal{S}$ is timelike or spacelike. Its signature will be $(+,-,-)$ in the former case and $(-,-,-)$ in the latter. The (unique) Levi-Civita connection of $\boldsymbol{h}$ will be denoted by $\boldsymbol{D}$. Alternatively, one can define the ***pull-back connection***

$$\varphi^*\boldsymbol{\nabla} : T(\mathcal{S}) \times T(\mathcal{S}) \to T(\mathcal{S})$$

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via

$$\varphi_*\big((\varphi^*\nabla)_{\boldsymbol{v}}\boldsymbol{u}\big) \equiv \nabla_{\varphi_*\boldsymbol{v}}(\varphi_*\boldsymbol{u}), \qquad \boldsymbol{u},\, \boldsymbol{v} \in T(\mathcal{S}). \tag{2.35}$$

It can be verified that $\varphi^*\boldsymbol{\nabla}$ as defined above is indeed a linear connection. Given a function $f \in \mathfrak{X}(\mathcal{M})$, the action of $\varphi^*\boldsymbol{\nabla}$ on the pull-back $\varphi^* f$ is defined by

$$(\varphi^*\nabla)_{\boldsymbol{v}}(\varphi^* f) \equiv \varphi^*(\nabla_{\varphi_*\boldsymbol{v}} f) \in T(\mathcal{M}). \tag{2.36}$$

In order to define the action of $\varphi^*\boldsymbol{\nabla}$ on covectors, one requires the Leibnitz rule

$$(\varphi^*\nabla)_{\boldsymbol{v}}\langle\varphi^*\boldsymbol{\omega}, \boldsymbol{u}\rangle = \langle(\varphi^*\nabla)_{\boldsymbol{v}}(\varphi^*\boldsymbol{\omega}), \boldsymbol{u}\rangle + \langle\varphi^*\boldsymbol{\omega}, (\varphi^*\nabla)_{\boldsymbol{v}}\boldsymbol{u}\rangle,$$

for $\boldsymbol{\omega} \in T^*(\mathcal{M})$ and $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{S})$. A calculation using this expression with the definitions (2.35) and (2.36) shows that for $\boldsymbol{\omega} \in T^*(\mathcal{M})$ one has

$$(\varphi^*\nabla)_{\boldsymbol{v}}\varphi^*\boldsymbol{\omega} \equiv \varphi^*(\nabla_{\varphi_*\boldsymbol{v}}\boldsymbol{\omega}).$$

In a natural way, the embedding $\varphi : \mathcal{S} \to \mathcal{M}$ takes the connection $\boldsymbol{\nabla}$ to the connection $\boldsymbol{D}$. More precisely, one has the following result:

Lemma 2.2 *Given* $\boldsymbol{u}$, $\boldsymbol{w} \in T(\mathcal{S})$

$$\varphi_*(D_{\boldsymbol{w}}\boldsymbol{u}) = \nabla_{\varphi_*\boldsymbol{w}}(\varphi_*\boldsymbol{u}). \tag{2.37}$$

Proof Given a function $f \in \mathfrak{X}(\mathcal{M})$ one has that

$$\begin{aligned}
(\varphi^*\nabla)_{\boldsymbol{u}}\left((\varphi^*\nabla)_{\boldsymbol{v}}(\varphi^* f)\right) &= (\varphi^*\nabla)_{\boldsymbol{u}}\left(\varphi^*(\nabla_{\varphi_*\boldsymbol{v}} f)\right) \\
&= \varphi^*\left(\nabla_{\varphi_*\boldsymbol{u}}\nabla_{\varphi_*\boldsymbol{v}} f\right) \\
&= \varphi^*\left(\nabla_{\varphi_*\boldsymbol{v}}\nabla_{\varphi_*\boldsymbol{u}} f\right) \\
&= (\varphi^*\nabla)_{\boldsymbol{v}}\left((\varphi^*\nabla)_{\boldsymbol{u}}(\varphi^* f)\right),
\end{aligned}$$

where to pass from the second to the third line it has been used that the connection $\boldsymbol{\nabla}$ is torsion-free. One thus concludes that the connection $\varphi^*\boldsymbol{\nabla}$ is indeed torsion free. Finally, it can be readily verified that one has compatibility with the metric $\boldsymbol{h}$. Indeed,

