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Siegel–Whittaker functions on $SO_o(2, q)$ for class one principal series representations

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Abstract

In this paper we study a kind of spherical function, which we call a Siegel–Whittaker function, related to Fourier expansions of automorphic forms on Hermitian symmetric domains of type IV. We obtain a multiplicity-one theorem and an integral representation of this spherical function.

1. Introduction

In this paper we study generalized spherical functions which are related to Fourier expansions of automorphic forms on the bounded symmetric domain of type IV. As is well-known Fourier expansions play a very important role in the theory of automorphic forms, for instance, in the construction of automorphic L-functions (cf. [And71, And74, Hor95, Sug85]). However, the theory is incomplete, especially at the archimedean places.

Before the formulation of our problems in representation theoretic terminology, let us explain the problem of Fourier expansion of wave forms more classically. We first recall the definition of wave forms for a Riemannian symmetric pair (G, K) and an arithmetic subgroup Γ of G which has cusps. A C^∞-function f on G is called a wave form if

i) f is left Γ-invariant and right K-invariant,

ii) f is a common eigenfunction of G-invariant differential operators on G/K, and

iii) f satisfies some growth conditions.

We notice that the conditions i and ii imply that f generates a class one principal series representation of G by right translation.

If we consider the case $G = SO_o(2, q)$, the symmetric space G/K is the Hermitian symmetric space of type IV. Let P_s be the Siegel parabolic subgroup of G with abelian unipotent radical N_s and the Levi part L_s associated to the zero-dimensional cusp of Γ. We investigate the Fourier expansion of a wave form f on G along P_s. Take a unitary character ξ of N_s and put

$$A_\xi(g) = \int_{N_s \cap \Gamma \backslash N_s} f(ng)\xi^{-1}(n)\, dn,$$

the Fourier coefficient corresponding to ξ. Then we have

$$f(ng) = \sum_{\xi \in (N_s \cap \Gamma \backslash N_s)^\wedge} A_\xi(g)\xi(n),$$

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for $(n, g) \in N_s \times G$. If we consider the system of partial differential equations of $A_\xi(g)$ deduced from the condition ii, the dimension of the space of solutions is infinite. Therefore we consider a further expansion of $A_\xi(g)$ with respect to a subgroup $SO(\xi)$ of L_s. If ξ is 'definite', $SO(\xi)$ is isomorphic to $SO(q-1)$. We fix a finite-dimensional irreducible representation (χ, V_χ) of $SO(\xi)$ and put

$$A_{\xi,\chi}(g) = \int_{SO(\xi)} A_\xi(sg) \cdot \chi(s)\, ds.$$

Hence we have

$$A_\xi(sg) = \sum_{\chi \in SO(\xi)^\wedge} \langle A_{\xi,\chi}(g), \chi^*(s) \rangle = \sum_{\chi \in SO(\xi)^\wedge} \sum_{i=1}^{\dim \chi} \langle A_{\xi,\chi,i}(g), \chi_i^*(s) \rangle,$$

for $(s, g) \in SO(\xi) \times G$. Here χ^* is the contragredient representation of χ and $\langle\ ,\ \rangle$ denotes the inner product on $V_\chi \times V_{\chi^*}$. The Fourier coefficient $A_{\xi,\chi,i}(g)$ is a V_χ-valued function on the two-dimensional space $SO(\xi)\backslash L_s / L_s \cap K$ and the space of solutions of the system of differential equations for $A_{\xi,\chi,i}(g)$ becomes finite dimensional. Further, combined with the growth conditions, $A_{\xi,\chi,i}$ is uniquely determined (up to a constant) and has an integral expression (Theorem 8.1). This leads us to the Fourier expansion of f for the 'definite' terms. In the case of 'indefinite' ξ, that is, $SO(\xi)$ is not a compact group, we do not have a satisfactory answer even for $G = \mathrm{Sp}(2, \mathbb{R})/\pm 1_2$ $(q = 3)$.

