BULL. AUSTRAL. MATH. SOC. 05B25
VOL. 19 (1978), 351-362.

A translation plane of order 25 and its full collineation group

M.L. Narayana Rao and K. Kuppuswamy Rao

Ostrom proposed classifications of translation planes on the basis of the action of the collineation group of the plane on the ideal points. There are examples of translation planes in which ideal points form a single orbit (flag transitive planes) and also several orbits (Hall, André, Foulser, and so forth, planes). In this paper the authors have constructed a translation plane in which the ideal points are divided into two orbits of lengths 18 and 8 respectively. A few collineations are computed together with their actions. The group of collineations G_1 which is transitive on the two sets of 18 and 8 lines separately is calculated. All the collineations that fix L_0 are also calculated and they form a group G_3. If G_2 is the group of translations then the full collineation group is shown to be $\langle G_1, G_2, G_3\rangle$.

A translation plane of order 25 is constructed which has the interesting property that its ideal points are divided into two orbits of lengths 18 and 8 respectively. Its full collineation group is computed.

1.

Ostrom proposed classification of translation planes on the basis of the action of the collineation group of the plane on the ideal points. The

Received 22 September 1978.

restriction of the collineation group of a plane to the ideal points may result in a single orbit or several orbits. The two flag transitive planes of order 25 by Foulser [3], the flag transitive plane of order 49, and a class of flag transitive planes of order q^2, q a prime power by one of the authors [5], [6], the flag transitive plane of order 27 of Hering [4], and a new flag transitive plane of order 27 by the authors [9] are some examples of planes in which all the ideal points form a single orbit. The other known translation planes are such that the ideal points form several orbits. Recently the authors constructed a new class of non-desargusian planes of order q^2, q a prime power with the property that they all admit a collineation group of order (q^2-1) [8].

2.

Let F be the set of all ordered pairs (a, b) over GF(5) and C the set of 2×2 matrices (Table 1) forming a t-spread set so that they satisfy:

(i) C contains $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(ii) C contains 25 matrices; and

(iii) if $M, N \in C$ and $M \neq N$, then $|M-N| \neq 0$ where $|X|$ denotes the determinant of the matrix X.

These conditions imply that corresponding to each ordered pair (a, b) in F, there is exactly one matrix of the form $\begin{pmatrix} a & b \\ p & q \end{pmatrix}$, which is denoted by $M(a, b)$. Addition and multiplication in F are defined by

$$(a, b) + (c, d) = (a+c, b+d),$$

$$(a, b) \cdot (c, d) = (c, d)M(a, b).$$

The set F with addition and multiplication defined as above is a left Veblen-Wedderburn system [1].

The projective plane π has (c), (a, d), and (∞) as points and $[k]$, $[m, b]$, and $[\infty]$ as lines where a, b, c, d, m, and $k \in F$, and $\infty \notin F$. Incidence in π is defined by $(x, y)I[m, b]$, if and only if $y = mx + b$, $(x, y)I[k]$, if and only if $x = k$.