$$(\varphi^*\nabla)_{\boldsymbol{v}}\boldsymbol{h} = (\varphi^*\nabla)_{\boldsymbol{v}}(\varphi^*\boldsymbol{g}) = \varphi^*(\nabla_{\varphi_*\boldsymbol{v}}\boldsymbol{g}) = 0,$$

where the last equality follows from the $\boldsymbol{g}$-compatibility of the connection $\boldsymbol{\nabla}$. As $\varphi^*\boldsymbol{\nabla}$ is torsion-free and $\boldsymbol{h}$-compatible, it follows from the fundamental theorem of Riemannian geometry that it must coincide with the connection $\boldsymbol{D}$. In other words, one has that $\varphi^*\boldsymbol{\nabla} = \boldsymbol{D}$, as given in Equation (2.37). □

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A frame formalism on hypersurfaces

The present discussion of the geometry of hypersurfaces is valid for both the spacelike and timelike case. To accommodate these two possibilities, all throughout, the following conventions concerning frame indices will be used: if the hypersurface is *timelike* so that $\epsilon = 1$, the frame indices $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}, \ldots$ take the values $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$; if the hypersurface is *spacelike* so that $\epsilon = -1$, the indices $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}, \ldots$ take the values $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$.

Following the conventions given in the previous paragraph, let $\{\boldsymbol{e_i}\} \subset T(\mathcal{S})$ denote a triad of $\boldsymbol{h}$-orthogonal vectors. If $\mathcal{S}$ is spacelike one has that $\boldsymbol{h}(\boldsymbol{e_i}, \boldsymbol{e_j}) = -\delta_{\boldsymbol{ij}}$, while in the timelike case $\boldsymbol{h}(\boldsymbol{e_i}, \boldsymbol{e_j}) = \mathrm{diag}(1, -1, -1)$. Using the push-forward $\varphi_* : T(\mathcal{S}) \to T(\mathcal{M})$ one obtains the vectors $\varphi_* \boldsymbol{e_i}$ defined on the restriction of $T(\mathcal{M})$ to $\mathcal{S}$. The triad $\{\boldsymbol{e_i}\}$ can be naturally extended to a tetrad $\{\boldsymbol{e_a}\}$ on the restriction of $T(\mathcal{M})$ to $\mathcal{S}$ by setting $\boldsymbol{e_0} = \boldsymbol{\nu}^\sharp$ in the spacelike case and $\boldsymbol{e_3} = \boldsymbol{\nu}^\sharp$ in the timelike case. In order to discuss these two cases simultaneously, the notation $\boldsymbol{e}_\perp$ will be used. Similarly, the notation $\boldsymbol{\omega}^\perp$ will be used to denote the normal element of the coframe, that is, $\boldsymbol{\omega}^{\mathbf{0}}$ or $\boldsymbol{\omega}^{\mathbf{3}}$. Given $\boldsymbol{v}$ and $\boldsymbol{\alpha}$ on the restriction of $T(\mathcal{M})$ and $T^*(\mathcal{M})$ to $\mathcal{S}$, their components along the normal will be denoted by $v^\perp$ and $\alpha_\perp$, respectively.

To simplify the presentation, the notation $\boldsymbol{e_i}$ will often be used to denote both the vectors of $T(\mathcal{S})$ and their push-forward to $T(\mathcal{M})$. The appropriate point of view should be clear from the context. In the cases where confusion may arise, it is convenient to make use of abstract index notation: given $\boldsymbol{e_i} \in T(\mathcal{S})$, we shall write $e_{\boldsymbol{i}}{}^i$; its push-forward $\varphi_* \boldsymbol{e_i} \in T(\mathcal{M})$ will be denoted by $e_{\boldsymbol{i}}{}^a$. Similarly, $\boldsymbol{\omega^i} \in T^*(\mathcal{M})$ will be written as $\omega^{\boldsymbol{i}}{}_a$, while the pull-back $\varphi^* \boldsymbol{\omega^i} \in T^*(\mathcal{S})$ will be denoted by $\omega^{\boldsymbol{i}}{}_i$. Given $\boldsymbol{u} \in T(\mathcal{S})$, one has that $u^{\boldsymbol{i}} \equiv \langle \varphi^* \boldsymbol{\omega^i}, \boldsymbol{u} \rangle = \langle \boldsymbol{\omega^i}, \varphi_* \boldsymbol{u} \rangle$. Written in index notation $u^a \omega^{\boldsymbol{i}}{}_a = u^i \omega^{\boldsymbol{i}}{}_i$; that is, the (spatial) components of $\boldsymbol{u}$ and its push-forward $\varphi_* \boldsymbol{u}$ coincide.

