DIRECT SUMS OF PARTIAL ALGEBRAS AND FINAL ALGEBRAIC STRUCTURES

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Słomiński (**9**), as well as the author (**8**), gave a descriptive, i.e., non-category-theoretic, definition of the direct sum of partial algebras, i.e., the co-product in the category of partial algebras (A, f), where $f = (f_i)_{i \in I}$, f_i: $\operatorname{dom} f_i \to A$, $\operatorname{dom} f_i \subset A^{K_i}$, of fixed type $\Delta = (K_i)_{i \in I}$. In the case of total absence of fundamental constants, i.e., $K_i \neq \emptyset$ for all indices $i \in \mathrm{I}$, the direct sum $(A, f), f = (f_i)_{i \in I}$, of partial algebras $(A_t, f_t), f_t = (f_{ti})_{i \in I}$, $t \in T$, may be described loosely as the disjoint union $A = SA_t = \bigcup \{t\} \times A_t$, where the sum operations f_i are only operating, in a self-evident manner, on argument sequences (of type K_i) all members of which are within one and only one of the classes $\{t\} \times A_t$. Even in the presence of constants, $K_i = \emptyset$ for some $i \in I$, this description of the partial direct sum may remain correct; in general, it becomes false. The formal reason: the empty argument sequence is a sequence in any of the classes $\{t\} \times A_t$, indeed in any set M. Hence, in the case that there are two different indices t such that the nullary operation f_{ti} is non-empty, i.e., $\operatorname{dom} f_{ti} = \{\text{empty sequence}\}$, or briefly, $f_{ti} \in A_t$ (by the usual identification of a nullary operation with its unique value), the nullary sum operation f_i (which should be an element of $A = SA_t$) does not know how to decide on one or the other of the possible values (t, f_{ti}). In other words, we have to identify (s, f_{si}), (t, f_{ti}) when $s \neq t$, and our description of direct sum as disjoint union no longer remains true. On the other hand, the direct sum of partial algebras, or partial direct sum (as we may call it), always exists. One might take this from general category theory assuring complete categories of some sort, e.g., categories of models, to be co-complete, i.e., their duals to be complete. Still, it is unnecessary to use a general argument of this kind that fails to give information on the concrete structure of direct sum in our relatively concrete case of partial algebras. In fact, algebra itself immediately remedies the failure described above.

This failure represents one more striking example of the anomalies of the empty set, which (far from being a purely dogmatic affair) are responsible for many significant mathematical facts (as is shown in this paper). The author had always been so certain that he would never overlook the perversities of $\emptyset$ that something like this was bound to happen. Thus it remains only to admit the fault and to correct it in the present paper.

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1. Final algebraic structure. We start from the specialization to partial algebras of a general concept introduced by Bourbaki (**1**).

THEOREM 1. *Let* (A_t, f_t) $(t \in T)$, (B, g) *be (partial) algebras of type* Δ, *and let* ϕ_t *map set* A_t *into set* B. *Then the following statements are equivalent*:

(i) *g is the poorest algebraic structure on set B such that all* ϕ_t: $(A_t, f_t) \rightarrow (B, g)$ *are homomorphisms*;

(ii) *for any index* $i \in I$, *any sequence* $\mathfrak{b} \in B^{K_i}$, *any element* $b \in B$: $g_i(\mathfrak{b}) = b$ *if and only if* $\phi_t(\mathfrak{a}_t) = \mathfrak{b}$, $\phi_t(a_t) = b$, *and* $f_{ti}(\mathfrak{a}_t) = a_t$, *for some* $t \in T$, *some sequence* $\mathfrak{a}_t \in A^{K_i}$, *some element* $a_t \in A_t$;

(iii) *for any (partial) algebra* (C, h) *and any map* ψ *of set* B *into set* C: ψ: $(B, g) \rightarrow (C, h)$ *is a homomorphism if and only if all* $\psi \circ \phi_t$: $(A_t, f_t) \rightarrow (C, h)$ *are homomorphisms.*

Proof. (i) $\Rightarrow$ (ii). Necessity follows by definition, in fact, is the definition of homomorphisms. For sufficiency, we introduce a new algebraic structure g^* of type Δ into set B by the definition:

$g_i^*(\mathfrak{b}) = b$ if and only if the condition of (ii) holds true, i.e., if $\phi_t(\mathfrak{a}_t) = \mathfrak{b}$, $\phi_t(a_t) = b$, and $f_{ti}(\mathfrak{a}_t) = a_t$, for some t, $\mathfrak{a}_t$, and a_t.

This definition is independent of the choice of t, $\mathfrak{A}_t$, and a_t since all ϕ_t: $(A_t, f_t) \rightarrow (B, g)$ are homomorphisms. Still, all ϕ_t: $(A_t, f_t) \rightarrow (B, g^*)$ are also homomorphisms, which gives $g \subset g^*$ by hypothesis, i.e., $g_i(\mathfrak{b}) = b$ implies $g_i^*(\mathfrak{b}) = b$.

(ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i) is the special case of the general conclusion drawn by Bourbaki (**1**).

According to Bourbaki, we call g the *final algebraic structure* on set B induced by the family of maps ϕ_t: $(A_t, f_t) \rightarrow B$. Condition (ii) gives an explicit description of this structure; it states that the ϕ_t are homomorphisms such that

$$\bigcup_{t \in T} \phi_t(\operatorname{dom} f_{ti}) = \operatorname{dom} g_i$$

(for general homomorphisms, only $\subset$ holds true). In the special case of a single map ϕ: $(A, f) \rightarrow B$, we may call ϕ: $(A, f) \rightarrow (B, g)$ a *final homomorphism*. In this case, $B - \operatorname{im} \phi$ has to be a discrete relative algebra of partial algebra (B, g); to make the notion of final homomorphism free from this undesirable relativity, we call ϕ a strong *homomorphism* if ϕ is a final homomorphism from algebra A onto relative algebra $\operatorname{im} \phi \subset B$ (Słomiński (**9**) also demands $\operatorname{im} \phi$ to be closed, i.e., a subalgebra of B). So in the case of onto maps, "final" and "strong" is the same: this is the obvious analogue of strongly continuous functions in the sense of Alexandroff-Hopf.

As an answer to the question of the existence of this final structure, we have the following corollary.

COROLLARY. *Let* ϕ_t *map algebra* (A_t, f_t) *into set* B, *for all* $t \in T$. *Then the following statements are equivalent*:

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(i) *the final algebraic structure* g *on* B *exists*;

(ii) *for any index* $i \in I$, *any indices* $s, t \in T$, *any sequences* $\mathfrak{a}_s \in A_s^{K_i}$, $\mathfrak{a}_t \in A_t^{K_i}$, *any elements* $a_s \in A_s$, $a_t \in A_t$: *if* $f_{si}(\mathfrak{a}_s) = a_s$, $f_{ti}(\mathfrak{a}_t) = a_t$, *and* $\phi_s(\mathfrak{a}_s) = \phi_t(\mathfrak{a}_t)$, *then* $\phi_s(a_s) = \phi_t(a_t)$;

(iii) *there is an arbitrary algebraic structure* h *on* B (*not necessarily the final one*) *such that all* ϕ_t: $(A_t, f_t) \to (B, h)$ *are homomorphisms.*

Condition (ii) is the independence of the choice of t, $\mathfrak{a}_t$, and a_t in the definition of g_i^* in the proof of Theorem 1 and, as has been used in this proof, it follows from the hypothesis that all ϕ_t are homomorphisms: (iii) $\Rightarrow$ (ii). By (ii), we may define an algebraic structure g^* on set B as in the proof of Theorem 1; then $g = g^*$ is the final structure by Theorem 1 (ii): (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (iii) is trivial.