TABLE 1

L_i	C	Action of α $L_i \to L_j$	Action of β $L_i \to L_m$	Action of γ $L_i \to L_n$	Action of δ $L_i \to L_k$
L_0	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	L_0	L_1	L_6	L_0
L_1	∞	L_1	L_2	L_{11}	L_5
L_2	$\begin{pmatrix}4&0\\0&4\end{pmatrix}$	L_2	L_3	L_{10}	L_4
L_3	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	L_3	L_4	L_9	L_3
L_4	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	L_4	L_5	L_8	L_2
L_5	$\begin{pmatrix}3&0\\0&3\end{pmatrix}$	L_5	L_0	L_7	L_1
L_6	$\begin{pmatrix}0&2\\2&4\end{pmatrix}$	L_{14}	L_7	L_0	L_7
L_7	$\begin{pmatrix}1&4\\4&4\end{pmatrix}$	L_{15}	L_8	L_5	L_6
L_8	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	L_{16}	L_9	L_4	L_{11}
L_9	$\begin{pmatrix}4&3\\3&0\end{pmatrix}$	L_{17}	L_{10}	L_3	L_{10}
L_{10}	$\begin{pmatrix}4&1\\1&1\end{pmatrix}$	L_{12}	L_{11}	L_2	L_9
L_{11}	$\begin{pmatrix}0&4\\4&3\end{pmatrix}$	L_{13}	L_6	L_1	L_8
L_{12}	$\begin{pmatrix}2&3\\1&3\end{pmatrix}$	L_7	L_{13}	L_{12}	L_{17}
L_{13}	$\begin{pmatrix}2&2\\4&1\end{pmatrix}$	L_8	L_{14}	L_{17}	L_{16}
L_{14}	$\begin{pmatrix}1&1\\2&3\end{pmatrix}$	L_9	L_{15}	L_{16}	L_{15}
L_{15}	$\begin{pmatrix}3&2\\4&2\end{pmatrix}$	L_{10}	L_{16}	L_{15}	L_{14}
L_{16}	$\begin{pmatrix}1&3\\1&2\end{pmatrix}$	L_{11}	L_{17}	L_{14}	L_{13}
L_{17}	$\begin{pmatrix}3&4\\3&1\end{pmatrix}$	L_6	L_{12}	L_{13}	L_{12}
L_{18}	$\begin{pmatrix}3&3\\2&1\end{pmatrix}$	L_{21}	L_{18}	L_{18}	L_{23}
L_{19}	$\begin{pmatrix}0&3\\1&4\end{pmatrix}$	L_{24}	L_{19}	L_{19}	L_{25}

Table 1 (continued)

L_i	C	Action of α $L_i \to L_j$	Action of β $L_i \to L_m$	Action of γ $L_i \to L_n$	Action of δ $L_i \to L_k$
L_{20}	$\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$	L_{18}	L_{20}	L_{20}	L_{24}
L_{21}	$\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$	L_{25}	L_{21}	L_{23}	L_{22}
L_{22}	$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$	L_{19}	L_{22}	L_{22}	L_{21}
L_{23}	$\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$	L_{22}	L_{23}	L_{21}	L_{18}
L_{24}	$\begin{pmatrix} 4 & 4 \\ 2 & 0 \end{pmatrix}$	L_{23}	L_{24}	L_{25}	L_{20}
L_{25}	$\begin{pmatrix} 4 & 2 \\ 4 & 0 \end{pmatrix}$	L_{20}	L_{25}	L_{24}	L_{19}

Alternatively π may also be considered as a four dimensional vector space over GF(5), the points of π being quadruples over GF(5) and the lines being two dimensional subspaces of V. The line corresponding to the equation $y = m \cdot x$ with m in F is given by

$$V_m = \{(a, b, c, d) \mid (a, b) \in F \text{ and } (c, d) = (a, b)M(m)\} ,$$

where $M(m)$ is the matrix from C corresponding to m. The line $x = 0$ corresponds to the subspace

$$V_\infty = \{(0, 0, c, d) \mid (c, d) \in F\} .$$

The line $y = m \cdot x + b$ corresponds to the appropriate translates of V_m for m in F or $m = \infty$. The group G_0 of all collineation fixing (0, 0) of π consists of all non-singular linear transformations of V which permute the subspaces V_m for m in F or $m = \infty$ among themselves.

Let R be a non-singular linear transformation partitioned as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are 2×2 matrices over GF(5). It is known that the non-singular linear transformation induces a collineation on π if and only if for each $M(m)$ there is a unique $N \in C$ such that $(A+MC)N = B + MD$ and a unique $T \in C$ such that $CT = D$. It is further known that if R induces a collineation then the matrices A, B, C, and D are zero matrices or non-singular.

3.

In this section we calculate some collineations of π.

LEMMA 3.1. *A linear transformation of the form* $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ *induces a collineation in* π *if and only if the set* C *is invariant under the mapping* $M \to A^{-1}MA$ *for* $M \in C$. *Further*

$$A \in \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix} \right\rangle .$$

Proof. Let $T = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ be a non-singular transformation and let $M, N \in C$. The vector space $\{(xy, xyM) \mid (x, y) \in F\}$ is transformed into the vector space $\{(xyA, xyMA) \mid (x, y) \in F\}$ by T. This will be identical with the vector space $\{(xy, xyN) \mid (x, y) \in F\}$ if and only if $AN = MA$ or $N = A^{-1}MA$. Hence the lemma. It may also be noted that under T the line corresponding to the matrix $M|L(M)$ is mapped onto the line with matrix $N|L(N)$.