As a consequence of the existence of two covariant derivatives, one also has two sets of directional covariant derivatives. Firstly, acting on spacetime objects, $\nabla_{\boldsymbol{a}} = e_{\boldsymbol{a}}{}^a \nabla_a$, so that in particular $\nabla_{\boldsymbol{i}} = e_{\boldsymbol{i}}{}^a \nabla_a$. Secondly, acting on hypersurface-defined objects, one has $D_{\boldsymbol{i}} = e_{\boldsymbol{i}}{}^i D_i$. The connection coefficients of $\boldsymbol{D}$ with respect to $\{\boldsymbol{e_i}\}$ are given by $\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \equiv \langle \boldsymbol{\omega^j}, D_{\boldsymbol{i}} \boldsymbol{e_k} \rangle$. Now, given $\boldsymbol{u} \in T(\mathcal{S})$ and $\boldsymbol{\alpha} \in T^*(\mathcal{S})$ and defining

$$D_{\boldsymbol{i}} u^{\boldsymbol{j}} \equiv \langle \boldsymbol{\omega^j}, D_{\boldsymbol{i}} \boldsymbol{u} \rangle, \qquad D_{\boldsymbol{i}} \alpha_{\boldsymbol{j}} \equiv \langle D_{\boldsymbol{i}} \boldsymbol{\alpha}, \boldsymbol{e_j} \rangle,$$

one has, by analogy to Equation (2.28), that

$$D_{\boldsymbol{i}} u^{\boldsymbol{j}} = \boldsymbol{e_i}(u^{\boldsymbol{j}}) + \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} u^{\boldsymbol{k}}, \qquad D_{\boldsymbol{i}} \alpha_{\boldsymbol{j}} = \boldsymbol{e_i}(\alpha_{\boldsymbol{j}}) - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} \alpha_{\boldsymbol{k}}.$$

To investigate relations between the directional covariant derivatives $\nabla_{\boldsymbol{i}}$ and $D_{\boldsymbol{i}}$ one makes use of the formula (2.37) with $\boldsymbol{w} = \boldsymbol{e_i}$, $\boldsymbol{u} = \boldsymbol{e_j}$ so that $\varphi_*(D_{\boldsymbol{i}} \boldsymbol{e_j}) = \nabla_{\boldsymbol{i}}(\varphi_* \boldsymbol{e_j}) = \nabla_{\boldsymbol{i}} \boldsymbol{e_j}$ – the last equality given in a slight abuse of notation as, strictly speaking, $\nabla_{\boldsymbol{i}}$ acts on spacetime objects. From the definition of connection coefficients one has that

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$$\begin{aligned}\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} &= \langle \boldsymbol{\omega}^{\boldsymbol{j}}, \nabla_{\boldsymbol{i}} \boldsymbol{e}_{\boldsymbol{k}} \rangle = \langle \boldsymbol{\omega}^{\boldsymbol{j}}, \varphi_*(D_{\boldsymbol{i}} \boldsymbol{e}_{\boldsymbol{k}}) \rangle \\ &= \langle \varphi^* \boldsymbol{\omega}^{\boldsymbol{j}}, D_{\boldsymbol{i}} \boldsymbol{e}_{\boldsymbol{k}} \rangle = \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}. \end{aligned} \tag{2.38}$$

Given a *spatial* vector $\boldsymbol{u} \in T(\mathcal{M})$ (i.e. $u^{\perp} \equiv n_a u^a = 0$) and recalling that $\nabla_{\boldsymbol{a}} u^{\boldsymbol{b}} \equiv e_{\boldsymbol{a}}{}^a \omega^{\boldsymbol{b}}{}_b \nabla_a u^b$, using Equation (2.28) one has that

$$\nabla_{\boldsymbol{a}} u^{\boldsymbol{b}} = \boldsymbol{e}_{\boldsymbol{a}}(u^{\boldsymbol{b}}) + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{k}} u^{\boldsymbol{k}}. \tag{2.39}$$

