ON A CONVERGENCE TEST OF HARDY-LITTLEWOOD'S TYPE FOR FOURIER SERIES

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(received 28 December 1961, revised 27 March 1963)

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Let $\varphi(t)$ be an even function integrable in the Lebesgue sense and periodic with period 2π. Let

$$\varphi(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt.$$

Write

$$\Phi(t) = \int_0^t \varphi(u)du.$$

By an indirect method based on the method of Riesz summability for the Fourier series, the author has established the following convergence test for the oscillating series $\sum a_n$. The theorem is as follows:

THEOREM A [1]. *If, for some* $\Delta > 0$

(i) $$\Phi(t) = o\left\{t\left(\log \frac{1}{t}\right)^{-\Delta}\right\}$$

as $t \to +0$ *and*

(ii) $$a_n > -Kn^{-1} (\log n)^{\Delta},$$

then $\sum a_n$ *converges to the sum* $s = 0$. *Here* K *is an absolute constant independent of* n.

In this note, we intend to show that it is essential to use the same Δ in the conditions (i) and (ii) of the theorem; i.e., we prove the following

THEOREM. *For each* $\Delta > 0$ *and each* $\eta > 0$, *there exists an even function* $\varphi(t)$ *satisfying* (i), *with its Fourier series diverging at* $t = 0$, *and such that*

(iii) $$a_n = o\left\{\frac{(\log n)^{\Delta+\eta}}{n}\right\}.$$

* The author expresses his hearty thanks to the referee for his elaborate help in simplifying the calculations of this paper and rewriting it in the present compact form.

https://doi.org/10.1017/S1446788700022734 Published online by Cambridge University Press

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Let $\{\lambda_i\}$ be a strictly increasing sequence of positive integers such that:

(a) $\lambda_i \to \infty$ as $i \to \infty$; and

(b) there exists a constant k and an i with

$$\frac{\lambda_i}{\lambda_{i+1}} < k < 1$$

for all $i \geqq i_0$.

Then

$$\frac{\lambda_i}{\lambda_n} < k^{n-i}$$

for all $i_0 \leqq i < n$.

Let $\Delta > 0$ and define

$$\alpha_i = \frac{(\log \lambda_i)^\Delta}{\lambda_i}.$$

It can easily be seen that there exists an i' such that $\alpha_{i'} < \pi$, $\alpha_i < \alpha_{i-1}$ for all $i \geqq i'+1$ and

$$\frac{\lambda_i}{\lambda_{i+1}} < k$$

for all $i \geqq i'-1$.

Let $\{c_i\}$ be a sequence of non-negative real numbers tending to zero as limit. Next, we define an even function:

$$\varphi(t) = \begin{cases} c_i \sin \lambda_i t & (\alpha_i \leqq t < \alpha_{i-1};\ i \geqq i'+1), \\ 0 & (\alpha_{i'} < t \leqq \pi). \end{cases}$$

We are going to prove that the Fourier series of $\varphi(t)$ diverges at the point $t = 0$ though $\varphi(t)$ and its Fourier coefficients (a_n) satisfy (i) and (iii) respectively. Denote by S_{λ_n} the λ_n-th partial sum of the series $a_0/2 + \sum_1^\infty a_n$, then

$$S_{\lambda_n} = \frac{2}{\pi} \int_0^\pi \varphi(t) \frac{\sin \lambda_n t}{t}\, dt + o(1).$$

Substitute the function $\varphi(t)$ defined above into this integral and write it in the form:

$$\begin{aligned} S_{\lambda_n} &= \frac{2}{\pi} \left\{ \sum_{i'+1 \leqq i < n} + \sum_{i > n} \right\} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t}\, dt + \frac{2}{\pi} c_n \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t}\, dt + o(1) \\ &= \frac{2}{\pi} \{\textstyle\sum_1 + \sum_2\} + \frac{2}{\pi} I + o(1). \end{aligned}$$

We require a number of lemmas.

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2.1 Lemma. *We have*

$$\sum\nolimits_1 \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Since, for every $i' \leqq i < n$, $\lambda_i/\lambda_n < k^{n-i} < k < 1$, thus we have

$$\sum\nolimits_1 = \frac{1}{\lambda_n} O\left\{\sum_{i'<i<n} \frac{c_i}{\alpha_i}\right\}.$$

Take $\varepsilon > 0$, and choose p sufficiently large that $p > i'+1$ and $c_i < \varepsilon$ for all $i > p$. Then

$$\begin{aligned}\sum_{i'<i<n} \frac{c_i}{\alpha_i} &= \sum_{i'<i\leqq p} \frac{c_i}{\alpha_i} + \sum_{p<i<n} \frac{c_i}{\alpha_i} \\ &\leqq O(1)+\varepsilon \sum_{p<i<n} \frac{\lambda_i}{(\log \lambda_i)^{\Delta}} \\ &\leqq O(1)+\varepsilon\lambda_n \sum_{p<i<n} \frac{\lambda_i}{\lambda_n} \\ &\leqq O(1)+\varepsilon\lambda_n \sum_{r=1}^{\infty} k^r.\end{aligned}$$

Thus $\sum_1 \to 0$ as $n \to \infty$.

