Λ-effect, Meridional Flow and the Differential Solar Rotation

G. Rüdiger[1] and I. Tuominen[2]

[1]Universitäts-Sternwarte Göttingen, Göttingen, F.R.G.
[2]Observatory and Astrophysics Laboratory, University of Helsinki, Helsinki, Finland

Abstract: Rotation laws for the solar convection zone are produced by the Λ-effect in rotating anisotropic turbulence fields. In this paper we use the structure of the turbulence parameters introduced by Rüdiger and Kichatinov (1990), based on a simplified turbulence model. When we neglect the meridional circulation, for small inverse Rossby numbers the angular velocity isolines are spherical, while for increasing inverse Rossby number they approach more and more the helioseismologically derived shape. This simple picture becomes more complicated if the meridional circulation is allowed to act as an angular momentum transporter.

1. Introduction

In a rotating turbulence without any preferred direction apart from the angular velocity vector, $\boldsymbol{\Omega}$, angular momentum is transported only according to the well-known Boussinesq relations

$$Q_{r\varphi} = -\nu_T \, r \frac{\partial \Omega}{\partial r} \sin\theta, \quad Q_{\theta\varphi} = -\nu_T \frac{\partial \Omega}{\partial \theta} \sin\theta \tag{1}$$

for the zonal off-diagonal components of the correlation tensor Q_{ij}

$$Q_{ij} = \langle u_i'(x,t) u_j'(x,t) \rangle. \tag{2}$$

ν_T is the eddy viscosity. For anisotropic turbulence, however, with the radial unit vector $\mathring{g} = \boldsymbol{r}/r$ as an additional preferred direction, new terms

$$\begin{aligned} Q_{r\varphi} &= \nu_T \left\{ -\frac{r\partial\Omega}{\Omega\partial r} + V^{(0)} + \sin^2\theta \, V^{(1)} \right\} \sin\theta \, \Omega, \\ Q_{\theta\varphi} &= \nu_T \left\{ -\frac{\partial\Omega}{\Omega\partial\theta} \frac{\sin\theta}{\cos\theta} + H^{(0)} + \sin^2\theta \, H^{(1)} \right\} \cos\theta \, \Omega \end{aligned} \tag{3}$$

appear – known as the Λ-effect. They act in a non-diffusive way as they do not vanish for $\Omega = \text{const.}$, thus preventing the existence of uniform rotation. For given

https://doi.org/10.1017/S0252921100079586 Published online by Cambridge University Press

coefficients $V^{(0)}, \ldots, H^{(1)}$ a mean-field flow system $\langle \boldsymbol{u} \rangle$ develops whose azimuthal (φ-) component reflects the differential rotation.

2. The turbulence model

The general theory of the Λ-effect is presented in Rüdiger (1989) using the so-called second-order correlation approximation. It works with the simplest truncation procedure, i.e. the restriction to a basically linear treatment of the coupling of all small-scale modes. The interaction of the laminar large-scale modes with the random small-scale modes is non-linear. As Vainshtein and Kichatinov (1983) have pointed out, the influence of higher-order correlations can be modelled by

$$\partial u_i'/\partial t - \langle (\boldsymbol{u}'\nabla)u_i' \rangle + (\boldsymbol{u}'\nabla)u_i' = u_i'/\tau \tag{4}$$

with τ as the relaxation time of the correlations (Orszag, 1970). It is tempting to apply this model to the general expressions for the turbulent angular momentum transport coefficients $V^{(0)}, \cdots, H^{(1)}$. One has then simply to use the relations

$$q(\boldsymbol{k}, w) = q(\boldsymbol{k})\, \delta(w)/\tau, \quad \nu k^2 = 1/\tau \tag{5}$$

for the dependence of the spectral function q on the wave-number $\boldsymbol{k}$ and the frequency w of the turbulence field.