Restricting the free frame indices in the above expression and using (2.38) one finds

$$\begin{aligned}\nabla_{\boldsymbol{i}} u^{\boldsymbol{j}} &= \boldsymbol{e}_{\boldsymbol{i}}(u^{\boldsymbol{j}}) + \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} u^{\boldsymbol{k}} \\ &= \boldsymbol{e}_{\boldsymbol{i}}(u^{\boldsymbol{j}}) + \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} u^{\boldsymbol{k}} = D_{\boldsymbol{i}} u^{\boldsymbol{j}}. \end{aligned}$$

The intrinsic curvature tensors on the hypersurface

In order to describe the intrinsic curvature of the submanifold $\mathcal{S}$, one considers the ***three-dimensional Riemann curvature tensor*** $r^k{}_{lij}$ of the Levi-Civita connection $\boldsymbol{D}$ of the intrinsic metric $\boldsymbol{h}$. Given $\boldsymbol{v} \in T(\mathcal{S})$, and recalling that $\boldsymbol{D}$ is torsion-free, one has by analogy to Equation (2.8) that

$$D_i D_j v^k - D_j D_i v^k = r^k{}_{lij} v^l.$$

As $r^k{}_{lij}$ is the Riemann tensor of a Levi-Civita connection one has the symmetries

$$\begin{aligned} &r_{klij} = r_{[kl]ij} = r_{kl[ij]} = r_{[kl][ij]}, \\ &r_{klij} = r_{ijkl}, \qquad r_{k[lij]} = 0. \end{aligned}$$

In what follows, let $r_{lj} \equiv r^k{}_{lkj}$ and $r \equiv h^{lj} r_{lj}$ denote, respectively, the Ricci tensors and scalars of $\boldsymbol{D}$. It is convenient to also consider the ***trace-free part of the three-dimensional Ricci tensor*** s_{ij} and the ***three-dimensional Schouten tensor*** l_{ij} given by

$$s_{ij} \equiv r_{\{ij\}} = r_{ij} - \frac{1}{3} r h_{ij}, \qquad l_{ij} \equiv s_{ij} + \frac{1}{12} r h_{ij}.$$

The three-dimensionality of the submanifold $\mathcal{S}$ leads to the decomposition

$$r_{klij} = 2h_{k[i} l_{j]l} + 2h_{l[j} l_{i]k}. \tag{2.40}$$

A computation using the above expressions shows that the second Bianchi identity $D_{[i} r_{jk]lm} = 0$ takes, in this case, the form

$$D^i s_{ij} = \frac{1}{6} D_j r.$$

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Given the $\boldsymbol{h}$-orthogonal triad $\{\boldsymbol{e_i}\}$ and its associated coframe basis $\{\boldsymbol{\omega^i}\}$ one defines the components $r^{\boldsymbol{k}}{}_{\boldsymbol{lij}} \equiv e_{\boldsymbol{i}}{}^i e_{\boldsymbol{j}}{}^j \omega^{\boldsymbol{k}}{}_k e_{\boldsymbol{l}}{}^l r^k{}_{lij}$. A computation similar to that leading to Equation (2.31) yields

$$\begin{aligned} r^{\boldsymbol{k}}{}_{\boldsymbol{lij}} &= \boldsymbol{e_i}(\gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) - \boldsymbol{e_j}(\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) + \gamma_{\boldsymbol{m}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}(\gamma_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{i}} - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{j}}) \\ &\quad + \gamma_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}} - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}}. \end{aligned} \tag{2.41}$$

Moreover, the definition of the torsion tensor implies:

$$[\boldsymbol{e_i}, \boldsymbol{e_j}] = \big(\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} - \gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{i}}\big)\boldsymbol{e_k}. \tag{2.42}$$

Remark. Equations (2.41) and (2.42) are the *three-dimensional analogue of the (Cartan) structure Equations* (2.30) and (2.31).