2.2. Lemma. *We have*

$$\sum\nolimits_2 \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Integrating by parts,

$$\int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt = -\left[\frac{\cos \lambda_i t}{\lambda_i} \cdot \frac{\sin \lambda_n t}{t}\right]_{\alpha_i}^{\alpha_{i-1}} + \frac{\lambda_n}{\lambda_i} \int_{\alpha_i}^{\alpha_{i-1}} \left(\cos \lambda_n t - \frac{\sin \lambda_n t}{\lambda_n t}\right) \frac{\cos \lambda_i t}{t} dt,$$

and noticing that the integrand in the second term of the right side is $O(\alpha_i^{-1})$, we obtain immediately

$$\begin{aligned}\sum\nolimits_2 &= \lambda_n O\left\{\sum_{i>n} \frac{c_i \alpha_{i-1}}{\alpha_i \lambda_i}\right\} \\ &= \lambda_n O\left\{\sum_{i>n} \frac{1}{\lambda_{i-1}}\right\} \\ &= o(1).\end{aligned}$$

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2.3. Lemma. *If* $\alpha_{n-1}/\alpha_n \to \infty$ *as* $n \to \infty$, *then*

$$\limsup_{n\to\infty} I = \tfrac{1}{2} \limsup_{n\to\infty} c_n \log \frac{\alpha_{n-1}}{\alpha_n}.$$

Proof. We have

$$\int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t}\, dt = \int_{\alpha_n}^{\alpha_{n-1}} \frac{1-\cos 2\lambda_n t}{2t}\, dt$$

and substitute $t = \alpha_n u$, we obtain

$$\int_{\alpha_n}^{\alpha_{n-1}} \frac{\cos 2\lambda_n t}{t}\, dt = \int_1^{\alpha_{n-1}/\alpha_n} \frac{\cos 2\lambda_n \alpha_n u}{u}\, du$$

$$= \int_1^{p_n} \frac{\cos q_n u}{u}\, du$$

(where $p_n \to \infty$, $q_n = (\log \lambda_n)^{\Delta} \to \infty$ as $n \to \infty$)

$$= \int_1^{r_n} \cos q_n u\, du$$

(by the second mean-value theorem, where $1 < r_n < p_n$)

$$= \frac{1}{q_n} \{\sin q_n r_n - \sin q_n\} \to 0$$

as $n \to \infty$, since $q_n \to \infty$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} I = \lim_{n\to\infty} c_n \int_{\alpha_n}^{\alpha_{n-1}} \frac{1}{2t}\, dt = \tfrac{1}{2} \limsup c_n \log \frac{\alpha_{n-1}}{\alpha_n}.$$

2.4. Lemma.

$$\frac{1}{t}\left(\log \frac{1}{t}\right)^{\Delta} \int_0^t \varphi(u)du \to 0$$

as $t \to 0$.

Proof. For $\alpha_n < t \leqq \alpha_{n-1}$,

$$\frac{1}{t}\left(\log \frac{1}{t}\right)^{\Delta} \int_0^t \varphi(u)du = O\left\{\frac{1}{\alpha_n}\left(\log \frac{1}{\alpha_n}\right)^{\Delta} \sum_{i\geqq n} \frac{c_i}{\lambda_i}\right\}$$

$$= O\left\{\frac{1}{\lambda_n \alpha_n}\left(\log \frac{1}{\alpha_n}\right)^{\Delta} \sum_{i\geqq n} c_i k^{i-n}\right\}$$

$$= O\left\{\left(\frac{\log 1/\alpha_n}{\log \lambda_n}\right)^{\Delta} \sum_{i\geqq n} c_i k^{i-n}\right\}$$

$$= O\left\{\left(\frac{\log 1/\alpha_n}{\log 1/\alpha_n + \log (\lambda_n \alpha_n)}\right)^{\Delta} \sum_{i\geqq n} c_i k^{i-n}\right\}$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

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2.5. LEMMA. *If $\eta > \eta' > 0$, if $\lambda_i/(\lambda_{i-1}+1) \leqq i^{\eta'}$ for all $i \geqq 2$ and if the series $\sum_i c_i$ converges, then*

$$\frac{na_n}{(\log n)^{\Delta+\eta}} \to 0$$

as $n \to \infty$.