In the case of a single map ϕ: $(A, f) \to B$, condition (ii) of this corollary states that the associated equivalence relation, $R = R_\phi = \phi^{-1} \circ \phi$, has to be a congruence relation of algebra (A, f). In the general case as well, all equivalence relations $R_t = \phi_t^{-1} \circ \phi_t$ have to be congruence relations; this is the special case $s = t$ of condition (ii), hence not sufficient for (ii) if $|T| \geqq 2$. Still, if

(*) the family of $\operatorname{im} \phi_t$ is pairwise disjoint,

condition (ii) implies that also

(**) the family of index-subsets $I_t := \{i |\, K_i = \emptyset, f_{ti}$ non-empty$\}$ is pairwise disjoint.

Conversely, if all R_t are congruence relations, (**), together with (*), implies condition (ii); hence, under the hypothesis (*), (ii) is equivalent to (**) and the condition that all R_t are congruence relations. Hence, under the hypothesis (*) and the injectivity of the ϕ_t, $R_t = \mathrm{id}_{A_t}$, (ii) is equivalent to (**). This is the phenomenon described in the introduction; let us again note that (**) holds in the special case if all I_t themselves are empty, i.e., if all algebras (A_t, f_t) are without constants and, in particular, if all K_i are non-empty, i.e., if type Δ is without constants.

The typical case of injective maps ϕ_t: $A_t \to B$ with pairwise disjoint images occurs if $B = SA_t$, $\phi_t = i_t$, the canonical injections $i_t(a) := (t, a)$ $(t \in T,\ a \in A_t)$. Hence, if in this case we have algebraic structures f_t on sets A_t, the final algebraic structure g on the disjoint union B exists if and only if (**) holds, and is defined (according to Theorem 1) by

$$g_i((t_\kappa, a_\kappa) \mid \kappa \in K_i) := (t, a) \text{ if and only if } t_\kappa = t, \text{ for all } \kappa \in K_i, \text{ and } f_{ti}(a_\kappa | \, \kappa \in K_i) = a;$$

(B, g) then is the *partial direct sum of algebras* (A_t, f_t) as defined by Słomiński (**9**) and the author (**8**).

2. The general partial direct sum. Let us consider the case of completely arbitrary summand algebras (A_t, f_t), i.e., (**) may or may not be

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fulfilled. We pass to the reduced type $\Delta^* := (K_j)_{j\in J}$, $J := \{j|\ j \in J, K_j \neq \emptyset\}$. The corresponding reduced algebras (A_t, f_t^*), where $f_t^* := (f_{tj})_{j\in J}$, fulfil (**), and we may construct their partial direct sum (B^0, g^0), with canonical injections $i^0{}_t: A_t \to B^0$, as described above. Let R be the congruence relation in (B^0, g^0) generated by the set of all pairs

$$((s, f_{si}), (t, f_{ti}))$$

such that $K_i = \emptyset, f_{si}, f_{ti}$ non-empty. There is (see Corollary above) a strong surjective homomorphism $\rho: (B^0, g^0) \to (B, g^*)$, where (B, g^*) is a partial algebra of type Δ^*, such that ρ induces R (to obtain this canonically, take $B = B^0/R$, ρ the associated canonical projection). Besides, (**) states that the above pairs $((s, f_{si}), (t, f_{ti}))$ belong to id_{B^0}, i.e., that $R = \mathrm{id}_{B^0}$; hence we may take $\rho = \mathrm{id}_{B^0}$, $(B, g^*) = (B^0, g^0)$: in case $J = I$, i.e., type Δ without constants, (B, g^*) is our old partial direct sum. In case $J \neq I$, with any index $i \in I - J$ such that there exists $t \in T$ with non-empty nullary operation $f_{ti} \in A_t$, we associate a non-empty nullary operation in B,

$$g_i := \rho(t, f_{ti}) := (\rho \circ i^0{}_t)(f_{ti})$$

(the definition being independent of the choice of index $t \in T$ by construction); in all other cases, nullary operation g_i shall be empty. (In case (**) and $\rho = \mathrm{id}_{B^0}$, this definition of g_i coincides with the old one given at the end of (**1**).) We consider algebra (B, g), where $g = (g_i)_{i\in I}$, and the maps $i_t := \rho \circ i^0{}_t: A_t \to B$, which are homomorphisms not only in the reduced sense $(A_t, f^*{}_t) \to (B, g^*)$, but also with respect to constants, $(A_t, f_t) \to (B, g)$. Clearly, the i_t need no longer be injective, but even if they were, they need not be strong as we shall show by a striking example. Let us first state the following theorem.

THEOREM 2. *The partial algebra (B, g) as constructed above, together with homomorphisms i_t: $(A_t, f_t) \to (B, g)$, is the direct sum of partial algebras (A_t, f_t) in the category of all partial algebras; g is the final structure for the i_t, and the i_t cover B, i.e., $\bigcup \mathrm{im}\, i_t = B$.*

The proof is an immediate consequence of the construction. Let χ_t: $(A_t, f_t) \to (C, h)$ $(t \in T)$ be homomorphisms. There is exactly one map ψ^0: $B^0 \to C$ such that $\psi^0 \circ i^0{}_t = \chi_t (t \in T)$, and since all χ_t: $(A_t, f^*{}_t) \to (C, h^*)$ are homomorphisms and (B^0, g^0) has the final structure for the $i^0{}_t$, ψ^0: $(B^0, g^0) \to (C, h^*)$ is a homomorphism. But if $K_i = \emptyset, f_{ti}$ non-empty, we have $\psi^0(t, f_{ti}) = \chi_t(f_{ti}) = h_i$; hence congruence relation R is contained in the one induced by ψ^0. So ψ^0 being surjective, there is exactly one map ψ: $B \to C$ such that $\psi \circ \rho = \psi^0$, and since ψ^0: $(B^0, g^0) \to (C, h^*)$ is a homomorphism and (B, g^*) has the final structure for ρ, ψ: $(B, g^*) \to (C, h^*)$ is a homomorphism. But ψ also respects constants: if $K_i = \emptyset$ and f_{ti} non-empty, we obtain $\psi(g_i) = \psi(\rho(t, f_{ti})) = \psi^0(t, f_{ti}) = h_i$; so ψ: $(B, g) \to (C, h)$ is a homomorphism; moreover, we have $\psi \circ i_t = \psi \circ \rho \circ i^0{}_t = \psi^0 \circ i^0{}_t = \chi_t$. Let ψ': $B \to C$ be an arbitrary map such that $\psi' \circ i_t = \chi_t$ (such that all

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$\psi' \circ i_t$: $(A_t, f_t) \to (C, h)$ are homomorphisms). From $\psi' \circ \rho \circ i^0{}_t = \chi_t$, we obtain $\psi' \circ \rho = \psi^0$ by the uniqueness of map ψ^0, hence $\psi' = \psi$ by the uniqueness of map ψ. So the homomorphism ψ: $(B, g) \to (C, h)$, such that $\psi \circ i_t = \chi_t$ $(t \in T)$, is unique; hence the family of the i_t: $(A_t, f_t) \to (B, g)$ $(t \in T)$ is the direct sum of the (A_t, f_t) in the category of all partial algebras; moreover, g is the final structure for the i_t. The covering property of the i_t is quite clear: $\bigcup i_t(A_t) = \rho(\bigcup i^0{}_t(A_t)) = \rho(B^0) = B$.