Now we give a precise formulation of our problem. Let G be a real reductive Lie group with the Lie algebra $\mathfrak{g}$ and K a maximal compact subgroup of G. Take a closed subgroup R of G and an irreducible smooth representation η of R. For an irreducible admissible representation π of G, consider the space of intertwining operators

$$\mathcal{I}(\pi, \eta) = \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C^\infty \operatorname{Ind}_R^G(\eta)).$$

For $\Phi \in \mathcal{I}(\pi, \eta)$, the realization $\mathrm{Im}(\Phi)$ of π is called the *spherical function for* π. The fundamental problems are

i) determining the dimension of the intertwining space (under some growth conditions),

ii) finding explicit formulas for the spherical functions.

If we take suitable R, the above problems are closely related to the local theory of automorphic forms. For example, when R is a maximal unipotent subgroup of G and η is a unitary character of R, the spherical functions are called Whittaker functions and have been studied by many authors (H. Jacquet, J. Shalika, B. Kostant, N. Wallach, etc.).

Let us explain our situation in this paper again. Let $G = SO_o(2, q)$ $(q \geqslant 3)$, $P_s = L_s \ltimes N_s$ and $\xi \in N_s^\wedge$ and consider the intertwining space $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C^\infty \operatorname{Ind}_{N_s}^G(\xi))$. If π is the holomorphic discrete series representation, this space is finite dimensional and the spherical functions for π can be expressed by using the exponential functions. This leads to the well-known Fourier expansions of holomorphic modular forms along P_s. However, when π is the (class one) principal series representation, the above space becomes infinite dimensional. Then we take a larger subgroup R containing N_s as follows. Denote by $SO(\xi)$ the identity component of the stabilizer of ξ in L_s and define $R = SO(\xi) \ltimes N_s$. Let us take an irreducible unitary representation χ of $SO(\xi)$ and set $\eta = \chi \cdot \xi$. We remark that the induced representation $\operatorname{Ind}_R^G(\eta)$ is a special case of the generalized Gel'fand–Graev representation studied by Yamashita [Yam88].

In the case where $G = \mathrm{Sp}(2, \mathbb{R})$ and ξ is 'definite' ($SO(\xi) \cong SO(2)$), Niwa [Niw91] obtained the multiplicity-one property for $\mathcal{I}(\pi, \eta)$ and found explicit formulas for the spherical functions, which we call *Siegel–Whittaker functions*, for the class one principal series representations. They appear in the Fourier expansions of Siegel wave forms of degree two. The main purpose

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of this paper is to extend his results to the $SO_o(2, q)$ case. Note that $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3)$ and $\mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4)$.

These kinds of spherical functions have been studied by Miyazaki [Miy00] in the case of $\mathrm{Sp}(2, \mathbb{R})$ and by Gon [Gon02] in the case of $SU(2, 2)$ for various discrete series representations. Among others, the author [Ish02] proved the multiplicity free theorem for the (general) principal series representations on $\mathrm{Sp}(2, \mathbb{R})$.

For the class one principal series representation of G, spherical functions are characterized by the invariant differential operators on G/K. We first determine the generators of this algebra by the classical method (§ 4) and write down the system of partial differential equations (Theorem 7.1) for 'definite' ξ. By considering the Mellin transformation of the system, we obtain the following theorem.

MAIN THEOREM (Theorem 8.1). *We assume that $\nu_1, \nu_2, \nu_1 \pm \nu_2$, the parameters of the class one principal series representation π_ν, are not integers. Let $\xi = \xi_0$ be the standard 'definite' unitary character of N_s and χ_λ be the irreducible finite-dimensional representation of $SO(\xi)$ ($\cong SO(q-1)$) with highest weight $\lambda = (\lambda_1, \dots, \lambda_{[(q-1)/2]})$.*

1) *If λ is not of the form $(\lambda_1, 0, \dots, 0)$, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta)) = 0.$$

2) *If $\lambda = (\lambda_1, 0, \dots, 0)$, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta)^{\mathrm{rap}}) = 1,$$

and the radial part of the spherical function has an explicit integral expression of Euler type. Here $C^\infty \mathrm{Ind}_R^G(\eta)^{\mathrm{rap}}$ means the space of rapidly decreasing functions in $C^\infty \mathrm{Ind}_R^G(\eta)$.

