PRIME POWER REPRESENTATIONS OF FINITE LINEAR GROUPS II

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1. Introduction. The aim of this paper is two-fold: first, to extend the results of **(4)** to the exceptional finite Lie groups recently discovered by Chevalley **(1)**, and, secondly, to give a construction which works simultaneously for the groups A_n, B_n, C_n, D_n, E_n, F_4 and G_2 (in the usual Lie group notation), and which depends only on intrinsic structural properties of these groups. It seems likely that the statements of this paper, especially (1) to (14) upon which the later work is based, also hold for the other known simple linear groups, namely the unitary and second orthogonal groups **(4)**.

Throughout this paper, the phrase "finite Lie group" and the symbol L refer to any of the groups in the first list above. We lean heavily on the properties of these groups developed by Chevalley **(1)**, and use his notations, slightly modified. The symbols u, h, etc., always denote elements of the groups U, H, etc., respectively, these groups being defined in §2.

2. Basic properties of finite Lie groups. In this section we set forth the properties of the groups $G = L$ to be used in the sequel. Statements (1) to (10) are extracted from **(1)**, and the others are proved in §4.

(1) *There exist two subgroups U and H of G such that $U \cap H = 1$, $U H$ is a group, and U is normal in $U H$* (**1**, *p.* 40, Lemma 9).

(2) *There exists a group W (the Weyl group) and for each $w \in W$ an element of G which is also denoted by w such that the union of the sets $H w$ is a group, H is a normal subgroup, and the quotient group is the isomorphic image of W under the map $w \to H w$* (**1**, p. 37 Lemma 3).

(3) *Corresponding to each $w \in W$, U has two subgroups U_w' and U_w* (this is U_w'' in **(1)**) *such that*

(4) $U = U_w' U_w$,

(5) $w U_w' w^{-1} \subseteq U$ *and*

(6) $U_{w_0} = U$ *for some* $w_0 \in W$ (**1**, pp. 41–43).

(7) *G is the union of the sets $U H w U_w$, and*

$$u\, h\, w_1\, u_{w_1} = \bar{u}\, \bar{h}\, w_2\, u_{w_2}$$

implies that $u = \bar{u}$, $h = \bar{h}$, $w_1 = w_2$, and

$$u_{w_1} = u_{w_2}$$

(**1**, p. 42, Theorem 2).

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(8) *W contains a set of elements $\{w_\alpha\}$ (the fundamental reflections) such that*

(9) *$w_\alpha^2 = 1$ for each α and*

(10) *$\{w_\alpha\}$ generates W* (**3**, p. 16–05, Theorem 1).

(11) *For each α, if $w = w_\alpha$, we set $U_w = U_\alpha$ and $U_w' = U_\alpha'$; then the union of $U_\alpha H$ and $U_\alpha H w_\alpha U_\alpha$ is a group.*

(12) *For each w and α, at least one of*

$$U_\alpha \subseteq U_w', \; U_\alpha \subseteq U_{ww_\alpha}'$$

holds.

(13) *There is a homomorphism ϵ of W onto the group $\{1, -1\}$ of two elements such that $\epsilon(w_\alpha) = -1$ for each α.*

(14) *There is an element u such that $u \notin U_w'$ for all $w \neq 1$.*

These properties are not independent. For example, the condition $U \cap H = 1$ of (1) follows from (7), and (9) follows from (4), (5), (7) and (11). The complete list is given here for ready reference in what follows.

3. The ideal I and the representation R. In this section, we assume that G is a group for which (1) to (14) hold. Let F be any field and A the group algebra of G over F. For any subset S of G, let the symbol S also denote the sum of the members of S considered as elements of A, and $|S|$ the cardinality of S. Let e be the element of A defined by

$$e = U H \Sigma \epsilon(w)\, w, \tag{15}$$

the summation being over the elements of W. Then we can state the following fundamental result.

LEMMA 1. *Let e be defined by* (15). *Then* (i) $e\, w_\alpha = -e$; (ii) *if $u_\alpha \neq 1$, then*

$$w_\alpha u_\alpha w_\alpha^{-1} = \tilde{u}_\alpha h\, w_\alpha \bar{u}_\alpha \tag{16}$$

for some $\tilde{u}_\alpha$, $\bar{u}_\alpha$ and h, and

$$e\,(u_\alpha w_\alpha^{-1} - \bar{u}_\alpha + 1) = 0. \tag{17}$$

Proof. Equation (i) follows from (2) and (13). By (7), $w_\alpha u_\alpha \notin U_\alpha H w_\alpha$. Thus $w_\alpha u_\alpha w_\alpha^{-1} \notin U_\alpha H$, and (11) implies (16). By (9) and (12), each right coset of W relative to the group $\{1, w_\alpha\}$ consists of two elements v and vw_α such that $U_\alpha \subseteq U_v'$. Hence, by (1), (2), (5), (9), (13) and (16), we get

$$U H v\, u_\alpha w_\alpha^{-1} = U H v\, w_\alpha, \tag{18}$$

$$U H v = U H v\, \bar{u}_\alpha \text{ and} \tag{19}$$

$$U H v\, w_\alpha u_\alpha w_\alpha^{-1} = U H v\, w_\alpha \bar{u}_\alpha. \tag{20}$$

Now to get (17), multiply (18), (19), and (20) by $\epsilon(v)$, $\epsilon(v)$ and $\epsilon(vw_\alpha)$, add, and then sum on the right cosets of W relative to the group $\{1, w_\alpha\}$.