The set C contains exactly 4 matrices with determinant 3 and trace 0. These are $\left\{\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\right\}$. This subset of C must be invariant under T. The action of T on $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ determines the action of T on the other two matrices because the other two are scalar multiples of these two matrices. Thus we need to consider A whose action on $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ is as follows:

(i) $A\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}A$ and $A\begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix}A$;

(ii) $A\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}A$ and $A\begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}A$;

(iii) $A\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}A$ and $A\begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix}A$;

(iv) $A\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}A$ and $A\begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}A$;

https://doi.org/10.1017/S000497270000890X Published online by Cambridge University Press

(v) $A\begin{pmatrix}2 & 3\\1 & 3\end{pmatrix} = \begin{pmatrix}1 & 4\\4 & 4\end{pmatrix}A$ and $A\begin{pmatrix}1 & 4\\4 & 4\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 3\end{pmatrix}A$;

(vi) $A\begin{pmatrix}2 & 3\\1 & 3\end{pmatrix} = \begin{pmatrix}1 & 4\\4 & 4\end{pmatrix}A$ and $A\begin{pmatrix}1 & 4\\4 & 4\end{pmatrix} = \begin{pmatrix}3 & 2\\4 & 2\end{pmatrix}A$;

(vii) $A\begin{pmatrix}2 & 3\\1 & 3\end{pmatrix} = \begin{pmatrix}4 & 1\\1 & 1\end{pmatrix}A$ and $A\begin{pmatrix}1 & 4\\4 & 4\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 3\end{pmatrix}A$;

(viii) $A\begin{pmatrix}2 & 3\\1 & 3\end{pmatrix} = \begin{pmatrix}4 & 1\\1 & 1\end{pmatrix}A$ and $A\begin{pmatrix}1 & 4\\4 & 4\end{pmatrix} = \begin{pmatrix}3 & 2\\4 & 2\end{pmatrix}A$.

These eight equations give the forms of A to be

$$\begin{pmatrix}a & 0\\0 & a\end{pmatrix}, \begin{pmatrix}a & 4a\\3a & 4a\end{pmatrix}, \begin{pmatrix}a & 4a\\4a & 4a\end{pmatrix}, \begin{pmatrix}a & 0\\2a & 4a\end{pmatrix}, \begin{pmatrix}a & 4a\\0 & 4a\end{pmatrix}, \begin{pmatrix}a & 0\\4a & 2a\end{pmatrix}, \begin{pmatrix}a & 0\\3a & 3a\end{pmatrix}, \begin{pmatrix}a & 4a\\2a & 4a\end{pmatrix},$$

where $a = 1, 2, 3$, and 4 .

Thus

$$A \in \left\langle \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\4 & 2\end{pmatrix}, \begin{pmatrix}1 & 4\\2 & 4\end{pmatrix} \right\rangle .$$

LEMMA 3.1. *Let* $A = \begin{pmatrix}1 & 0\\3 & 3\end{pmatrix}$ *and let* $\alpha = \begin{pmatrix}A & 0\\0 & A\end{pmatrix}$. *Let* $P = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $Q = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$, $R = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$, *and* $\beta = \begin{pmatrix}0 & P\\Q & R\end{pmatrix}$. *Let* $X = \begin{pmatrix}1 & 4\\4 & 4\end{pmatrix}$, $Y = \begin{pmatrix}3 & 3\\3 & 4\end{pmatrix}$, $Z = \begin{pmatrix}2 & 1\\1 & 4\end{pmatrix}$, $S = \begin{pmatrix}4 & 1\\1 & 1\end{pmatrix}$, *and* $\gamma = \begin{pmatrix}X & Y\\Z & S\end{pmatrix}$. *Then the actions of* α, β , *and* γ *are given by*

$$\alpha = (6, 14, 9, 17)(7, 15, 10, 12)(8, 16, 11, 13)(18, 21, 25, 20)(19, 24, 23, 22) ,$$

$$\beta = (0, 1, 2, 3, 4, 5)(6, 7, 8, 9, 10, 11)(12, 13, 14, 15, 16, 17) ,$$

$$\gamma = (0, 6)(1, 11)(2, 10)(3, 9)(4, 8)(5, 7)(13, 17)(14, 16)(21, 23)(24, 25) ,$$

where $(r, s, \ldots)$ *indicate that the ideal point corresponding to the line* L_r *is mapped onto the ideal point corresponding to the line* L_s .