We comment on the possible application of our result to automorphic L-functions. As is well known, Andrianov [And71, And74] constructed the (spinor) L-function for holomorphic Siegel cusp forms of degree two and proved the functional equation. Sugano [Sug85] extended Andrianov's result to the $SO(2, q)$ case. Meanwhile, Hori [Hor95] studied the wave form version of Andrianov's L-function and proved the functional equation by using Niwa's explicit formulas for Siegel–Whittaker functions. Generalizing this further, we already obtained a kind of zeta integral for the (general) principal series representation of $\mathrm{Sp}(2, \mathbb{R})$ in the previous paper [Ish02]. In the same way as in [Sug85] and [Hor95], it may be possible to prove the functional equation for the L-function of wave forms on $SO(2, q)$.

2. Structure theory for $SO_o(2, q)$ and its Lie algebra

We recall some basic facts on our Lie groups and algebras. Let $q \geqslant 3$ and $G = SO_o(2, q)$ be the identity component of the special orthogonal group of signature $(2+, q-)$:

$$SO(2, q) = \left\{ g \in SL(2+q, \mathbb{R}) \,\middle|\, {}^{\mathrm{t}}g 1_{2,q} g = 1_{2,q} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_q \end{pmatrix} \right\}.$$

Here we denote by 1_n the unit matrix of degree n. A maximal compact subgroup K of G is given by

$$K = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \,\middle|\, k_1 \in SO(2), k_2 \in SO(q) \right\}.$$

The Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \mathfrak{so}(2, q) = \{ X \in M(2+q, \mathbb{R}) \mid {}^{\mathrm{t}}X 1_{2,q} + 1_{2,q} X = 0 \}.$$

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The subspaces

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \middle| X_1 = -{}^{\mathrm{t}}X_1 \in M(2,\mathbb{R}), X_2 = -{}^{\mathrm{t}}X_2 \in M(q,\mathbb{R}) \right\},$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ {}^{\mathrm{t}}X & 0 \end{pmatrix} \middle| X \in M(2,q,\mathbb{R}) \right\}$$

give a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ as

$$\mathfrak{a} = \mathbb{R}A_1 \oplus \mathbb{R}A_2$$

with $A_1 = E_{1,q+2} + E_{q+2,1}$, $A_2 = E_{2,q+1} + E_{q+1,2}$. Here $E_{i,j}$ is a matrix with 1 in the (i,j) entry and 0 elsewhere. Put

$$\begin{aligned} A = \exp(\mathfrak{a}) &= \{\exp(\log a_1 A_1 + \log a_2 A_2) \mid a_1, a_2 > 0\} \\ &= \left\{ \left(\begin{array}{cc|c|cc} c(a_1) & & & & s(a_1) \\ & c(a_2) & & s(a_2) & \\ \hline & & 1_{q-2} & & \\ \hline & s(a_2) & & c(a_2) & \\ s(a_1) & & & & c(a_1) \end{array} \right) \middle| a_1, a_2 > 0 \right\} \end{aligned}$$

with $c(a) = (a + a^{-1})/2$ and $s(a) = (a - a^{-1})/2$. Define linear forms e_1, e_2 on $\mathfrak{a}$ by $e_i(a_1A_1 + a_2A_2) = a_i (i = 1, 2)$ and set $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X,\ \forall H \in \mathfrak{a}\}$. Then the restricted root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is given by $\Delta = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ and we fix a positive system Δ^+ as $\Delta^+ = \{e_1, e_2, e_1 \pm e_2\}$. Then each root space $\mathfrak{g}_\alpha (\alpha \in \Delta^+)$ is given as