Again, we call (B, g) the *partial direct sum of algebras* (A_t, f_t). Concerning the homomorphism ψ: $(B, g) \to (C, h)$ as constructed in the proof of Theorem 2, we have the following.

ADDITION 1. *The following equivalences hold true*:

(i) *for any single index* $t \in T$: $\psi|\text{im}\, i_t$ *is injective if and only if* i_t *and* χ_t *induce the same congruence relation*;

(ii) ψ *is surjective if and only if the* χ_t *cover* C, $\bigcup \text{im}\, \chi_t = C$;

(iii) ψ *is strong if and only if relative algebra* $\bigcup \text{im}\, \chi_t \subset C$ *has the final structure for the* χ_t.

(i) is nothing but a simple fact from general set theory: note that map i_t: $A_t \to \text{im}\, i_t$ is surjective, and $(\psi|\,\text{im}\, i_t) \circ i_t = \chi_t$. (ii) and (iii) rest upon the equation

$$\text{im}\, \psi = \psi(B) = \psi(\bigcup i_t(A_t)) = \bigcup \chi_t(A_t) = \bigcup \text{im}\, \chi_t.$$

In particular, if we restrict considerations to subset $\text{im}\, \psi \subset C$, the algebraic structures k of $\text{im}\, \psi$, such that ψ: $(B, g) \to (\text{im}\, \psi, k)$ is a homomorphism, are precisely those k for which all χ_t: $(A_t, f_t) \to (\text{im}\, \psi, k)$ are homomorphisms, as (B, g) has the final structure for the i_t; moreover, ψ: $(B, g) \to (\text{im}\, \psi, k_0)$ as well as χ_t: $(A_t, f_t) \to (\text{im}\, \psi, k_0)$ is a homomorphism, where k_0 is the relative algebraic structure (restriction) of h to $\text{im}\, \psi$.

We have already proved the main part of the following.

ADDITION 2. *The following statements concerning the partial direct sum are equivalent*:

(*) *the* $\text{im}\, i_t$ *are pairwise disjoint*;

(**) *the index-subsets* $I_t = \{i|\, K_i = \emptyset, f_{ti}$ *non-empty*$\}$ *are pairwise disjoint. If this is the case, then the* i_t *are injective and strong.*

It only remains to note that, more generally, if some partial algebra (B, g) has the final structure for some family of homomorphisms i_t: $(A_t, f_t) \to (B, g)$ and (*) holds, then the i_t are strong.

Let us consider the example $A_0 = A_1 = B = \{0, 1\}$; let f: $B \to B$ be the non-trivial permutation, $f_t := f|\{t\}$ $(t = 0, 1)$. Then

$$(B, (0, f_t, f)) \xrightarrow{i_t := \text{id}_B} (B, (0, f, f))$$

is a direct sum representation of partial algebras (of type $\Delta = (0, 1, 1)$) $(B, (0, f_t, f))$ $(t = 0, 1)$; for if χ_t: $(B, (0, f_t, f)) \to (C, (c, g, h))$ $(t = 0, 1)$ are

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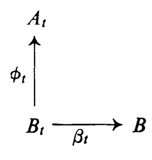
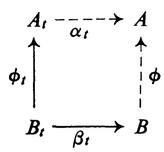
homomorphisms, then $\chi_0 = \chi_1$ $(=\psi)$. Note that both universal homomorphisms i_t are bijective, but fail to be strong.

3. Generalized amalgamated direct sums. In this section, we wish to solve the problem when the homomorphisms $i_t: A_t \to B$ in a partial direct sum representation are injective. To make the situation clear, let us compare our problem with the analogous problem for full (complete) algebras. As is well known (cf. Kerkhoff (**5**)), and may easily be derived from the unrestricted existence of the partial direct sum (cf. Słomiński (**9**) and Schmidt (**8**)), the direct sum in the category of full algebras, which we may call the *full* or *complete direct sum* (Kerkhoff (**5**): *absolut freies Produkt*), (also) always exists. Let $i_t: A_t \to B$ be a full direct sum of full algebras A_t $(t \in T)$, then, as is well known (**5**), the $i_t: A_t \to B$ are injective if and only if their restrictions $i_t|O_t: O_t \to O$ also are, where O_t and O are the smallest subalgebras of A_t and B respectively, i.e., the subalgebras generated by the empty set $\emptyset$. In fact, in this case the $i_t|O_t: O_t \to O$ are isomorphisms, and $i_t: A_t \to B$ may be considered as an amalgamated direct sum (with amalgam O) in the classical sense, i.e., as a co-fiberproduct of the $(i_t|O_t)^{-1}: B \to A_t$. Still in the case of partial algebras, the situation is more complicated, because here the $i_t|O_t: O_t \to O$, even if injective, need neither be bijective nor strong (cf. the example given above). Hence, a more general concept of amalgamated direct sum than the classical one becomes necessary; we may take it as well from the theory of general categories.

The general situation then is as follows. Instead of the inclusions $O_t \subset A_t$, we consider completely arbitrary (homo)morphisms $\phi_t: B_t \to A_t$; instead of the restricted natural maps: $i_t|O_t: O_t \to O$, we consider completely arbitrary (homo)morphisms $\beta_t: B_t \to B$, A_t, B_t, B being arbitrary objects of a category (e.g., that of partial algebras) $\mathfrak{A}$. If $\mathfrak{A}$ is co-complete (right-complete), i.e., if $\mathfrak{A}$-direct sums (co-products) as well as co-equalizers (difference co-kernels) exist, then, as is well known, direct limits exist for all "small" diagrams [cf. Freyd (**4**), Mitchell (**6**), and Felscher (**3**)]. In particular, this is the case for the diagram



arising in our situation: there is an object A and (homo)morphisms $\alpha_t: A_t \to A$ as well as $\phi: B \to A$ such that, for all $t \in T$, the diagram