Proof. The proof is clear from Table 1. Further the group $\langle\alpha, \beta, \gamma\rangle$ is transitive on lines L_i , $0 \le i \le 17$ and L_j , $18 \le j \le 25$, separately.

4.

In this section we wish to investigate *all* the collineations that fix L_0 .

LEMMA 4.1. $\delta = \begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$ *is a collineation fixing* L_0 . *Further, the action of* δ *is given by*

δ : (0)(3)(1, 5)(2, 4)(6, 7)(8, 11)(9, 10)(12, 17)(13, 16)
(14, 15)(18, 23)(19, 25)(20, 24)(21, 22) .

Proof. If δ maps L_i with a matrix M onto L_j with matrix N then $N = (I+3M)^{-1}\cdot 4M$. The proof of the lemma follows from Table 1.

LEMMA 4.2. *Any collineation that fixes* L_0 *and* L_1 *fixes* L_2, L_3, L_4 , *and* L_5 *also.*

Proof. Any collineation fixing L_0 and L_1 is of the form $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A and B are non-singular 2×2 matrices. If T maps the line with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ onto a line with matrix M_1 then $B = AM_1$, so that $T = \begin{pmatrix} A & 0 \\ 0 & AM_1 \end{pmatrix}$ where $M_1 \in C$. Similarly if M_2, M_3 , and M_4 are the images of the lines with matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, and $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ respectively under T , then $AM_i = iAM_1$, $i = 1, 2, 3, 4$. This implies that iM_1 for $i = 1, 2, 3, 4$ are matrices in C . But from the table, $M_i = \mu I$, $\mu = 1, 2, 3, 4$. Then $T = \begin{pmatrix} A & 0 \\ 0 & \mu A \end{pmatrix}$, $\mu = 1, 2, 3, 4$. Further, if T maps $L(M)$ onto $L(N)$, then $AN = \mu MA$ or $N = \mu A^{-1}MA$. Then $|N| = \mu^2|M|$. If μ is either 2 or 3 then a line with a matrix whose determinant is σ is mapped onto a line with determinant 4σ . This is not possible since there are 8 matrices with determinant 2 and 4 matrices with determinant 3 . If $\mu = 4$ then the line with matrix $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$ will be mapped onto a line with matrix M where $AMA^{-1} = \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix} \notin C$. Thus

https://doi.org/10.1017/S000497270000890X Published online by Cambridge University Press

Thus $\mu = 1$ and $T = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ which fixes L_2, L_3, L_4, and L_5.

LEMMA 4.3. *$T = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}$ is a collineation only if either*

(a) $A = 0$, $B = I$, or

(b) $A \neq 0$, $A = -N^{-1}$, $B = -N^{-1}M$ where $N, M \in C$.

In that case $N + M = I$.

Proof. Let $T = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}$ be a collineation. If $A = 0$, then $B = I$ follows from Lemma 4.2. So let $A \neq 0$. Then there exist matrices M and N in C such that a line with a matrix N is mapped onto L_1 which in turn is mapped onto a line with a matrix M. Then $I + NA = 0$ and $AM = B$. Therefore $A = -N^{-1}$ and $B = -N^{-1}M$. Then $T = \begin{pmatrix} I & 0 \\ -N^{-1} & -N^{-1}M \end{pmatrix}$.

Suppose $T_1 = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$ is a collineation with $A = -N^{-1}$ for some $N \in C$ and $I = \left(-N^{-1}\right)(-N)$ implies $4N \in C$. Thus if $T_1 = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$ is a collineation, then N and $4N \in C$. Let T_1, T_1^2, T_1^3 map $4N$ onto lines with matrices M_2, M_3, and M_4, respectively. Then

$$\left(I+4N\left(-N^{-1}\right)\right)M_2 = 4N \Rightarrow M_2 = 2N ,$$

$$\left(I+2N\left(-N^{-1}\right)\right)M_3 = 2N \Rightarrow M_3 = 3N ,$$

and

$$\left(I+3N\left(-N^{-1}\right)\right)M_4 = 3N \Rightarrow M_4 = N .$$

Thus N, $2N$, $3N$, and $4N$ all belong to C. Then $T_1 = \begin{pmatrix} I & 0 \\ \mu I & I \end{pmatrix}$ where $\mu = 1, 2, 3, 4$. Then we can find an integer k such that $T_1^k = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} = S$ and some integer m such that $S^m = T_1$. Then T_1 is a collineation if and only if S is. But S is not a collineation. For, if S maps $L\begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}$ onto $L(M)$, then $M = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix} \notin C$.