Extrinsic curvature

The discussion in Section 2.7.2 concerning the Weingarten map can be specialised to the case of the tangent space of a hypersurface. This leads to the notion of ***extrinsic curvature*** or ***second fundamental form*** of the hypersurface $\mathcal{S}$. The latter is defined via the map $\boldsymbol{K} : T(\mathcal{S}) \times T(\mathcal{S}) \to \mathbb{R}$ given by

$$\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) \equiv \langle \nabla_{\boldsymbol{u}} \boldsymbol{\nu}, \boldsymbol{v} \rangle = \boldsymbol{g}(\nabla_{\boldsymbol{u}} \boldsymbol{\nu}^\sharp, \boldsymbol{v}). \tag{2.43}$$

From the discussion of the Weingarten map it follows that $\boldsymbol{K}$ as defined above is a symmetric three-dimensional tensor. In abstract index notation the latter will be written as K_{ij}.

Now, given an orthonormal frame $\{\boldsymbol{e_i}\}$ on $\mathcal{S}$ and choosing $\boldsymbol{v} = \boldsymbol{e_i}$ and $\boldsymbol{u} = \boldsymbol{e_j}$ in formula (2.43) one finds that the components $K_{\boldsymbol{ij}}$ are given by

$$K_{\boldsymbol{ij}} = \nabla_{\boldsymbol{i}} \nu_{\boldsymbol{j}} \equiv \langle \nabla_{\boldsymbol{j}} \boldsymbol{\nu}, \boldsymbol{e_i} \rangle = \langle \nabla_{\boldsymbol{j}} \boldsymbol{\omega}^\perp, \boldsymbol{e_i} \rangle \tag{2.44}$$

so that, comparing Equation (2.44) with the definition of the connection coefficients one finds that

$$\begin{aligned} K_{\boldsymbol{ij}} &= \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{a}}{}_\perp \eta_{\boldsymbol{aj}} \\ &= -\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{a}}{}_{\boldsymbol{j}} \eta_{\boldsymbol{a}\perp} = -\epsilon \Gamma_{\boldsymbol{i}}{}^\perp{}_{\boldsymbol{j}}. \end{aligned} \tag{2.45}$$

Now, looking again at Equation (2.39) and setting $\boldsymbol{a} \mapsto \boldsymbol{i}$, $\boldsymbol{b} \mapsto \perp$ one obtains

$$\begin{aligned} \nabla_{\boldsymbol{i}} v^\perp &= \boldsymbol{e_i}(v^\perp) + \Gamma_{\boldsymbol{i}}{}^\perp{}_{\boldsymbol{k}} v^{\boldsymbol{k}} \\ &= \Gamma_{\boldsymbol{i}}{}^\perp{}_{\boldsymbol{k}} v^{\boldsymbol{k}} = -\epsilon K_{\boldsymbol{ik}} v^{\boldsymbol{k}}, \end{aligned} \tag{2.46}$$

as $\boldsymbol{v} \in T(\mathcal{S})$ so that $v^\perp = 0$.

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The Gauss-Codazzi and Codazzi-Mainardi equations

The curvature tensors of the connections $\boldsymbol{\nabla}$ and $\boldsymbol{D}$ are related to each other by means of the ***Gauss-Codazzi equation***

$$R_{\boldsymbol{ijkl}} = r_{\boldsymbol{ijkl}} + K_{\boldsymbol{ik}}K_{\boldsymbol{jl}} - K_{\boldsymbol{il}}K_{\boldsymbol{jk}}, \tag{2.47}$$

and the ***Codazzi-Mainardi equation***

$$R_{\boldsymbol{i}\perp\boldsymbol{jk}} = D_{\boldsymbol{j}}K_{\boldsymbol{ki}} - D_{\boldsymbol{k}}K_{\boldsymbol{ji}}. \tag{2.48}$$

The proof of the Gauss-Codazzi equation follows by considering the commutator of $\boldsymbol{\nabla}$, Equation (2.8), acting on the frame vectors $\boldsymbol{e_l}$:

$$\nabla_a\nabla_b e_{\boldsymbol{l}}{}^c - \nabla_b\nabla_a e_{\boldsymbol{l}}{}^c = R^c{}_{dab}e_{\boldsymbol{l}}{}^d \equiv R^c{}_{\boldsymbol{l}ab}.$$

