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A DEFORMATION LEMMA AND SOME CRITICAL POINT THEOREMS

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A new deformation lemma for functions satisfying the Palais-Smale condition on a real Banach space is obtained. This is used to deduce some critical point theorems which are extensions of some well known results.

1. A DEFORMATION LEMMA

Throughout this paper, we assume that X is a real Banach space, $f \in C^1(X, R^1)$ satisfies the Palais-Smale condition: any sequence (x_n) for which $f(x_n)$ is bounded and $f'(x_n) \to 0$ possesses a converging subsequence. For a real number c, define

$$f_c = \{x \in X : f(x) \leqslant c\},\ f^c = \{x \in X : f(x) \geqslant c\}$$

and

$$K_c = \{x \in X : f(x) = c \text{ and } f'(x) = 0\}.$$

LEMMA 1.1. *Let A, B be two closed sets in X with $A \subset f^c$, $B \subset f_c$ and $A \cap B = \emptyset$. Then, if $B \cap K_c = \emptyset$, we can find an $\varepsilon > 0$ and a homeomorphism η of X onto X such that*

(i) *$f(\eta x) \leqslant f(x)$ for all $x \in X$;*
(ii) *$\eta x = x$ and hence $f(\eta x) = f(x)$ for $x \in A$;*
(iii) *$f(\eta x) \leqslant c - \varepsilon$ for $x \in B$.*

PROOF: Since f satisfies the Palais-Smale condition, K_c is a compact set or an empty set. Hence from $B \cap K_c = \emptyset$ we can find a $\delta_1 > 0$ such that $\overline{U}_{3\delta_1}(B) \cap K_c = \emptyset$, where $U_\tau(B)$ denotes the τ-neighbourhood of B and $\overline{U}_{3\delta_1}(B)$ is the closure of $U_{3\delta_1}(B)$. Using the Palais-Smale condition again we find $\varepsilon_1 > 0$, $\sigma_1 > 0$ such that

$$\|f'(x)\| \geqslant \sigma_1 \text{ for } x \in \overline{U}_{3\sigma_1}(B) \cap f^{c-3\varepsilon_1} \cap f_{c+3\varepsilon_1}.$$

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After seeing the original manuscript, Professor P.H. Rabinowitz sent me some suggestions which motivated my further work on this line; he also sent me his work [10] by which I could clear up several mistakes in the original manuscript. I wish to express my gratitude here to Professor P.H. Rabinowitz for his kind help. I also wish to thank Mr. Guijie Qi for some helpful conversations.

Now for $x \in X$, define

$$\phi(x) = \frac{d\big(x, X \setminus \big(U_{2\delta_1} \cap f^{c-2\varepsilon_1} \cap f_{c+2\varepsilon_1}\big)\big)}{d(x, X \setminus (U_{2\delta_1} \cap f^{c-2\varepsilon_1} \cap f_{c+2\varepsilon_1})) + (d(x, U_{\delta_1} \cap f^{c-\varepsilon_1} \cap f_{c+\varepsilon_1}))}.$$

It is easy to check that $\phi \colon X \to [0, 1]$ is locally Lipschitz continuous, $\phi(x) = 0$ when $x \in X \setminus \big(U_{2\delta_1}(B) \cap f^{c-2\varepsilon_1} \cap f_{c+2\varepsilon_1}\big)$, and $\phi(x) = 1$ when $x \in U_{\delta_1}(B) \cap f^{c-\varepsilon_1} \cap f_{c+\varepsilon_1}$.

Let

$$h(s) = \begin{cases} 1, & \text{when } 0 \leqslant s \leqslant 1 \\ s^{-1}, & \text{when } s > 1. \end{cases}$$

$$w_1(x) = \begin{cases} -\delta_1 \phi(x) h(\|v(x)\|) v(x) & \text{for } x \in \overline{U}_{3\delta_1}(P) \cap f^{c-3\varepsilon_1} \cap f_{c+3\varepsilon_1} \\ \theta & \text{for } x \in X \setminus \big(U_{3\delta_1}(B) \cap f^{c-3\varepsilon_1} \cap f_{c+3\varepsilon_1}\big), \end{cases}$$

