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JOINS OF ALMOST SUBNORMAL SUBGROUPS

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Following (1) we say that a subgroup H of a group G is almost subnormal in G if H is of finite index in some subnormal subgroup of G, or, equivalently, if $|H_n : H|$ is finite for some n, where H_n is the n-th term of the normal closure series of H in G. The aim of this article is to prove, in answer to a question of R. Baer, the following analogue of the well known result of Roseblade and Stonehewer (3) that in any group the join of a pair of finitely generated subnormal subgroups is always subnormal:

Theorem A. *In any group the join of a pair of finitely generated almost subnormal subgroups is almost subnormal.*

It follows at once, of course, that in any group the join of finitely many finitely generated almost subnormal subgroups is almost subnormal.

In order to prove Theorem A we need an analogue of the fact, established by Robinson in (2) that the join of a permutable pair of subnormal subgroups of a group is almost subnormal.

Theorem B. *The join of a permutable pair of almost subnormal subgroups of any group is almost subnormal.*

In what follows we shall abbreviate 'H is of finite index in K' to 'H fi K', 'finitely generated' to 'f.g.' and 'subnormal' to 'sn'.

Proof of Theorem B. Suppose that H and K are almost subnormal subgroups of a group G and that $HK = KH = J$, say. Then for some n, m we have H fi H_n and K fi K_m. By (3, Theorem D) there is a subnormal subgroup X of G such that

$$J = HK \leqslant X \leqslant H_nK_m.$$

But X is subnormal in G so that both H_r and K_r are contained in X for sufficiently large r. Thus if we assume, as we may, that $H_n = H_{n+1}$ and $K_m = K_{m+1}$ we have at once that $X = H_nK_m$.

It is enough to prove that J fi H_nK_m and in order to do this we may clearly assume that $G = H_nK_m = X$. Theorem B is then a direct consequence of the

Lemma. *If $G = H_nK_m$, where H fi H_n and K fi K_m, and if $J = HK = KH$, then J fi G.*

Proof. We proceed by induction on n. If $n = 1$ then H fi $H_1 \lhd G$ and hence HK fi H_1K fi H_1K_m, from which the result follows. Suppose that $n > 1$. Now $H_1 =$

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$H_n(K_m \cap H_1)$ and $H_1 \cap (HK) = H(K \cap H_1)$. Also H fi H_n and $K \cap H_1$ fi $K_m \cap H_1$ sn H_1. Hence by the natural induction hypothesis we have that

$$H(K \cap H_1) \text{ fi } H_n(K_m \cap H_1) = H_1.$$

Now the normal closure of $H(K \cap H_1)$ in G is H_1 so that if we put $I = H(K \cap H_1)$ we have that I fi I_1 and $G = I_1K_m$ and the case $n = 1$ yields $J = IK$ fi G, as required.

Proof of Theorem A. Suppose that H and K are f.g. subgroups of a group G, that H fi H_n and K fi K_m. We wish to show that $J = \langle H, K \rangle$ is almost subnormal in G.

We proceed by induction on n.

If $n = 1$, then H fi $H_1 \triangleleft G$ and H_1K_m sn G. Now H_1 is f.g. so that there is a characteristic subgroup of H_1 which is contained in H and is of finite index in H_1. Thus we may assume that H_1 is finite. Hence K_m has finite index in H_1K_m, therefore so have K and J. Thus J is almost subnormal in G.

Suppose that $n > 1$ and assume the natural induction hypothesis on n. Clearly $K \leq K_m \cap J$ sn J. By (3, Lemma 5) we have that $J = H^*(K_m \cap J)$, where H^* is generated by finitely many conjugates $H^{k_1}, \ldots, H^{k_r}$, of H under K. Working inside H_1 and applying the induction hypothesis on n, a simple induction on r yields that H^* is almost subnormal in H_1 and therefore H^* is almost subnormal in G.

But now J is the product of the permutable almost subnormal subgroups H^* and $K_m \cap J$ and we may apply Theorem B to produce the desired result.

A counterexample. In (2, Theorem 6.1) Robinson has given an example of a group G which is a split extension of an infinite abelian group M by a group J where J is the join of a pair of subnormal subgroups H and K of G and where $J^G = G$. Thus H and K are trivially almost subnormal in G. However if J were almost subnormal in G then we would have from the condition $J^G = G$ that J fi G. But this would contradict the fact that M is infinite. Hence J is not almost subnormal in G and this demonstrates that the join of almost subnormal subgroups is not in general almost subnormal. Robinson also shows how to embed his group G as a normal subgroup of a f.g. group and hence we have that the join of almost subnormal subgroups of a f.g. group is not in general almost subnormal.

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