3. The Λ-terms

Rüdiger and Kichatinov (1990) presented the Λ-coefficients $V^{(0)}, \ldots, H^{(1)}$ for the turbulence model (5):

$$\begin{aligned} V^{(0)} &= \frac{1}{\nu_{\mathrm{T}}}\left(\frac{8}{15} - \frac{64}{35}\tau^2 \Omega^2\right) \int k^2 q(k)\, d\boldsymbol{k} \\ V^{(1)} = H^{(1)} &= -\frac{32}{35}\frac{\tau^2 \Omega^2}{\nu_{\mathrm{T}}} \int k^2 q(k)\, d\boldsymbol{k}, \end{aligned} \tag{6}$$

$H^{(0)}$ being zero. It is easy to show that

$$\int k^2 q(k)\, d\boldsymbol{k} = \frac{\tau}{3}\left(\langle u_\theta'^2 \rangle - \langle u_r'^2 \rangle\right), \tag{7}$$

which gives the relation for the anisotropy in the turbulence field. Thus, ***positive*** q implies dominance of the ***horizontal*** motions and *v.v.* All our micro-scale expressions are correct to order Ω^3. That is a minimal constraint as the inverse Rossby number $\mathrm{Ro}^{-1} = 2\tau\Omega$ is expected to be of order unity at the bottom of the convection zone. According to (6), a strongly reduced $V^{(0)}$ and maximal values of $V^{(1)}$ and $H^{(1)}$ are the consequence of high Ro^{-1} values. At the solar surface, however, we have the opposite situation: because of the very small Ro^{-1}, maximal

$V^{(0)}$ occurs with minimal $V^{(1)}$ and $H^{(1)}$. From the observations of the horizontal motions of sunspots we obtain $H^{(1)} \cong 1$. If this is correct, Eq. (6) leads directly to $q < 0$, so that a preference for vertical turbulent motions must be assumed.

With the anisotropy parameter δI,

$$\delta I = -\frac{1}{2\nu_{\mathrm{T}}} \int k^2 q(k) dk, \tag{8}$$

we write

$$V^{(0)} = -2\left(\frac{8}{15} - \frac{64}{35}\tau^2 \Omega^2\right) \delta I, \qquad V^{(1)} = H^{(1)} = \frac{64}{35}\tau^2 \Omega^2 \, \delta I. \tag{9}$$

We consider δI as *depth-independent*, while, on the other hand, the correlation time τ can be parameterized by

$$\tau = \tau_i (x_i/x)^{\lambda/2} \tag{10}$$

with $x = r/R$, so that simply

$$V^{(0)} \cong \left(-\frac{16}{15} + \frac{32}{35}(x_i/x)^{\lambda}\,\mathrm{Ro_i^{-2}}\right) \delta I,\ V^{(1)} = H^{(1)} \cong \frac{16}{35}(x_i/x)^{\lambda}\,\mathrm{Ro_i^{-2}}\delta I, \tag{11}$$

with the inner rotation parameter

$$\mathrm{Ro_i}^{-1} = 2\tau_i \Omega. \tag{12}$$

According to its construction, the expression in brackets in Eq. (11) is not allowed to change sign. For smaller Rossby numbers only $V^{(0)}$ is quenched. In principle, three parameters describe the non-diffusive part of the turbulent angular momentum transport: Rossby number, Ro, anisotropy parameter, δI, and the radial dependence of the correlation time, λ. The observations are the surface profile of the angular velocity, $\Omega(\theta)$, the surface value of $H^{(1)}$ and the surface characteristics of the meridional circulation.

We ask whether there is a range for these parameters and the Taylor number Ta for which the helioseismologically derived smooth rotation law in the convection zone appears and whether these values are in agreement with usual estimates for the Sun. Using a representation

$$\Omega = \Omega_0 \sum \omega_{n-1} P_n^1(\cos\theta)/\sin\theta, \tag{13}$$

the "observed rotation law" corresponds to

$$\omega_0(1) = 1, \quad \omega_2(1) = -0.033, \qquad \omega_0(x_i) \cong 1, \quad \omega_2(x_i) \cong 0. \tag{14}$$

The Taylor number, Ta$= \Omega^2 R^4/\nu_{\mathrm{T}}^2$ (with $\nu_{\mathrm{T}} = C_\nu L^2/\tau$), can be rewritten as

$$\mathrm{Ta} = \frac{\mathrm{Ro_i^{-2}}}{C_\nu^2 \xi_i^4}, \tag{15}$$

https://doi.org/10.1017/S0252921100079586 Published online by Cambridge University Press

so that additionally the (normalized) mixing length $\xi_i = L_i/R$ at the base of the convection zone enters our formulation. With $\xi = 0.1$

$$\mathrm{Ta} \cong 10^{5\ldots6}\mathrm{Ro}_\mathrm{i}^{-2}. \tag{16}$$

It remains to determine the rotationally created terms in the eddy heat transport tensor. Again we restrict ourselves to the isotropic part of the turbulence and find

$$VV^{(1)} = HV^{(1)} = Q/\chi_\mathrm{T}, \tag{18}$$

with Q taken from Rüdiger and Tuominen (1989), so that

$$VV^{(1)} = HV^{(1)} = \frac{1}{5}(x_i/x)^\lambda\,\mathrm{Ro}_\mathrm{i}^{-2}. \tag{19}$$