where $v(x)$ is a pseudo-gradient of $f(x)$. It is easily seen that $w_1(x)$ is locally Lipschitz continuous, $\|w_1(x)\| \leqslant \delta_1$ for all $x \in X$, and that for $x \in \overline{U}_{\delta_1}(B) \cap f^{c-\varepsilon_1} \cap f_{c+\varepsilon_1}$, $w_1(x) = -\delta_1 h(\|v(x)\|) v(x)$ and

$$\text{(1)} \qquad \begin{aligned} -(f'(x), w_1(x)) &= \delta_1 h(\|v(x)\|)(f'(z), v(x)) \\ &\geqslant \frac{1}{2}\delta_1 h(2\|f'(x)\|) 2\|f'(x)\| \cdot \|f'(x)\| \\ &\geqslant \frac{1}{2}\delta_1 \min(1, \sigma_1)\sigma_1 \triangleq \sigma > 0. \end{aligned}$$

Let $\xi(t) = \xi(t, x)$ be the unique solution of the problem

$$\begin{cases} d\xi/dt = w_1(\xi), \\ \xi(0) = x. \end{cases}$$

It is known from the theory of differential equations in Banach spaces that $\xi(t, x)$ is defined for all $(t, x) \in R^1 \times X$. Let $B_1 = \{\xi(t, x) \colon x \in B, t \in [0, 1]\}$. We show that (a) B_1 is a closed set and (b) $A \cap B_1 = \emptyset$.

In fact, suppose $y_n \in B_1$ and $y_n \to y_0$. Then we can find $x_n \in B$ and $t_n \in [0, 1]$ such that $y_n = \xi(t_n, x_n)$. Now $\{t_n\}$ must have a converging subsequence: $t_{n_k} \to t_0 \in [0, 1]$, so let $z_k = \xi\big(t_0, x_{n_k}\big)$ and we have

$$\begin{aligned} \big\|z_k - y_{n_k}\big\| &= \big\|\xi\big(t_0, x_{n_k}\big) - \xi\big(t_{n_k}, x_{n_k}\big)\big\| \\ &= \left\| \int_{t_{n_k}}^{t_0} w_1\big(\xi\big(t, x_{n_k}\big)\big) dt \right\| \\ &\leqslant \delta_1 \big|t_0 - t_{n_k}\big| \to 0. \end{aligned}$$

Hence $z_k \to y_0$ and $x_{n_k} = \xi(-t_0, z_k) \to \xi(-t_0, y_0)$ due to the continuous dependence of solutions on initial values. Let $x_0 = \xi(-t_0, y_0)$; then since $x_{n_k} \to x_0$, $x_{n_k} \in B$ and B is closed, we see that $x_0 \in B$ and thus $y_0 = \xi(t_0, x_0) \in B_1$. This proves the closedness of B_1.

Now we prove assertion (b). From the definition of $w_1(x)$, it is easy to check that $(d/dt)f(\xi(t, x)) \leqslant 0$ for $x \in X$ and $t \in R^1$. This implies that $f(\xi(t, x))$ is a decreasing function of t. Suppose by contradiction that $A \cap B_1 \neq \emptyset$; then there are $x_1 \in B$ and $t_1 \in [0, 1]$ such that $\xi(t_1, x_1) \in A$. Since $A \cap B = \emptyset$, we must have $t_1 > 0$. Using $A \subset f^c$ and $B \subset f_c$ we get for $t \in [0, t_1]$ that

$$c \leqslant f(\xi(t_1, x_1)) \leqslant f(\xi(t, x_1)) \leqslant f(\xi(0, x_1)) = f(x_1) \leqslant c$$

which implies

$$f(\xi(t, x_1)) = c \text{ for } t \in [0, t_1]. \tag{2}$$

On the other hand, we have for $t \in [0, 1]$ that

$$\|\xi(t, x_1) - x_1\| = \|\xi(t, x_1) - \xi(0, x_1)\| = \left\| \int_0^t w_1(\xi(s, x_1))ds \right\| \leqslant \delta_1.$$