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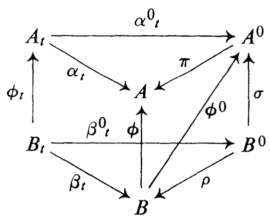
commutes and is universal with respect to this property. In fact, this special direct limit, which we may call the *generalized amalgamated direct sum*, may be constructed as follows: Let $\alpha^0{}_t$: $A_t \to A^0$ and ϕ^0: $B \to A^0$ be the direct sums of the A_t and B. Then there is π: $A^0 \to A$ such that

$$(+) \qquad \pi \circ \alpha^0{}_t \circ \phi_t = \pi \circ \phi^0 \circ \beta_t \quad \text{for all } t \in T,$$

and π is universal with respect to this property. To obtain this π directly from direct sums and co-equalizers (of pairs and morphisms), one might take the direct sum $\beta^0{}_t$: $B_t \to B^0$ of the B_t. Then there is exactly one ρ: $B^0 \to B$ such that $\rho \circ \beta^0{}_t = \beta_t$ for all $t \in T$, and exactly one σ: $B^0 \to A^0$ such that $\sigma \circ \beta^0{}_t = \alpha^0{}_t \circ \phi_t$ for all $t \in T$. Then for completely arbitrary π: $A^0 \to A$, (+) is equivalent to

$$(++) \qquad \pi \circ \sigma = \pi \circ \phi^0 \circ \rho,$$

showing that we may only select π as the co-equalizer of the pair of (homo)-morphisms σ and $\phi^0 \circ \rho$: $B^0 \to A^0$. In the concrete case of partial algebras, one might simply take $A := A^0/R$, where R is the congruence relation in A^0



generated by the set

$$\{((\alpha^0{}_t \circ \phi_t)(b), (\phi^0 \circ \beta_t)(b)) \mid t \in T, b \in B_t\},$$

and $\pi :=$ the natural strong surjective homomorphism onto this quotient algebra A^0/R. Finally, defining

$$\alpha_t := \pi \circ \alpha^0{}_t, \qquad \phi := \pi \circ \phi^0,$$

we obtain the universal (homo)morphisms of our generalized amalgamated direct sum.

In the classical case (as quoted above), the β_t: $B_t \to B$ are isomorphisms. Then $\alpha_t \circ \phi_t = \phi \circ \beta_t$ if and only if $\alpha_t \circ (\phi_t \circ \beta_t^{-1}) = \phi$, making clear that the universal (homo)morphisms α_t of the generalized amalgamated direct sum constitute nothing but the co-intersection ("pushout", co-fiberproduct) of the $\phi_t \circ \beta_t^{-1}$: $B \to A_t$, i.e., the classical amalgamated direct sum (with amalgam B). Concerning partial algebras, we have the following theorem.

THEOREM 3. *The generalized amalgamated partial direct sum A has the final structure for the universal homomorphisms* α_t: $A_t \to A$ *and* ϕ: $B \to A$. *If "amalgam" B has the final structure for the given homomorphisms* β_t: $B_t \to B$, *then so has A for the* α_t: $A_t \to A$ *alone.*

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The first statement is an immediate consequence of the construction given above, since partial direct sum A^0 has the final structure for the $\alpha^0{}_t$: $A_t \to A^0$ and ϕ^0: $B \to A^0$ (Theorem 2), and A has the final structure for π: $A^0 \to A$. Let us now consider an arbitrary map ψ: $A \to C$, C a partial algebra, such that $\psi \circ \alpha_t$: $A_t \to C$ is a homomorphism for all $t \in T$. Since $\psi \circ \phi \circ \beta_t = \psi \circ \alpha_t \circ \phi_t$, if B has the final structure for the β_t: $B_t \to B$, $\psi \circ \phi$: $B \to C$ is a homomorphism; so ψ: $A \to C$ is a homomorphism as A has the final structure for the α_t: $A_t \to A$ together with ϕ: $B \to A$, showing that the latter statement remains true if we disregard ϕ.

If we call the properties of the β_t: $B_t \to B$ local ("in the small"), those of the α_t: $A_t \to A$ global ("in the large"), the second part of Theorem 3 is a typical conclusion from the small to the large (whether one may draw the opposite conclusion in general remains an open question). Another simple conclusion of this kind is the following.

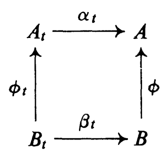
ADDITION. *If the small homomorphisms β_t: $B_t \to B$ cover B, $\bigcup \operatorname{im} \beta_t = B$, then so do the large homomorphisms α_t: $A_t \to A$, $\bigcup \operatorname{im} \alpha_t = A$.*

For by Theorem 2, $A = \pi A^0 = \pi(\phi^0 B \cup \bigcup \alpha^0{}_t A_t) = \phi B \cup \bigcup \alpha_t A_t = \phi \bigcup \beta_t B_t \cup \bigcup \alpha_t A_t = \bigcup \alpha_t \phi_t B_t \cup \bigcup \alpha_t A_t = \bigcup \alpha_t A_t$.

(In an arbitrary category $\mathfrak{A}$, one can conclude: if family $(\beta_t)_{t \in T}$ is epimorphic in the sense that for all morphisms χ, χ': $B \to C$ into arbitrary objects C, $\chi \circ \beta_t = \chi' \circ \beta_t$ for all $t \in T$ implies $\chi = \chi'$, then family $(\alpha_t)_{t \in T}$ is epimorphic.) Moreover, if family $(\beta_t)_{t \in T}$ is an $\mathfrak{A}$-direct sum, then family $(\alpha_t)_{t \in T}$ is an $\mathfrak{A}$-direct sum.

4. Injectivity of the α_t. An important conclusion from the small to the large is closely connected with an internal characterization of the generalized amalgamated partial direct sum, which gives complete insight into its interior structure in this special case. (A complete internal characterization of the generalized amalgamated partial direct sum in the general case remains an open problem.) We begin with the following theorem.

THEOREM 4. *Given commutative diagrams of homomorphisms of partial al-*



gebras, where the β_t cover B, the α_t cover A, ϕ as well as the restrictions $\alpha_t|(A_t - \phi_t B_t)$ are injective, and A has the final structure for the α_t and ϕ. Then, if

(i) $\alpha_t(A_t - \phi_t B_t) \subset A - \phi B \qquad (t \in T)$,

(ii) $\alpha_s A_s \cap \alpha_t A_t \subset \phi B \qquad (s \neq t,\ s, t \in T)$,

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then $\alpha_t: A_t \to A$, $\phi: B \to A$ *is the generalized amalgamated partial direct sum of* $\beta_t: B_t \to B$, $\phi_t: B_t \to A_t$.

Proof. We consider homomorphisms $\gamma_t: A_t \to C$, $\chi: B \to C$ such that $\chi \circ \beta_t = \gamma_t \circ \phi_t$ for all $t \in T$; we have to find a unique homomorphism $\psi: A \to C$ such that $\psi \circ \alpha_t = \gamma_t$ for all $t \in T$, and $\psi \circ \phi = \chi$. The uniqueness of ψ is clear since the α_t cover A. We now define $\psi := \bigcup \gamma_t \circ \alpha_t^{-1}$, i.e.,

$$\psi(a) = \gamma_t(a_t) \quad (\text{if } a = \alpha_t(a_t)) \quad \text{for some } t \in T,\ a_t \in A_t,$$

where dom $\psi = A$ since the α_t cover A, and $\psi(a)$ is unique by (i), (ii), and the injectivity of the $\alpha_t|(A_t - \phi_t B_t)$ and ϕ. By definition, $\psi \circ \alpha_t = \gamma_t$, hence also $\psi \circ \phi = \chi$ since the β_t cover B. In particular, the $\psi \circ \alpha_t$ as well as $\psi \circ \phi$ are homomorphisms, so, since A has the final structure for the α_t and ϕ, ψ is a homomorphism, concluding the proof.