THEOREM 1. *Let e be given by* (15), *and let I be the right ideal of A generated by e. Then the set $B = \{e\,u, u \in U\}$ is a vector space basis for I. The dimension of I is $|U|$.*

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Proof. For a fixed u, by (4), (5), (6), and (7), the coset $U\,H\,w_0\,u$ is present in $e\,u$ with a coefficient $\epsilon(w_0)$ and is not present in $e\,u_1$ if $u_1 \neq u$. Thus B is linearly independent. Let I_0 be the linear span of B. Then $e \in I_0 \subseteq e\,A = I$. Thus $I_0 = I$ if it can be shown that I_0 is an ideal. For any u and h, $I_0\,u \subseteq I_0$ and $I_0\,h \subseteq I_0$, by (1) and (2). Also for any u and α, by (4) we can write $u = u_\alpha\,u_\alpha'$ and then $e\,u\,w_\alpha^{-1} = (e\,u_\alpha\,w_\alpha^{-1})\ (w_\alpha\,u_\alpha'\,w_\alpha^{-1}) \in I_0$, using Lemma 1 and (5); hence $I_0\,w_\alpha^{-1} \subseteq I_0$. By (2), (7), and (10), A is generated by elements of the form u, h and w_α^{-1}. Thus $I_0\,A \subseteq I_0$ and Theorem 1 is proved.

COROLLARY. *If $s \in I$ and $s = \Sigma\,\gamma_u\,e\,u$, $\gamma_u \in F$, then γ_u is $\epsilon(w_0)$ times the coefficient of $U\,H\,w_0\,u$ in the right coset decomposition of s relative to $U\,H$, and $\Sigma\,\gamma_u$ is the coefficient of $U\,H$. If $x \in G$ and $e\,x = \Sigma\,\gamma_u\,e\,u$, then $\gamma_u = 1, -1$ or 0, and at most $|W|$ of the γ_u are non-zero.*

Proof. The first statement follows from (6), (7) and Theorem 1. Since e is the signed sum of $|W|$ right cosets relative to $U\,H$, the same is true of $e\,x$, and the last statement now follows from the first.

LEMMA 2. *Let m be the index of $U\,H$ in G: $m = \Sigma|U_w|$. Then*

$$e\;U\;\Sigma\;\epsilon\;(w)\;w = m\;e. \tag{21}$$

Proof. For each $x \in G$, write $e\,x = \Sigma\,\gamma(x;\,u)\;e\,u$ so that

$$e = \Sigma\;\gamma(x;\;u)\;\;e\,u\,x^{-1}. \tag{22}$$

Here $\gamma(x;\,u)$ is $\epsilon(w\,w_0)$ if

$$U\,H\,w_0\,u = U\,H\,w\,x \tag{23}$$

for some w and is 0 otherwise, by the corollary to Theorem 1. Now (23) is equivalent to $u\,x^{-1} \in w_0^{-1}\;U\,H\,w$. Thus summation of (22) on $x \in G$ gives

$$|G|\;e = \Sigma_u\;\Sigma_w\epsilon(w\,w_0)\;e\,w_0^{-1}\;U\,H\,w = \Sigma_u e\;U\,H\;\Sigma_w\epsilon(w)\;w = |U\,H|e\;U\;\Sigma\epsilon(w)w$$

by (1), (2) and (13). If the base field F has characteristic 0, division by $|\,U\,H\,|$ yields (21). Since only integral coefficients occur in (21), it remains valid for any field.

We now state the principal result of this section.