$$\mathfrak{g}_{e_1} = \bigoplus_{i=1}^{q-2} \mathbb{R}X_i, \quad \mathfrak{g}_{e_2} = \bigoplus_{i=1}^{q-2} \mathbb{R}Y_i, \quad \mathfrak{g}_{e_1-e_2} = \mathbb{R}Z_1, \quad \mathfrak{g}_{e_1+e_2} = \mathbb{R}Z_2,$$

with root vectors

$$\begin{aligned} X_i &= E_{1,i+2} + E_{i+2,1} - E_{i+2,q+2} + E_{q+2,i+2}, \\ Y_i &= E_{2,i+2} + E_{i+2,2} - E_{i+2,q+1} + E_{q+1,i+2}, \\ Z_1 &= (-E_{1,2} - E_{1,q+1} + E_{2,1} - E_{2,q+2} - E_{q+1,1} + E_{q+1,q+2} - E_{q+2,2} - E_{q+2,q+1})/2, \\ Z_2 &= (-E_{1,2} + E_{1,q+1} + E_{2,1} - E_{2,q+2} + E_{q+1,1} - E_{q+1,q+2} - E_{q+2,2} + E_{q+2,q+1})/2. \end{aligned}$$

For $\alpha \in \Delta \backslash \Delta^+$, the elements $X_{-i} = {}^{\mathrm{t}}X_i$, $Y_{-i} = {}^{\mathrm{t}}Y_i$, $Z_{-1} = {}^{\mathrm{t}}Z_1$, $Z_{-2} = {}^{\mathrm{t}}Z_2$ are generators of $\mathfrak{g}_\alpha$. If we put $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Further put $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then we have

$$\mathfrak{m} = \left\{ \left(\begin{array}{c|c|c} 1_2 & & \\ \hline & M & \\ \hline & & 1_2 \end{array} \right) \middle| M \in \mathfrak{so}(q-2) \right\},$$

and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$. We take an $\mathbb{R}$-basis of $\mathfrak{m}$ by

$$K_{i,j} = E_{i+2,j+2} - E_{j+2,i+2} \quad (1 \leqslant i < j \leqslant q-2).$$

Set $N = \exp(\mathfrak{n})$ and $M = \exp(\mathfrak{m})$.

3. Definition of Siegel–Whittaker functions

3.1 Class one principal series representations of G

We recall the definition of the class one principal series representations of G. Let $P_0 = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of G. For $\nu = (\nu_1, \nu_2) \in \mathfrak{a}_{\mathbb{C}}^*$, let H_{π_ν}

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be the space of smooth functions ϕ on G such that

$$\phi(mang) = a^{\nu+\rho}\phi(g),$$

for $m \in M$, $a \in A$, $n \in N$ and $g \in G$. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}(\dim \mathfrak{g}_\alpha)\alpha = (q/2, q/2-1)$. G acts on H_{π_ν} by right translation:

$$\pi_\nu(g)\phi(x) = \phi(xg)$$

for $x, g \in G$ and $\phi \in H_{\pi_\nu}$. We call the induced representation π_ν the *class one principal series representation* and denote by $H_{\pi_\nu,K}$ the subspace of K-finite vectors in H_{π_ν}. Define an element ϕ_0 in $H_{\pi_\nu,K}$ by

$$\phi_0(g) = a(g)^{\nu+\rho}$$

with the Iwasawa decomposition $g = n(g)a(g)k(g)$ of $g \in G$ ($n(g) \in N$, $a(g) \in A$ and $k(g) \in K$). Then $\phi_0(g)$ is a K-fixed vector in H_{π_ν}.

3.2 Definition of Siegel–Whittaker functions

In this section we introduce a kind of spherical function, which we call a Siegel–Whittaker function. This type of spherical function was introduced by Yamashita [Yam88] in a more general setting, and studied by Miyazaki [Miy00], by the author [Ish02] for $\mathrm{Sp}(2,\mathbb{R})$ and by Gon [Gon02] for $SU(2,2)$.