https://doi.org/10.1017/S000497270000890X Published online by Cambridge University Press

If $T = \begin{pmatrix} I & 0 \\ -N^{-1} & -N^{-1}M \end{pmatrix}$ is a collineation, so is $T^{-1} = \begin{pmatrix} I & 0 \\ -M^{-1} & -M^{-1}N \end{pmatrix}$.
Since $\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$ is a collineation so must be

$$\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}\begin{pmatrix} I & 0 \\ -N^{-1} & -N^{-1}M \end{pmatrix}\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}\begin{pmatrix} I & 0 \\ -M^{-1} & -M^{-1}N \end{pmatrix} .$$

But this is $\begin{pmatrix} I & 0 \\ 3I+2N^{-1}+3N^{-1}M & I \end{pmatrix} = T$. Thus T is a collineation only if T is. But this is possible only when $3I + 2N^{-1} + 3N^{-1}M = 0$; that is $M + N = I$.

COROLLARY 4.4. *$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}$ is a collineation only if $A = 3I$ and $B = 4I$.*

Proof. In view of Lemma 4.3, there exist matrices N and M in C such that $A = -N^{-1}$, $B = -N^{-1}M$, and $N + M = I$. An inspection of Table 1 reveals that the only matrices for N such that $N + M = I$ are when $N = 2I, 3I$, or $4I$. Then the possible collineations are $U = \begin{pmatrix} I & 0 \\ 2I & 3I \end{pmatrix}$, $V = \begin{pmatrix} I & 0 \\ I & 2I \end{pmatrix}$, and $W = \begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$. However U and V are not collineations. For let $L(M)$ be the image of $L\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$ under U . Then $\left(I+2\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}\right)M = 3\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$. This gives $M = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix} \notin C$. A similar argument shows that V is not a collineation. That W is a collineation follows from Lemma 4.1.

LEMMA 4.5. *If $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ is a collineation, then $A + B = C$.*

Proof. Let $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ B_1 & C_1 \end{pmatrix}$ be two distinct collineations, and let the inverse of $\begin{pmatrix} A & 0 \\ B_1 & C_1 \end{pmatrix}$ be $\begin{pmatrix} A^{-1} & 0 \\ B_2 & C_2 \end{pmatrix}$. Then $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ B_2 & C_2 \end{pmatrix}$ is a collineation and different from the identity, and hence must be of the form $\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$. Similarly $\begin{pmatrix} A^{-1} & 0 \\ B_2 & C_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$. Further $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ commutes with $\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$. Therefore $\begin{pmatrix} A & 0 \\ 3A+4B & 4C \end{pmatrix} = \begin{pmatrix} A & 0 \\ B+3C & 4C \end{pmatrix}$. Thus $A + B = C$.

LEMMA 4.6. $\begin{pmatrix} A & 0 \\ B & A+B \end{pmatrix}$ *is a collineation only if* $B = 3A$.

Proof. If $\begin{pmatrix} A & 0 \\ B & A+B \end{pmatrix}$ is a collineation, then there is a matrix $N \in C$ such that $B = -N^{-1}A$. If this collineation maps $L(M)$ onto L_1 , then $M = I - A^{-1}NA$. Inspection of Table 1 reveals that $N = \lambda I$, $\lambda = 1, 2, 3, 4$. If $\lambda = 4$ then T is singular. Therefore $N = \lambda I$ where $\lambda = 1, 2, 3$. Then the possible collineations are $T_1 = \begin{pmatrix} A & 0 \\ A & 2A \end{pmatrix}$, $T_2 = \begin{pmatrix} A & 0 \\ 2A & 3A \end{pmatrix}$, and $T_3 = \begin{pmatrix} A & 0 \\ 3A & 4A \end{pmatrix}$. If T_1 is a collineation, then $T_1^2 = \begin{pmatrix} A^2 & 0 \\ 3A^2 & 4A^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}$ must also be a collineation. But since $\begin{pmatrix} I & 0 \\ 3I & 4I \end{pmatrix}$ is a collineation, T_1 is a collineation if and only if $\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}$ is. But the possible matrices for A^2 are