THEOREM 2. *Let G be a group for which* (1) *to* (13) *hold. Let I and B be as in Theorem* 1, *let m be the index of $U\,H$ in G, and let R be the representation of G on I by right multiplication. Then*

(i) *R restricted to U is equivalent to the right regular representation of U; the degree of R is $|\,U\,|$;*

(ii) *relative to B, R is represented by a set of matrices each of which has only entries of* 1, -1 *or* 0 *with at most $|\,W\,|$ non-zero entries in each row;*

(iii) *if the base field F has characteristic* 0 *or prime to m, then R is irreducible.*

Proof. Theorem 1 and its corollary imply (i) and (ii). We prove (iii) by showing that any element $s \neq 0$ of I generates all of I under right multiplica-

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tion by the elements of A, the linear span of G. If $s \neq 0$, there is a coset $U\,H\,x$ represented in s with a coefficient $\gamma \neq 0$. Then $U\,H$ has the coefficient γ in $s\,x^{-1}$. By the corollary to Theorem 1, $s\,x^{-1} = \Sigma\,\gamma_u\,e\,u$ with $\Sigma\,\gamma_u = \gamma$. Hence, by (1) and Lemma 2,

$$s\,x^{-1}\,(\gamma\,m)^{-1}\,U\,\Sigma\,\epsilon(w)\,w = (\gamma\,m)^{-1}\,\Sigma_u\,\gamma_u\,e\,u\,U\,\Sigma_w\,\epsilon(w)\,w = e.$$

Thus $I = e\,A \subseteq s\,A \subseteq I$, and (iii) is proved.

We use the condition (14) for the first time in the proof of the following converse to Theorem 2.

THEOREM 3. *In the notation of Theorem* 2, *if* (1) *to* (14) *hold for G and if the characteristic of F divides m, then R is reducible.*

Proof. By (6) and (7), $e\,U \neq 0$ so that $e\,U$ generates a non-zero invariant subspace I_1 of I. We complete the proof by showing that $I_1 \neq I$ or, equivalently, that $e \notin I_1$. Lemma 2, with the elements of G replaced by their inverses, implies, because of (1), (2), and (13), that $\Sigma\,\epsilon(w)\,w$ is a left annihilator of $e\,U$ and hence of I_1. But it is not a left annihilator of e. Indeed we show that an element u for which (14) holds appears in $\Sigma\,\epsilon(w)\,w\,e$ with a coefficient 1. Now $u \in w_1\,U\,H\,w_2$ implies that $w_1^{-1}\,u \in U\,H\,w_2$. By (2) (7), and (14), $w_1 = 1$ and then $w_2 = 1$. Again by (2) and (7), u can be written uniquely as $1\,\bar{u}\,\bar{h}\,1$, so that all assertions are proved.

4. Representations of finite Lie groups. In order to apply the results of §3 to the groups L, we first prove:

LEMMA 3. *Each group L has properties* (1) *to* (14).

Proof. As has already been stated in §2, (1) to (10) are extracted from **(1)**. In the proof of (11) to (14), we use the standard Lie algebra terminology **(2)**, and note that the index set $\{\alpha\}$ of (8) refers to a simple system of positive roots and the w_α are the reflections in these roots. It is proved in **(3**, p. 19–01, Lemma 1**)** that w_α maps α upon $-\alpha$ and permutes the remaining positive roots. Then (11) follows by **(1**, pp. 36, 41**)**. Also

$$(w\,w_\alpha)(\alpha) = w(-\alpha) = -w(\alpha).$$

Thus either $(w\,w_\alpha)(\alpha)$ or $w(\alpha)$ is positive, and (12) is true. For each w, let $n(w)$ be the number of roots r such that $r > 0$ and $w(r) < 0$. Then $\epsilon(w) = (-1)^{n(w)}$ fulfils the requirements of (13). Finally, if $w \neq 1$, then $w(\alpha) < 0$ for at least one α **(3**, p. 16–08, Theorem 2**)**, and so, by the same reasoning as above, $U_w' \subseteq U_\alpha'$ for some α, and similarly, $U_\beta \subseteq U_\gamma'$ if $\beta \neq \gamma$. Thus, if we choose $u_\alpha \neq 1$ for each α and set $u = \Pi u_\alpha$, then $u \notin U_\alpha'$ for each α, hence $u \notin U_w'$ for each $w \neq 1$, and (14) is proved.

If the group L is defined over a field of q elements and of characteristic p, it is further proved in **(1)** that U can be chosen as a p-Sylow subgroup of L,

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UH as the normalizer of U, and then $|U_w| = q^{n(w)}$, with $n(w)$ as above. Thus we have:

THEOREM 4. *Let L be a finite Lie group over a field of q elements and of characteristic p. Let U be a p-Sylow subgroup of L, and let m be the index of the normalizer of U in L, this number being given by $m = \Sigma q^{n(w)}$. Let I be any vector space of dimension $|U|$ over a field of characteristic 0 or prime to m. Then L has an irreducible representation R of degree $|U|$ which has I as the representation space and for which the results of Theorem 2 are valid.*

REFERENCES

1. C. Chevalley, *Sur certains groupes simples*, Tohoku Math. J., *7* (1955), 14–66.
2. E. B. Dynkin, *The structure of semi-simple algebras*, Amer. Math. Soc. Translation No. 17.
3. Séminaire "Sophus Lie" (Paris, 1954–5).
4. R. Steinberg, *Prime power representations of finite linear groups*, Can. J. Math., *8* (1956), 580–591.

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