PROOF.

$$a_n = \frac{2}{\pi} \sum_{i=i'+1}^{\infty} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos nt\, dt,$$

hence

$$\begin{aligned} \frac{na_n}{(\log n)^{\Delta+\eta}} &= \frac{2n}{\pi(\log n)^{\Delta+\eta}} \sum_{\substack{\lambda_n \leqq 2n,\\ i>i'}} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos nt\, dt \\ &\quad + \frac{2n}{\pi (\log n)^{\Delta+\eta}} \sum_{\lambda_i > 2n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos nt\, dt \\ &= A_n + B_n. \end{aligned}$$

But

$$\begin{aligned} |A_n| &\leqq \frac{2n}{\pi (\log n)^{\Delta+\eta}} \sum_{\lambda_i \leqq 2n} c_i \left| \left[\frac{\sin \lambda_i t \sin nt}{n} \right]_{\alpha_i}^{\alpha_{i-1}} \right. \\ &\qquad \left. - \frac{\lambda_i}{n} \int_{\alpha_i}^{\alpha_{i-1}} \cos \lambda_i t \sin nt\, dt \right| \\ &\leqq \frac{2n}{\pi (\log n)^{\Delta+\eta}} \sum c_i \left(\frac{2}{n} + \frac{\lambda_i \alpha_{i-1}}{n} \right) \\ &= \frac{2}{\pi (\log n)^{\Delta+\eta}} \sum c_i \left\{ 2 + \frac{\lambda_i}{\lambda_{i-1}} (\log \lambda_{i-1})^{\Delta} \right\} \\ &\leqq \frac{2}{\pi (\log n)^{\Delta+\eta}} \sum c_i \left\{ 2 + \frac{\lambda_i}{\lambda_{i-1}} (\log 2n)^{\Delta} \right\} \\ &\leqq \frac{2 (\log 2n)^{\Delta}}{\pi (\log n)^{\Delta+\eta}} \sum K c_i i^{\eta'} \end{aligned}$$

for large n.

Let $\lambda_{p(n)} \leqq 2n < \lambda_{p(n)+1}$. Then

$$\begin{aligned} |A_n| &\leqq \frac{2(\log 2n)^{\Delta}}{\pi(\log n)^{\Delta+\eta}} \sum K \{p(n)\}^{\eta'} \\ &\leqq K' \left(1 + \frac{\log 2}{\log n} \right)^{\Delta} \frac{\{p(n)\}^{\eta'}}{(\log n)^{\eta}}. \end{aligned}$$

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But

$$\log n + \log 2n \geqq \log \lambda_{p(n)}$$
$$= \log \frac{\lambda_{p(n)}}{\lambda_{i'}} + \log \lambda_{i'}$$
$$\geqq \{p(n) - i'\} \log \frac{1}{K} + \log \lambda_{i'},$$

hence

$$|A_n| \leqq K' \left(1 + \frac{\log 2}{\log n}\right)^{\Delta} \cdot \frac{\{p(n)\}^{\eta'}}{\{(p(n) - i') \log 1/K + \log \lambda_{i'} - \log 2\}^{\eta}}$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

Also

$$|B_n| = \left| \frac{2n}{\pi(\log n)^{\Delta+\eta}} \sum_{\lambda_i > 2n} \frac{c_i}{2} \left[-\frac{\cos(\lambda_i + n)t}{\lambda_i + n} - \frac{\cos(\lambda_i - n)t}{\lambda_i - n} \right]_{\alpha_i}^{\alpha_{i-1}} \right|$$
$$\leqq \frac{4n}{\pi(\log n)^{\Delta+\eta}} \sum_{\lambda_i > 2n} \frac{c_i}{\lambda_i - n}$$
$$\leqq \frac{4}{\pi(\log n)^{\Delta+\eta}} \sum c_i$$
$$\to 0$$

as $n \to \infty$.

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Let $\eta > 0$. Take η' such that $0 < \eta' < \eta$ and define

$$\lambda_i = [(i!)^{\eta'}].$$

Then $\lambda_i/\lambda_{i+1} \to 0$ as $i \to \infty$, hence k exists. Also, $\lambda_i \leqq (i!)^{\eta'}$ and $\lambda_{i-1}+1 \geqq \{(i-1)!\}^{\eta'}$, so that

$$\frac{\lambda_i}{\lambda_{i-1}+1} \leqq i^{\eta'}.$$

Now,

$$\frac{\alpha_{i-1}}{\alpha_i} = \left(\frac{\log \lambda_{i-1}}{\log \lambda_i}\right)^{\Delta} \frac{\lambda_i}{\lambda_{i-1}} \cong i^{\eta} \to \infty$$

as $i \to \infty$, since

$$\log \lambda_i \cong \eta' \log i! \cong \eta'(i+\tfrac{1}{2}) \log i \to \infty$$

as $i \to \infty$ by Stirling's approximation formula for $i!$. Define

$$e_i = \left(\log \frac{\alpha_{i-1}}{\alpha_i}\right)^{-\frac{1}{2}}$$

Then $e_i \to 0$ as $i \to \infty$. Let $c_i = e_i$ for an infinite number of i's, but put $c_i = 0$ for enough values of i to make $\sum c_i$ converge. Thus by 2.3

$$\limsup_{n\to\infty} I = \infty,$$

so that by 2.1 and 2.2, the Fourier series of φ diverges at $t = 0$. By 2.4, (i) is satisfied and it follows from 2.5 that (iii) is satisfied. Thus the theorem is completely established.

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https://doi.org/10.1017/S1446788700022734 Published online by Cambridge University Press