Theorem 4 will be used in the proof of Theorem 5.

THEOREM 5. *Let* $\alpha_t: A_t \to A$, $\phi: B \to A$ *be the generalized amalgamated partial direct sum of* $\beta_t: B_t \to B$, $\phi_t: B_t \to A$, *where the* β_t *are injective and cover* B, *the* ϕ_t *are isomorphisms onto* (*closed*) *subalgebras* $\phi_t B_t \subset A_t$. *Then the* α_t *are injective and cover* A, ϕ *is an isomorphism onto the* (*closed*) *subalgebra* $\phi B \subset A$, *and* (i), (ii) *of Theorem* 4 *hold.*

Proof by construction of a partial algebra A' *and homomorphisms* α'_t. $A_t \to A'$, $\phi': B \to A'$ such that all hypotheses of Theorem 4 hold true, even in the stronger form that α'_t (not only the restriction to $A_t - \phi_t B_t$) is injective for all $t \in T$, and that ϕ' is not only injective but an isomorphism onto a subalgebra $\phi' B \subset A'$. Then by Theorem 4, $\alpha'_t: A_t \to A'$, $\phi': B \to A'$ will be another model of the generalized amalgamated partial direct sum; hence there will be an (unique) isomorphism $\omega: A \to A'$ such that $\omega \circ \alpha_t = \alpha'_t$ for all $t \in T$, also $\omega \circ \phi = \phi'$, showing that the properties established by construction for the α'_t and ϕ' also hold true for the α_t and ϕ of the present theorem.

Writing A instead of A', α_t instead of α'_t, ϕ instead of ϕ', we start from a disjoint union A of sets $A_t - \phi_t B_t$ and B, with associated injections $\gamma_t: A_t - \phi_t B_t \to A$ and $\phi: B \to A$. One may even construct A in such a manner that $\phi: B \to A$ is the inclusion. Defining $\alpha_t := \gamma_t \cup (\phi \circ) \beta_t \circ \phi_t^{-1}$ $(t \in T)$, already the non-algebraic statements as listed above hold true: the α_t are injective since the β_t and ϕ_t are, the α_t cover A since the β_t cover B, ϕ is injective, the diagrams of Theorem 4 commute, and we have (i) and (ii).

In order to show that in A the final structure for the α_t and ϕ exists, we have to verify condition (ii) of the corollary of Theorem 1. First, let

$$\alpha_s(\mathfrak{a}_s) = \alpha_t(\mathfrak{a}_t), \qquad f_{si}(\mathfrak{a}_s) = a_s, \qquad f_{ti}(\mathfrak{a}_t) = a_t$$

($s, t \in T$, $i \in I$, $\mathfrak{a}_s$, $\mathfrak{a}_t$ sequences of type K_i in A_s, A_t, respectively, f_{si}, f_{ti} the fundamental operations in algebras A_s, A_t); we have to show that

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$\alpha_s(a_s) = \alpha_t(a_t)$. If $s = t$, $\mathfrak{a}_s = \mathfrak{a}_t$ since α_t is injective; hence $a_s = a_t$. If $s \neq t$, sequence $\mathfrak{a}_s$ is in $\phi_s(B_s)$, $\mathfrak{a}_t$ in $\phi_t(B_t)$, $\mathfrak{a}_s = \phi_s(\mathfrak{b}_s)$, $\mathfrak{a}_t = \phi_t(\mathfrak{b}_t)$, where $\mathfrak{b}_s, \mathfrak{b}_t$ are sequences in B_s and B_t, respectively. Since $\phi_s(B_s)$ and $\phi_t(B_t)$ are closed subsets of A_s, A_t, respectively, we have $a_s \in \phi_s(B_s)$, $a_t \in \phi_t(B_t)$; $a_s = \phi_s(b_s)$, $a_t = \phi_t(b_t)$, where $b_s \in B_s$, $b_t \in B_t$. Moreover, $g_{si}(\mathfrak{b}_s) = b_s$, $g_{ti}(\mathfrak{b}_t) = b_t$ (g_{si}, g_{ti} the fundamental operations in algebras B_s, B_t, respectively), since ϕ_s, ϕ_t are isomorphisms onto $\phi_s(B_s)$, $\phi_t(B_t)$. Hence

$$g_i(\beta_s(\mathfrak{b}_s)) = \beta_s(b_s), \qquad g_i(\beta_t(\mathfrak{b}_t)) = \beta_t(b_t)$$

(g_i the fundamental operation algebra B) since β_s, β_t are homomorphisms. From

$$\beta_s(\mathfrak{b}_s) = (\phi \circ \beta_s)(\mathfrak{b}_s) = (\alpha_s \circ \phi_s)(\mathfrak{b}_s) = \alpha_s(\mathfrak{a}_s) = \alpha_t(\mathfrak{a}_t) = \ldots = \beta_t(\mathfrak{b}_t),$$

we obtain

$$\alpha_s(a_s) = (\alpha_s \circ \phi_s)(b_s) = \beta_s(b_s) = g_i(\beta_s(\mathfrak{b}_s)) = g_i(\beta_t(\mathfrak{b}_t)) = \ldots = \alpha_t(a_t).$$

Hence the final structure f^0 for the α_t exists:

> $f^0{}_i(\mathfrak{a}) = a$ if and only if $\alpha_t(\mathfrak{a}_t) = \mathfrak{a}$, $\alpha_t(a_t) = a$, $f_{ti}(\mathfrak{a}_t) = a_t$, for some $t \in T$, some sequence $\mathfrak{a}_t$, some element a_t in A_t.

If, in particular, $\mathfrak{a}$ and a are in B $(=\phi(B))$, again $\mathfrak{a}_t$ and a_t are in $\phi_t(B_t)$, $\mathfrak{a}_t = \phi_t(\mathfrak{b}_t) a_t = \phi_t(b_t)$; again $g_{ti}(\mathfrak{b}_t) = b_t$, and hence

$$g_i(\mathfrak{a}) = g_i(\alpha_t(\mathfrak{a}_t)) = g_i((\alpha_t \circ \phi_t)(\mathfrak{b}_t)) = g_i(\beta_t(\mathfrak{b}_t)) = \alpha_t(a_t) = a,$$

showing that $f^0{}_i(\mathfrak{a}) = a$, $\mathfrak{a}$, a in B implies $g_i(\mathfrak{a}) = a$. So by the definition

$$f_i := f^0{}_i \cup g_i \qquad (i \in I),$$

we obtain the final structure f for the α_t, together with inclusion $\phi: B \to A$, and the given algebra (B, g) becomes a relative algebra of (A, f), i.e., inclusion ϕ is a strong homomorphism.