Hence $\{\xi(t, x_1) : t \in [0, t_1]\} \subset \overline{U}_{\delta_1}(B) \cap f^{c-\varepsilon_1} \cap f_{c+\varepsilon_1}$. Now using (1) we see that

$$\begin{aligned}
&(f'(\xi(t, x_1)), w_1(\xi(t, x_1))) \leqslant -\sigma \text{ for } t \in [0, t_1]; \\
&\qquad f(\xi(t_1, x_1)) = f(\xi(0, x_1)) + \int_0^{t_1} \frac{d}{dt} f(\xi(t, x_1))dt \\
&\qquad\qquad = f(x_1) + \int_0^{t_1} (f'(\xi(t, x_1)), w_1(\xi(t, x_1)))dt \\
&\qquad\qquad \leqslant f(x_1) - t_1\sigma < c;
\end{aligned}$$

which contradicts (2). Thus assertion (b) is proved.

For $x \in X$, define

$$\psi_1(x) = \frac{d(x, A)}{d(x, A) + d(x, B_1)}.$$

It is clear that $\psi_1 : X \to [0, 1]$ is continuous, so $A_1 = \{x \in X : \psi_1(x) \leqslant \frac{1}{2}\}$ is a closed set. Evidently $A \subset \mathring{A}_1$ and $A_1 \cap B_1 = \emptyset$. Now for $x \in X$ define

$$\psi(x) = \frac{d(x, A_1)}{d(x, A_1) + d(x, B_1)}.$$

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Clearly, $\psi\colon X \to [0, 1]$ is locally Lipschitz continuous and $\psi(x) = \begin{cases} 0 & \text{for } x \in A_1, \\ 1 & \text{for } x \in B_1. \end{cases}$
Let

$$w(x) = \psi(x) w_1(x) \text{ for all } x \in X,$$

then $w(x)$ is locally Lipschitz continuous; $\|w(x)\| \leqslant \delta_1$ for all $x \in X; w(x) = \theta$ for $x \in A_1$ and $w(x) = w_1(x)$ for $x \in B_1$.

Let $\eta(t) = \eta(t, x)$ be the unique solution of the problem

$$\begin{cases} d\eta/dt = w(\eta), \\ \eta(0) = x, \end{cases} \tag{3}$$

and then $\eta(t, x)$ is defined for all $(t, x) \in R^1 \times X$. Define

$$\varepsilon = \min(\sigma, \varepsilon_1) \text{ and } \eta x = \eta(1, x) \text{ for all } x \in X.$$

We show that ε, η meet our requirements (i) – (iii).

In fact, it is known from the theory of differential equations in Banach spaces that η is a homeomorphism of X onto X. We can easily check that $(d/dt)f(\eta(t, x)) \leqslant 0$, so $f(\eta(t, x))$ is a decreasing function of t. This gives $f(\eta(1, x)) \leqslant f(\eta(0, x))$, that is, $f(\eta x) \leqslant f(x)$ and (i) is satisfied. Since $w(x) = \theta$ for $x \in A_1$, we have $\eta(t, x) = x$ for all t when $x \in \mathring{A}_1$; in particular, $\eta(1, x) = x$ for $x \in A$, that is, $\eta x = x$ for $x \in A$ and (ii) is proved.

Suppose by contradiction that (iii) is not satisfied. Then we find $x_0 \in B$ such that $f(\eta x_0) > c - \varepsilon$. Let $y_0 = \eta x_0$; then $y_0 = \eta(1, x_0)$ and $f(y_0) > c - \varepsilon$. Since $\xi(t, x_0) \in B_1$ for $t \in [0, 1]$ and $w(x) = w_1(x)$ when $x \in B_1$, we see that $w(\xi(t, x_0)) = w_1(\xi(t, x_0))$ for $t \in [0, 1]$. Hence $\xi(t) = \xi(t, x_0)$ satisfies (3) for $t \in [0, 1]$ and $x = x_0$. This implies $\xi(t, x_0) = \eta(t, x_0)$ for $t \in [0, 1]$.