$$\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix} .$$

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$. This gives $b = 0$ or $a + d = 0$. If $a + d = 0$, then $c(a+d) = 0 = 4a$, which is not possible. If $b = 0$, then $a^2 = 2$, and this also is not true. The other cases can similarly be disposed of.

(ii) Similarly if $T_2 = \begin{pmatrix} A & 0 \\ 2A & 3A \end{pmatrix}$ is a collineation, then $T_2^2 = \begin{pmatrix} I & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}$ should also be a collineation, and a similar argument as in (i) can be used to show that T_2 can not be a collineation. Thus if $\begin{pmatrix} A & 0 \\ B & A+B \end{pmatrix}$ is a collineation, then $B = 3A$.

CONCLUSION. The collineations that fix L_0 belong to the group

$$\left\langle \begin{pmatrix} 10 & 00 \\ 42 & 00 \\ 00 & 10 \\ 00 & 42 \end{pmatrix}, \begin{pmatrix} 14 & 00 \\ 24 & 00 \\ 00 & 14 \\ 00 & 24 \end{pmatrix}, \begin{pmatrix} 10 & 00 \\ 01 & 00 \\ 30 & 40 \\ 03 & 04 \end{pmatrix} \right\rangle .$$

The order of this group is 64 .

5.

The group $G_1 = \langle \alpha, \beta, \gamma \rangle$ is transitive on the lines $0 \le L_i \le 17$ and $18 \le L_j \le 25$, separately. Further there is no collineation that maps a line of the first set onto a line of the second. For, if there is a collineation T that maps L_{18} onto L_0 (say) , then $T^{-1}\beta T$ fixes L_0 and has 3 cycles of length six each, and hence its order is a multiple of 6 . Since the order of the group of collineations that fix L_0 is 64 , it can not possibly have an element of order 6 . If x is a collineation that fixes L_{18} and maps L_0 onto L_r , $0 \le r \le 17$, then there is a collineation y such that xy^{-1} fixes L_{18} and L_0 . Thus it suffices to consider only those collineations that fix both L_0 and L_{18} . These are all contained in G_1 . Let G_2 be the group of translations and G_3 the group of all collineations that fix L_0 . Then the full collineation group is $\langle G_1, G_2, G_3 \rangle$.

References

[1] R.H. Bruck and R.C. Bose, "The construction of translation planes from projective spaces", *J. Algebra* 1 (1964), 85-102.

[2] P. Dembowski, *Finite geometries* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 44. Springer-Verlag, Berlin, Heidelberg, New York, 1968).

[3] David A. Foulser, "Solvable flag transitive affine groups", *Math. Z.* 86 (1964), 191-204.

[4] Christoph Hering, "Eine nicht-desarguessche zweifach transitive affine Ebene der Ordnung 27", *Abh. Math. Sem. Univ. Hamburg* **34** (1970), 203-208.

[5] M.L. Narayana Rao, "A flag transitive plane of order 49", *Proc. Amer. Math. Soc.* **32** (1972), 256-262.

[6] M.L. Narayana Rao, "A class of flag transitive planes", *Proc. Amer. Math. Soc.* **39** (1973), 51-56.

[7] M.L. Narayana Rao and E.H. Davis, "Construction of translation planes from t-spread sets", *J. Combinatorial Theory Ser. A* **14** (1973), 201-208.

[8] M.L. Narayana Rao and K. Kuppuswamy Rao, "A class of non-Desargusian planes", *J. Combinatorial Theory Ser. A* **19** (1975), 247-255.

[9] M.L. Narayana Rao and K. Kuppuswamy Rao, "A new flag transitive affine plane of order 27", *Proc. Amer. Math. Soc.* **59** (1976), 337-345.

Department of Mathematics,
Osmania University,
Hyderabad,
Andhra Pradesh,
India.