Finally, $B = \phi(B)$ is a closed subset, i.e., (B, g) is a subalgebra of algebra (A, f). For let $i \in I$, $\mathfrak{a} \in B^{K_i}$, and $f_i(\mathfrak{a}) = a \in A$. If $g_i(\mathfrak{a}) = a$, trivially $a \in B$. If $f^0{}_i(\mathfrak{a}) = a$, i.e., $\alpha_t(\mathfrak{a}_t) = \mathfrak{a}$, $\alpha_t(a_t) = a$, $f_{ti}(\mathfrak{a}_t) = a_t$, for some $t \in T$, etc., $\mathfrak{a}_t$ again is a sequence in subset $\phi_t(B_t) \subset A_t$, and since this subset is closed, we have $a_t \in \phi_t(B_t)$, $a_t = \phi_t(b_t)$, with $b_t \in B_t$, hence $a = \alpha_t(a_t) = (\alpha_t \circ \phi_t)(b_t) = \beta_t(b_t) \in B$, completing the proof of Theorem 5.

Note that the closure hypothesis for the $\phi_t B_t \subset A_t$ is indispensable. (Cf. the trivial but striking counter-example: Let $\alpha_t: A_t \to A$ be the partial direct sum of partial algebras A_t, and define $B_t = B = \emptyset$, $\beta_t = \phi_t = \phi =$ empty homomorphism (viz., the identical automorphism of partial algebra $\emptyset$, the inclusion homomorphism of the empty relative algebra into A_t or A, respectively); then $\alpha_t: A_t \to A$ together with $\phi: \emptyset \to A$ is the generalized amalgamated partial direct sum of the $\beta_t: \emptyset \to \emptyset$ and $\phi_t: \emptyset \to A_t$.) Naturally, the α_t need not be injective (this was just the point where our problem

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arose), and this can only happen if $\emptyset$ is not closed in at least one of the A_t, since all other assumptions are fulfilled in our example.

From Theorems 3, 4, and 5, we obtain the following corollary.

COROLLARY. *Let β_t: $B_t \to B$ be injective homomorphisms that cover B, and let ϕ_t: $B_t \to A_t$ be isomorphisms onto (closed) subalgebras $\phi_t B_t \subset A_t$. Then α_t: $A_t \to A$, ϕ: $B \to A$ is the corresponding generalized amalgamated partial direct sum if and only if*

(1) *A has the final structure for the α_t and ϕ;*

(2) *the α_t are injective and cover A, ϕ is injective;*

(3) *the diagrams of Theorem 4 commute;*

(4) (i), (ii) *of Theorem 4 hold.*

In this case, ϕ is an isomorphism onto a (closed) subalgebra $\phi B \subset A$.

Here we have an internal characterization (description) of the generalized amalgamated partial direct sum, at least in a particular situation. Note that at least in this particular situation one may conclude, conversely to Theorem 3, that if A has the final structure for the α_t alone, then amalgam B has the final structure for the β_t; the easy proof is left to the reader. Another addition follows.

ADDITION. *For arbitrary $s \in T$, α_s: $A_s \to A$ is strong if and only if β_s: $B_s \to B$ is.*

For if α_s: $A_s \to A$ is strong, then so is $\phi \circ \beta_s = \alpha_s \circ \phi_s$: $B_s \to A$, hence also β_s: $B_s \to B$, since all homomorphisms are injective, ϕ and ϕ_s even strong. Conversely, let β_s: $B_s \to B$ be strong. Assume $f_i(\alpha_s(\mathfrak{a}_s)) = \alpha_s(a_s)$ for some $i \in I$, some sequence $\mathfrak{a}_s$, some element a_s in A_s; as α_s is injective, we have to show that $f_{si}(\mathfrak{a}_s) = a_s$. A has the final structure for the α_t and ϕ; so $\alpha_s(\mathfrak{a}_s) = \alpha_t(\mathfrak{a}_t)$, $\alpha_s(a_s) = \alpha_t(a_t)$, and $f_{ti}(\mathfrak{a}_t) = a_t$, for some $t \in T$, some sequence $\mathfrak{a}_t$, some element a_t in A_t, or $\alpha_s(\mathfrak{a}_s) = \phi(\mathfrak{b})$, $\alpha_s(a_s) = \phi(b)$, and $g_i(\mathfrak{b}) = b$, for some sequence $\mathfrak{b}$, some element b in B. In the first case, if $s = t$, our proof is complete since α_s is injective. If $s \neq t$, $\alpha_s(\mathfrak{a}_s)$ and $\alpha_s(a_s)$ are in $\phi(B)$, again $\alpha_s(\mathfrak{a}_s) = \phi(\mathfrak{b})$, $\alpha_s(a_s) = \phi(b)$, for some sequence $\mathfrak{b}$, some element b in B. Moreover, since ϕ is strong and injective, $g_i(\mathfrak{b}) = b$, and we are in the second case. In this case, $\mathfrak{a}_s = \phi_s(\mathfrak{b}_s)$, $a_s = \phi_s(b_s)$, for some sequence $\mathfrak{b}_s$, some element b_s in B_s. We obtain

$$\phi(\beta_s(b_s)) = \alpha_s(\phi_s(\mathfrak{b}_s)) = \alpha_s(\mathfrak{a}_s) = \phi(\mathfrak{b});$$

hence $\beta_s(\mathfrak{b}_s) = \mathfrak{b}$ since ϕ is injective, and equally $\beta_s(b_s) = b$. Moreover, since β_s is injective and strong by assumption, $g_{si}(\mathfrak{b}_s) = b_s$, which gives $f_{si}(\mathfrak{a}_s) = a_s$ since ϕ_s is a homomorphism, completing the proof of the addition.

5. Application to partial direct sums (the application we wanted).

THEOREM 6. *Let α_t: $A_t \to A$ be the partial direct sum of the A_t, and let the $B_t \subset A_t$ be (closed) subalgebras. Then the following statements are equivalent:*

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(i) *the* α_t *are injective;*

(ii) *the* $\beta_t := \alpha_t|B_t$ *are injective.*

Moreover, for an arbitrary partial direct sum $\beta^0{}_t\colon B_t \to B^0$ *of the* B_t, *the above statements are equivalent to*

(iii) *the* $\beta^0{}_t$ *are injective.*

In this case, $\beta_t\colon B_t \to B := \bigcup \operatorname{im} \beta_t$ *is a partial direct sum of the* B_t, *and relative algebra* $B \subset A$ *is closed, i.e., a subalgebra of* A. *Moreover,*

(iv) $\alpha_t(A_t - B_t) \subset A - B$, $\alpha_s A_s \cap \alpha_t A_t \subset B$ *for all* $s, t \in T$, $s \neq t$.

Proof. (i) $\Rightarrow$ (ii) is trivial, equally, because of the universality property of the partial direct sum, (ii) $\Rightarrow$ (iii). It remains to prove (iii) $\Rightarrow$ (i) as well as the additional statements. Let $\phi_t\colon B_t \to A_t$ be the inclusion homomorphisms; then there is a (unique) homomorphism $\phi^0\colon B^0 \to A$ such that $\phi^0 \circ \beta^0{}_t = \alpha_t \circ \phi_t = \beta_t$ for all $t \in T$, and $\alpha_t\colon A_t \to A$ together with $\phi^0\colon B^0 \to A$ becomes a generalized amalgamated partial direct sum of the $\beta^0{}_t\colon B_t \to B^0$, $\phi_t\colon B_t \to A_t$. Since the ϕ_t are isomorphisms onto subalgebras $B_t \subset A_t$, the α_t are injective by Theorem 5. By Theorem 5, ϕ^0 is an isomorphism onto subalgebra

$$\phi^0 B^0 = \phi^0 \bigcup \beta^0{}_t B_t = \bigcup \beta_t B_t = B.$$

Hence also $\beta_t\colon B_t \to B$ is a partial direct sum, and $\alpha_t\colon A_t \to A$, together with inclusion homomorphism $\phi\colon B \to A$, is a generalized amalgamated partial direct sum of the $\beta_t\colon B_t \to B$, $\phi_t\colon B_t \to A_t$. Hence by Theorem 5, (i) and (ii) of Theorem 4, i.e., (iv) of the present theorem holds, completing its proof.