Now from $B \subset f_c$ we have for $t \in [0, 1]$ that

$$c \geqslant f(x_0) = f(\eta(0, x_0)) \geqslant f(\eta(t, x_0)) \geqslant f(\eta(1, x_0)) = f(y_0) > c - \varepsilon. \tag{4}$$

So $\{\eta(t, x_0)\colon t \in [0, 1]\} \subset f^{c-\varepsilon} \cap f_c$. Moreover, for $t \in [0, 1]$,

$$\begin{aligned} \|\eta(t, x_0) - x_0\| &= \|\eta(t, x_0) - \eta(0, x_0)\| \\ &= \left\| \int_0^t (d/ds)\eta(s, x_0)\,ds \right\| = \left\| \int_0^t w(\eta(s, x_0))\,ds \right\| \leqslant \delta_1. \end{aligned}$$

Hence $\{\eta(t, x_0)\colon t \in [0, 1]\} \subset \overline{U}_{\delta_1}(B) \cap f^{c-\varepsilon_1} \cap f_{c+\varepsilon_1}$. Now using (1) and $w(\eta(t, x_0)) =$

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$w_1(\eta(t, x_0))$ for $t \in [0, 1]$ we get

$$\begin{aligned} f(x_0) - f(y_0) &= f(\eta(0, x_0)) - f(\eta(1, x_0)) \\ &= -\int_0^1 \frac{d}{dt} f(\eta(t, x_0))dt \\ &= -\int_0^1 (f'(\eta(t, x_0))), w(\eta(t, x_0))dt \geqslant \sigma \geqslant \varepsilon, \end{aligned}$$

which contradicts (4). This proves assertion (iii) and the proof is complete. □

2. Some critical point theorems

In this section, we give some applications of Lemma 1.1.

Let $Q \subset X$ be a Banach manifold with boundary ∂Q and let S be a closed set in X. We say that ∂Q is linking with S if $\partial Q \cap S = \emptyset$ and $\phi(Q) \cap S \neq \emptyset$ for any $\phi \in C(Q, X)$ with $\phi|_{\partial Q} = \mathrm{id}|_{\partial Q}$.

Theorem 2.1. *Suppose that ∂Q is linking with S, $\sup_{x \in Q} f(x) < +\infty$, and that there is an $\alpha \in R^1$ such that*

$$f(x) \leqslant \alpha \text{ for } x \in \partial Q; \; f(x) \geqslant \alpha \text{ for } x \in S. \tag{5}$$

Define
$$c = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)),$$

where $\Gamma = \{\phi \in C(Q, X) : \phi|_{\partial Q} = id|_{\partial Q}\}$. Then we have

(i) $c \geqslant \alpha$;

(ii) $K_c \setminus \partial Q \neq \emptyset$;

(iii) $K_c \cap S \neq \emptyset$ *if* $c = \alpha$.

Proof: $c \geqslant \alpha$ is evident. If $c > \alpha$, from the well-known deformation lemma ([**4**]; see also [**1**], [**3**]), we can easily prove that $K_c \setminus \partial Q = K_c \neq \emptyset$ (see [**3**]). If $c = \alpha$, clearly it is enough to prove $K_c \cap S \neq \emptyset$. Suppose by contradiction that $K_c \cap S = \emptyset$. Then using Lemma 1.1 to $-f(x)$ with $c = -c$, $A = \partial Q$ and $B = S$, we find an $\varepsilon > 0$ and a homeomorphism η of X onto X such that

$$\eta x = x \text{ for } x \in \partial Q: \; -f(\eta x) \leqslant -c - \varepsilon \text{ for } x \in S. \tag{6}$$

From the definition of c we can find a $\phi_0 \in \Gamma$ such that

$$f(\phi_0(x)) < c + \varepsilon \text{ for } x \in Q. \tag{7}$$

Now let $\phi_1(x) = \eta^{-1}(\phi_0(x))$; evidently $\phi_1 : Q \to X$ is continuous. Since $\phi_0(x) = x$ for $x \in \partial Q$, we have $\phi_1(x) = \eta^{-1}(\phi_0(x)) = \eta^{-1}x = x$ for $x \in \partial Q$. Since ∂Q is linking with S, we have $\phi_1(Q) \cap S \neq \emptyset$. Hence there exists $x_0 \in Q$ such that $\phi_1(x_0) \in S$. This and (6) yield $f(\phi_0(x_0)) = f(\eta\phi_1(x_0)) \geqslant c + \varepsilon$, a contradiction to (7). The proof is complete. □