Contracting the previous equation with $e_{\boldsymbol{i}}{}^a e_{\boldsymbol{j}}{}^b \omega^{\boldsymbol{k}}{}_c$, and using

$$\nabla_b e_{\boldsymbol{l}}{}^c = \omega^{\boldsymbol{b}}{}_b \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{l}} e_{\boldsymbol{a}}{}^c,$$

together with formulae (2.38) and (2.46) and the expression for the components of the three-dimensional Riemann tensor in terms of the connection coefficients, Equation (2.41), yields (2.47). The proof of the Codazzi-Mainardi Equation (2.48) involves less computation. In this case one evaluates the commutator of covariant derivatives on the covector $\boldsymbol{\nu}$. Contracting with $e_{\boldsymbol{i}}{}^a e_{\boldsymbol{j}}{}^b e_{\boldsymbol{k}}{}^c$ one readily finds that

$$\nabla_{\boldsymbol{i}}\nabla_{\boldsymbol{j}}\nu_{\boldsymbol{k}} - \nabla_{\boldsymbol{j}}\nabla_{\boldsymbol{i}}\nu_{\boldsymbol{k}} = -R^{\perp}{}_{\boldsymbol{kij}},$$

where $R^{\perp}{}_{\boldsymbol{kij}} \equiv R^d{}_{cab}\nu_d e_{\boldsymbol{k}}{}^c e_{\boldsymbol{i}}{}^a e_{\boldsymbol{j}}{}^b$. Now, using Equation (2.44) one finds that

$$\nabla_{\boldsymbol{i}}K_{\boldsymbol{jk}} - \nabla_{\boldsymbol{j}}K_{\boldsymbol{ik}} = -R^{\perp}{}_{\boldsymbol{kij}}.$$

Formula (2.48) follows from the above expression by noticing that $\nabla_{\boldsymbol{i}}K_{\boldsymbol{jk}} = D_{\boldsymbol{i}}K_{\boldsymbol{jk}}$ as $K_{\boldsymbol{jk}}$ corresponds to the spatial components of a spatial tensor.

A remark concerning foliations

The discussion in the previous subsections was restricted to a single hypersurface $\mathcal{S}$. However, it can be readily extended to a foliation $\{\mathcal{S}_t\}$. In this case the contravariant version of the normal $\boldsymbol{\nu}^\sharp$ and the unit vector $\boldsymbol{t}$ generating the congruence coincide. Moreover, one has a distribution which is integrable so that the Weingarten tensor $\boldsymbol{\chi} \in \big(\langle\boldsymbol{t}\rangle^\perp \otimes \langle\boldsymbol{t}\rangle^\perp\big)|_p$, for $p \in \mathcal{M}$ can be identified with the second fundamental form $\boldsymbol{K} \in T|_p(\mathcal{S}_{t(p)}) \otimes T|_p(\mathcal{S}_{t(p)})$ where $t(p) \in \mathbb{R}$ is the only value of the time function such that $p \in \mathcal{S}_{t(p)}$. In particular one has that $\chi_{ij} = \chi_{(ij)}$.

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2.8 Further reading

There is a vast choice of books on differential geometry ranging from introductory texts to comprehensive monographs. An introductory discussion geared towards applications in general relativity can be found in the first chapter of Stewart (1991) or the second and third chapters of Wald (1984). A more extensive introduction with broader applications in physics is Frankel (2003). A more advanced discussion, again aimed at applications in physics, is the classical textbook by Choquet-Bruhat et al. (1982). A systematic and coherent discussion of the theory from a modern mathematical point of view covering topological manifolds, smooth manifolds and differential geometry can be found in Lee (1997, 2000, 2002). A more concise alternative to the latter three books is given in Willmore (1993). A monograph on Lorentzian geometry with applications to general relativity is O'Neill (1983). Readers who like the style of this reference will also find the brief summary of differential geometry given in the first chapter of O'Neill (1995) useful. The present discussion of differential geometry has avoided the use of the language of fibre bundles. Readers interested in the latter are referred to Taubes (2011).

Books on numerical relativity like Baumgarte and Shapiro (2010) and Alcubierre (2008) also provide introductions to the 3+1 decomposition of general relativity. In these references, the reader will encounter an approach to this topic based on the so-called *projection formalism.* A more detailed discussion, also aimed at numerical relativity, can be found in Gourgoulhon (2012).

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