ADDITION. *Let* $\alpha_t\colon A_t \to A$ *be the partial direct sum of the* A_t, *all* α_t *injective. Then for arbitrary* $s \in T$, α_s *is strong if and only if* $\alpha_s|B_s$ *is, where* B_s *is some (closed) subalgebra of* A_s.

Proof. Select, for all $t \neq s$, subalgebras $B_t \subset A_t$. By Theorem 6, $\beta_t := \alpha_t|B_t\colon B_t \to B := \bigcup \alpha_t B_t \subset A$ is a partial direct sum. Again, $\alpha_t\colon A_t \to A$, $\phi\colon B \to A$ is a generalized amalgamated partial direct sum of the $\beta_t\colon B_t \to B$, $\phi_t\colon B_t \to A_t$, where ϕ and the ϕ_t are the inclusion homomorphisms. By the addition to Theorem 5, the asserted equivalence holds.

Again, together with Theorems 2 and 4, Theorem 6 leads to a corollary similar to that of Theorem 5.

COROLLARY. *Let* $\alpha_t\colon A_t \to A$ *be homomorphisms such that their restrictions* $\beta_t := \alpha_t | B_t$ *to certain (closed) subalgebras* $B_t \subset A_t$ *are injective. Then* $\alpha_t\colon A_t \to A$ *is a partial direct sum of the* A_t *if and only if*

(1) A *has the final structure for the* α_t;

(2) *the* α_t *are injective and cover* A;

(3) $\beta_t\colon B_t \to B := \bigcup \operatorname{im} \beta_t$ *is a partial direct sum of the* B_t;

(4) (iv) *of Theorem* 6 *holds.*

Note that if A has the final structure for the α_t, then this trivially remains true if we add an arbitrary homomorphism into A, e.g., the inclusion homomorphism $\phi\colon B \to A$. So by Theorem 4, if the conditions of our corollary

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hold, $\alpha_t: A_t \to A$, together with $\phi: B \to A$, becomes a generalized amalgamated partial direct sum of the $\beta_t: B_t \to B$ and the inclusion homomorphisms $\phi_t: B_t \to A_t$; hence, as $\beta_t: B_t \to B$ is a partial direct sum, so is $\alpha_t: A_t \to A$.

Unfortunately, this corollary does not give any real insight into the structure of the partial direct sum (as the corollary to Theorem 5 did for the generalized amalgamated partial direct sum); rather than giving an internal description, the present corollary only localizes the partial direct sum property, and as the example below shows, not much can be done about it.

Naturally, this localization obtains its highest possible efficiency if we select as subalgebras B_t the smallest subalgebras O_t, those generated by the empty set $\emptyset$. If we do so, relative algebra $B = \bigcup \alpha_t O_t$ (which ought to be a subalgebra in the case under consideration) will have to pass into the smallest subalgebra $O \subset A$, since, trivially, $B \subset O$ in general. On the other hand, if we assume that $A_t = O_t$, $A = 0$, i.e., all of the partial algebras A_t, A without proper subalgebras, then the localization criteria as given in Theorem 6 and its corollary becomes absolutely worthless (a rose is a rose is a rose!). In particular, it seems impossible to obtain a complete internal description of the partial direct sum of partial algebras O_t (without proper subalgebras), even in the special case when the universal homomorphisms happen to become injective. (Cf. the most simple and striking example of a one-element set $A_0 = A_1 = A = \{a\}$ supplied with three different partial algebraic structures of type $\Delta = (0, 0)$:

$$(A_0, (a, \emptyset)), \quad (A_1, (\emptyset, a)), \quad (A, (a, a)),$$

where a is the only non-empty nullary operation in set $A_0 = A_1 = A$, whereas $\emptyset$ ($\neq a$!) denotes the empty nullary operation.) Clearly, the $i_t := \mathrm{id}_A: A_t \to A$ are injective, even bijective, and cover A; moreover, A has the final structure; nevertheless, the $i_t: A_t \to A$ fail to represent a partial direct sum (cf. Addition 2 to Theorem 2).

There is a still more special case in which we find a completely satisfying description of the partial direct sum:

THEOREM 7. *Let the A_t ($t \in T$) be partial algebras with isomorphic smallest subalgebras O_t. Then $\alpha_t: A_t \to A$ is a partial direct sum of the A_t if and only if:*

(1) *A has the final structure for the α_t;*

(2) *the α_t are injective and cover A;*

(3) $\mathrm{im}\, \alpha_s \cap \mathrm{im}\, \alpha_t \subset 0$ *for all* $s \neq t$, $s, t \in T$.

In this case, the α_t are strong, hence isomorphisms onto relative algebras $\mathrm{im}\, \alpha_t \subset A$, and their restrictions $\alpha_t \mid O_t$ are isomorphisms onto smallest subalgebra $O \subset A$.

Observe that a family of isomorphisms $\beta^0{}_t: O_t \to B^0$ is necessarily a partial direct sum of the O_t; for as B^0 is generated by the empty set, there is at most one homomorphism $\psi: B^0 \to C$, C an arbitrary partial algebra, so for arbitrary homomorphisms $\chi_t: O_t \to C, \psi := \chi_t \circ (\beta^0{}_t)^{-1}: B^0 \to C$ is independent

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of index t. Hence, if the $\alpha_t: A_t \to A$ constitute a partial direct sum of the A_t, they are injective and strong by Theorem 6 and its addition; moreover, $\operatorname{im} \alpha_s \cap \operatorname{im} \alpha_t \subset B$ for all $s \neq t$, where $B := \bigcup \alpha_t O_t \subset 0$, even $B = 0$ since B is a subalgebra of O. Again by Theorem 2, A has the final structure for the α_t and is covered by them.