4. Models

The aim of our calculations is to explain the results for the angular velocity in the solar convection zone presented by Stenflo (these Proceedings, Fig. 8), derived from helioseismology and from the surface pattern of magnetic fields. The given isolines (strictly surfaces) of the angular velocity, Ω = const., are cylindrical or radial in the equatorial region, and rather disk-like in the polar region, while the isolines are radially directed in middle latitudes.

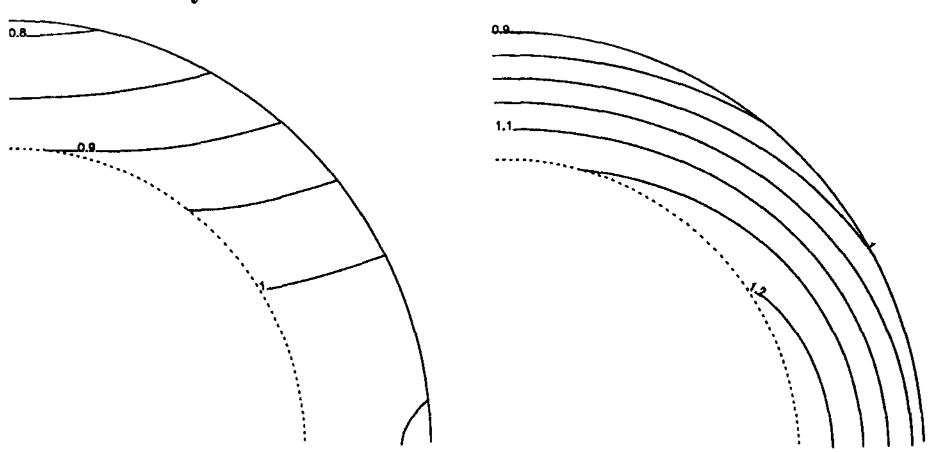

Fig. 1. Ω-contours for models in which the meridional circulation has been set to zero. a) Uniform Λ-effect ($\lambda = 0$). b) Λ-effect concentrated to the base of the convection zone ($\lambda = 5$).

Let us first focus on the angular momentum balance. Neglect of the meridional flow gives the simplest possible model. The Λ-effect alone then produces a characteristic profile of the angular velocity $\Omega(r, \theta)$. We present here models with the

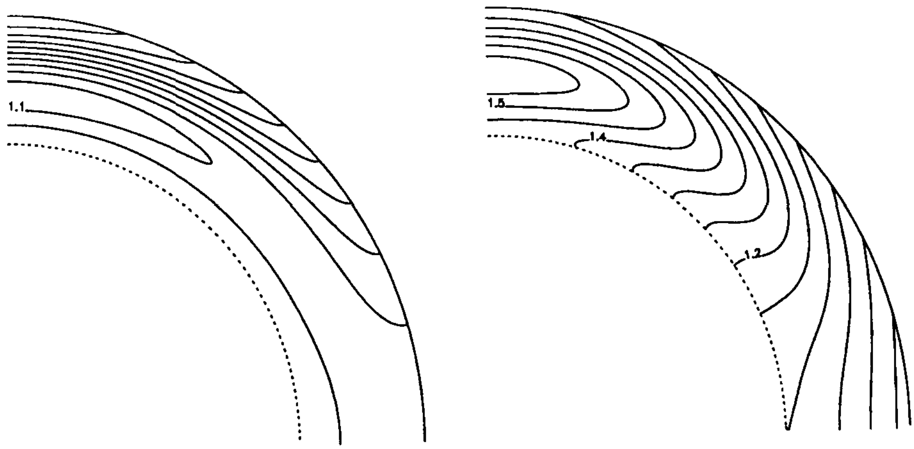

Fig. 2. Ω-contours for models in which the meridional circulation acts. The cases a) and b) as in Fig. 1.

parameter values Ta $= 10^5$ and Pr $= 0.33$, and with a uniform Λ-effect ($\lambda = 0$) and with Λ-effect more concentrated to the base of the convection zone ($\lambda = 5$).