REMARK 2.1. (a) When condtion (5) is replaced by

$$f(x) \leqslant \alpha \text{ for } x \in \partial Q; \ f(x) \geqslant \beta \text{ for } x \in S; \ \alpha < \beta, \tag{8}$$

Theorem 2.1 becomes a well-known result (see, for example, [3] and [7]) which contains in particular the Mountain Pass Lemma ([1]) as well as some other useful critical point theorems (see [10]) as special cases.

(b) In the Mountain Pass setting, Theorem 2.1 was proved independently in [10] and [6]; see also [5], [8] and [9] for related results.

(c) If Q is a compact set, Theorem 2.1 can be proved entirely by Lemma 1.1. More precisely, the case $c > \alpha$ can also be proved by using Lemma 1.1 instead of the old one. In fact, let $S_1 = f^{-1}(c)$; we can easily prove that ∂Q is linking with S_1 in this situation. Then since $\partial Q \subset f_c$ and $S_1 \subset f^c$, we can show as above that $K_c \cap S_1 \neq \emptyset$.

THEOREM 2.2. *Let the conditions of Theorem 2.1 be satisfied. Define*

$$d = \sup_{h \in \Gamma^*} \inf_{x \in S} f(h(x)).$$

where $\Gamma^* = \{h \in C(X, X)\colon h$ *is a homeomorphism of* X *onto* X *with* $h|_{\partial Q} = id|_{\partial Q}\}$. *Then we have*

(i) $\alpha \leqslant d \leqslant c$;
(ii) $K_d \setminus \partial Q \neq \emptyset$;
(iii) $K_d \cap S \neq \emptyset$ *if* $d = \alpha$;

where c *is defined as in Theorem 2.1.*

PROOF: It is easy to see that $\alpha \leqslant d \leqslant c$. If $d > \alpha$, it can be proved by the well-known deformation lemma that $K_d = K_d \setminus \partial Q \neq \emptyset$ (see, for example, [3]). If $d = \alpha$ and $K_d \cap S = \emptyset$, then use Lemma 1.1 to $-f(x)$ with $c = -d$, $A = \partial Q$ and $B = S$; we can find an $\varepsilon > 0$ and a homeomorphism η of X onto X such that

$$\eta x = x \text{ for } x \in \partial Q \text{ and } -f(\eta x) \leqslant -d - \varepsilon \text{ for } x \in S.$$

Let $h_0(x) = \eta x$; evidently $h_0 \in \Gamma^*$ and $\inf_{x \in S} f(h_0(x)) = \inf_{x \in S} f(\eta x) \geqslant d + \varepsilon$, and this contradicts the definition of d. Hence $K_d \cap S \neq \emptyset$ if $d = \alpha$. This completes the proof. ∎

REMARK 2.2. (a) Under condition (8), Theorem 2.2 was proved in [3].

(b) Theorem 2.1 can be easily deduced from Theorem 2.2. In fact, if $c = \alpha$, we must have $d = c = \alpha$ and hence $K_c \cap S = K_d \cap S \neq \emptyset$ by Theorem 2.2.

COROLLARY 2.1. *Under the assumptions of Theorem 2.1, if condition (5) is replaced by*

$$f(x) \leqslant \alpha \text{ for } x \in \partial Q \text{ and } f(x) > \alpha \text{ for } x \in S,$$

then we have

(i) $\alpha < d \leqslant c$,

(ii) $K_c \neq \emptyset$, $K_d \neq \emptyset$,

where c and d are defined as before.

REMARK 2.3. Corollary 2.1 improves a result in [3] where the conclusion was proved under the condition that $f \in C^{2-0}$ and that every critical value of f corresponds to finitely many critical points.

REMARK 2.4. The idea of this paper can be used to improve several critical point theorems of even functionals as well. This and some variants of Theorem 2.1 can be used to deal with some new situations in elliptic boundary value problems. These will be done in a forthcoming paper.

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