Conversely, let the three conditions above hold true. We show that the $\alpha_t: O_t \to B$ are isomorphisms. In fact, $\alpha_s O_s = \alpha_t O_t = B$ for all $s, t, \in T$. For, by hypothesis, there is an isomorphism $\omega_{ts}: O_t \to O_s$, so

$$\alpha_s \circ \omega_{ts} \quad \text{and} \quad \alpha_t | O_t: O_t \to B$$

are homomorphisms which, coinciding on generating subset $\emptyset$, must be equal; in particular, $\alpha_s O_s = \alpha_s \omega_{ts} O_t = \alpha_t O_t$. So the $\alpha_t | O_t: O_t \to B$ are surjective. Moreover, they are strong. For let $f_i \alpha_t \mathfrak{b}_t = \alpha_t b_t$, $\mathfrak{b}_t$, b_t in O_t (f_i the fundamental operation in A). As A has the final structure, $\alpha_t \mathfrak{b}_t = \alpha_s \mathfrak{a}_s$, $\alpha_t b_t = \alpha_s a_s, f_{si} \mathfrak{a}_s = a_s$, for some $s \in T$, $\mathfrak{a}_s$, a_s in A_s. But then $\alpha_s \omega_{ts} \mathfrak{b}_t = \alpha_s \mathfrak{a}_s$; hence $\mathfrak{a}_s = \omega_{ts} \mathfrak{b}_t$ is in O_s since α_s is injective; similarly, $a_s = \omega_{ts} b_t \in O_s$, and applying the converse isomorphism ω_{ts}^{-1}, we obtain $f_{ti} \mathfrak{b}_t = b_t$. As remarked above, $\alpha_t | O_t: O_t \to B$ becomes a partial direct sum of the O_t. Moreover, from the injectivity of the α_t, we obtain

$$\alpha_t(A_t - O_t) = \alpha_t A_t - \alpha_t O_t \subset A - \alpha_t O_t = A - B.$$

So the conditions of the corollary of Theorem 6 hold, and the present theorem is proved.

6. Application to full direct sums. This becomes possible by means of the following theorem.

THEOREM 8. *Let $\alpha_t: A_t \to A$ be homomorphisms of full algebras A_t into partial algebra A. Then the following statements are equivalent*:

(i) *$\alpha_t: A_t \to A$ is a full direct sum of the A_t;*

(ii) *A is a free completion of relative algebra $A^0 := \bigcup \operatorname{im} \alpha_t \subset A$, and $\alpha_t: A_t \to A^0$ is a partial direct sum of the A_t.*

To prove (ii) $\Rightarrow$ (i), let $\chi_t: A_t \to C$ be homomorphisms into some full algebra C. Then there is a unique homomorphism $\chi^0: A^0 \to C$ such that $\chi^0 \circ \alpha_t = \chi_t$ for all $t \in T$. By definition of free completion (cf. Burmeister-Schmidt (**2**)), χ^0 has a unique homomorphic extension $\chi: A \to C$, which is the wanted unique homomorphism such that $\chi \circ \alpha_t = \chi_t$ for all $t \in T$. To prove (i) $\Rightarrow$ (ii), we construct a partial direct sum $\beta_t: A_t \to B^0$ of the A_t. B^0 has a free completion B (which, in particular, is an extension of B^0, i.e., contains B^0 as a relative algebra). According to (ii) $\Rightarrow$ (i) (as just proved), $\beta_t: A_t \to B$ is a full direct sum of the A_t, as, by assumption, is the given family $\alpha_t: A_t \to A$. Hence, there is a (unique) isomorphism $\omega: B \to A$ such that $\alpha_t = \omega \circ \beta_t$ for all $t \in T$, by means of which the properties of B^0 are transported to $A^0 = \omega B^0$, the properties of B to A; also $\alpha_t: A_t \to A$ has property (ii), completing the proof of Theorem 8.

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Note that relative algebras $\operatorname{im} \alpha_t \subset A$ are complete, hence closed in A. Now, if type $\Delta = (K_i)_{i \in I}$ is at most unary, i.e., $|K_i| \leqq 1$ for all $i \in I$, and if $T \neq \emptyset$, then also $A^0 = \bigcup \operatorname{im} \alpha_t$ is closed, hence $A^0 = A$.

COROLLARY 1. *For a non-empty family of at most unary full algebras A_t, partial and full direct sums coincide.*

Without restriction of type Δ, we have the following corollary.

COROLLARY 2. *Let α_t: $A_t \to A$ be the full direct sum of full algebras A_t. Then the following statements are equivalent*:

(i) *the α_t are injective*;

(ii) *the $\alpha_t \mid O_t$ are injective, hence isomorphisms onto O, where the O_t and O are the smallest subalgebras of the A_t and A, respectively*;

(iii) $O_s \cong O_t$ *for all $s, t \in T$.*

Proof. (i) $\Rightarrow$ (ii) is trivial; note that, because of the completeness of the A_t, $\operatorname{im} \alpha_t = 0$. So (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i) follows immediately from Theorems 7 and 8 or, if one prefers, Theorems 6 and 8 (for another proof cf. Kerkhoff (**5**)).

In (**7**), the author has defined full algebras A_s, A_t such that $O_s \cong O_t$ to be of *equal* 0-*characteristic.* Full algebras of equal 0-characteristic are also treated in the analogue to Theorem 7:

THEOREM 9. *Let the A_t $(t \in T)$ be full algebras with isomorphic smallest subalgebras (equal 0-characteristic). Then α_t: $A_t \to A$ (A some full algebra) is a full direct sum of the A_t if and only if*:

(1) *the α_t are injective and generate A*;

(2) $\operatorname{im} \alpha_s \cap \operatorname{im} \alpha_t \subset 0$ *for all $s \neq t$, $s, t \in T$*;

(3) *if $g_i\mathfrak{a} \in T A^0 := \bigcup \operatorname{im} \alpha_t$, then sequence $\mathfrak{a}$ is in* $\operatorname{im} \alpha_t$, *for some $t \in T$*;

(4) *if $g_i\mathfrak{a} = g_j\mathfrak{b} \notin A^0$, then $i = j$ and $\mathfrak{a} = \mathfrak{b}$.*

Here the g_i are the fundamental operations of algebra A, O its smallest subalgebra.

Proof. If α_t: $A_t \to A$ is a full direct sum, then the α_t are injective; moreover, $\operatorname{im} \alpha_s \cap \operatorname{im} \alpha_t \subset 0$ for $s \neq t$ by Theorems 7 and 8. But A is the free completion of A^0 by Theorem 8. So by the internal characterization of the free completion by the *Generalized Peano Axioms* FC1–FC3 (cf. Burmeister-Schmidt (**2**)), A^0 generates A (*Axiom of Induction*, FC3). Moreover, $g_i\mathfrak{a} = g_j\mathfrak{b} \notin A^0$ implies $i = j$ and $\mathfrak{a} = \mathfrak{b}$(FC2). Finally, $g_i\mathfrak{a} \in A^0$ implies $\mathfrak{a}$ in A^0 (FC1). But since A^0 has the final structure for the α_t (Theorems 7 and 8), $\mathfrak{a}$ is even in some $\operatorname{im} \alpha_t$.

Conversely, let the four conditions of our theorem hold. In particular, the Generalized Peano Axioms FC1–FC3 as described above hold: A is the free completion of A^0. It remains to show that α_t: $A_t \to A^0$ is a partial direct sum, i.e., that the three conditions of Theorem 7 hold true. The only thing to show is that relative algebra $A^0 \subset A$ has the final structure for the α_t. So

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let $g_i\mathfrak{a} = a$, where $\mathfrak{a}$, a are in A^0. Hence $\mathfrak{a}$ is in some $\operatorname{im}\alpha_\iota$, as is a, since $\operatorname{im}\alpha_\iota$ is closed. This completes the proof, since $\alpha_\iota: A_\iota \to \operatorname{im}\alpha_\iota$ is an isomorphism.

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