Fig. 1 describes the consequences of the pure Λ-effect. We find already here essential differences. With increasing $\mathrm{Ro}_\mathrm{i}^{-1}$, the Ω-contours change from spherical to cylindrical, being close to the observations with $\mathrm{Ro}_\mathrm{i}^{-1} = 0.90$ (Fig. 1a). The case of non-uniform Λ differs drastically (Fig. 1b). If the Λ-effect is concentrated at the bottom of the convection zone, then any differential rotation at the surface can only follow from a strong pole-equator difference of Ω deep in the SCZ. The observed radially aligned isolines in middle latitudes are then always missing. If this result is of some generality then the Λ-effect must be depth-independent which, however, is not compatible with the mixing-length theory.

Let us now consider the effects of a meridional flow. Its inclusion requires the simultaneous solution of the temperature equation. It is, of course, formulated for a stratified medium, but simplifying the behaviour of the eddies, the medium is assumed to be adiabatic. Fig. 2 gives the solutions. For the uniform Λ-effect the angular velocity contours are less modified. The particular turbulence model adopted here does not give a reasonable solution for the non-uniform Λ-effect: there is an equatorial deceleration. The two circulation drivers – non-conservative centrifugal force and buoyancy – change the simple picture, given in Fig. 1. Clearly a proper turbulence model and the appropriate Λ-parameters may be sought, combining the "inverse" method, attempted in Tuominen and Rüdiger (1989), and their derivation by a direct numerical simulation, as in Pulkkinen *et al.* (these Proceedings).

5. The key question

Our numerical results look convincing despite the relatively simple turbulence model used. The free parameters are Prandtl number and Taylor number. It is, of course, an interesting task to study their influence on the resulting flow pattern. Doing so, we find a very unexpected behaviour of the solutions since they do not depend continuously on Ta or Pr. The pole-equator differences of the angular velocity and/or the temperature do not remain finite for all values of the dimensionless parameters. If one of them is fixed, critical values of the other exist for which the solution loses its relevance.

Mathematically speaking we have established an inhomogeneous system as a natural description of the over-all problem. That is in contrast to the dynamo theory, which essentially deals with a self-excitation problem. We are here confronted with the numerical fact that for certain values of Pr or Ta the characteristic determinant vanishes so that at these points self-excitation occurs.

Similar results have already been obtained by Gierasch (1974), Schmidt (1982) and Chan *et al.* (1987). It simply means that the full system of equations, in which the influence of the turbulence only enters via the eddy diffusivities, possesses an eigensolution for certain eigenvalues. Chan *et al.* (1987) propose that these solutions – which already appear for very simplified equations – give the desired answer for the problem of the maintenance of the solar differential rotation. Nonlinear calculations are necesssary to clarify the reality of this important phenomenon. Without the answer to this "key question of the theory of differential rotation", all results presented from linearized equations must be considered with care.

References

Chan, K.L., Sofia, S., Mayr, H.G.: 1987, in *The internal solar angular velocity*, eds. B.R. Durney, S. Sofia, Reidel, p. 347

Gierasch, P.J.: 1974, *Astrophys. J.* **190**, 199

Orszag, S.A.: 1970, *J. Fluid Mech.* **41**, 363

Rüdiger, G.: 1989, *Differential rotation and stellar convection, Sun and solar-type stars*, Gordon and Breach

Rüdiger, G., Kichatinov, L.L.: 1990, *Astron. Astrophys.* **236**, 503

Rüdiger, G., Tuominen, I.: 1990, in *Solar Photosphere: structure, convection and magnetic fields*, ed. J.O. Stenflo, Kluwer, p. 315

Schmidt, W.: 1982, *Geophys. Astrophys. Fluid Dyn.* **21**, 27

Tuominen, I., Rüdiger, G.: 1989, *Astron. Astrophys.* **217**, 217

Vainshtein, S.I., Kichatinov, L.L.: 1983, *Geophys. Astrophys. Fluid Dyn.* **24**, 273