The Siegel parabolic subgroup P_s of G is a maximal parabolic subgroup corresponding to the short root with abelian unipotent radical. Its Levi decomposition is given by $P_s = L_s \ltimes N_s$, where

$$L_s = \left\{ \left(\begin{array}{c|c|c} a & & b \\ \hline & g_0 & \\ \hline c & & d \end{array}\right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO_o(1,1),\ g_0 \in SO_o(1,q-1) \right\}$$

and

$$\begin{aligned} N_s &= \exp(\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1-\alpha_2}) \\ &= \left\{ n_s(x) := \exp\left((x_1 - x_q)Z_1 + (x_1 + x_q)Z_2 + \sum_{i=1}^{q-2} x_{i+1}X_i\right) \right. \\ &\qquad = \left. \left(\begin{array}{c|c|c} 1 + x_0 & \tilde{x} & -x_0 \\ \hline {}^{\mathrm{t}}x & 1_q & -{}^{\mathrm{t}}x \\ \hline x_0 & \tilde{x} & 1 - x_0 \end{array}\right) \,\middle|\, x = (x_1, \ldots, x_q) \in \mathbb{R}^q \right\} \end{aligned}$$

with $x_0 = \frac{1}{2}(-x_1^2 + \sum_{i=2}^q x_i^2)$ and $\tilde{x} = (-x_1, x_2, \ldots, x_q)$.

Fix a unitary character ξ of N_s defined by

$$\xi(n_s(x)) = \exp\left(2\pi\sqrt{-1}\sum_{i=1}^q \xi_i x_i\right)$$

with real numbers ξ_i. We assume that $\xi_1^2 - \sum_{i=2}^q \xi_i^2 \neq 0$.

Consider the action of L_s on N_s by conjugation, and also the induced action of L_s on the character group $\hat{N}_s$. The identity component of the stabilizer subgroup of the character ξ is denoted by $SO(\xi)$. Then

$$\begin{aligned} SO(\xi) &= \mathrm{Stab}_{L_s}(\xi)^\circ \\ &= \left\{ \left(\begin{array}{c|c|c} 1 & & \\ \hline & g_0 & \\ \hline & & 1 \end{array}\right) \,\middle|\, (\xi_1, \ldots, \xi_q)g_0 = (\xi_1, \ldots, \xi_q),\ g_0 \in SO_o(1,q-1) \right\}. \end{aligned}$$

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Hence we can verify that

$$SO(\xi) \cong \begin{cases} SO(q-1) & \text{if } \xi_1^2 - \sum_{i=2}^q \xi_i^2 > 0, \\ SO_{\mathrm{o}}(1, q-2) & \text{if } \xi_1^2 - \sum_{i=2}^q \xi_i^2 < 0. \end{cases}$$

We say that ξ is *definite* if $\xi_1^2 - \sum_{i=2}^q \xi_i^2 > 0$ and *indefinite* if $\xi_1^2 - \sum_{i=2}^q \xi_i^2 < 0$.

Remark 3.1. Let ξ_0 be the standard definite character of N_s with $\xi_1 = 1, \xi_2 = \cdots = \xi_q = 0$. We can easily check that

$$SO(\xi_0) = \left\{ \left(\begin{array}{c|c|c} 1_2 & & \\ \hline & g_0 & \\ \hline & & 1 \end{array} \right) \;\middle|\; g_0 \in SO(q-1) \right\}.$$

From now on we only treat the case of definite ξ. (We may assume $\xi = \xi_0$ without loss of generality.) Now we take an irreducible representation (χ, V_χ) of $SO(\xi) \cong SO(q-1)$ and define the subgroup R of L_s by

$$R = SO(\xi) \ltimes N_s.$$

Then we can construct a well-defined representation $\eta = \chi \cdot \xi$ of R and consider the C^∞-induced representation from R to G,

$$C^\infty \operatorname{Ind}_R^G(\eta) = \{f : G \to V_\chi, C^\infty \mid f(rg) = \eta(r) f(g),\ \forall (r, g) \in R \times G\}.$$

This is a special case of the reduced generalized Gel'fand–Graev representation studied in [Yam88].

Definition 3.2. Let (π, H_π) be an irreducible admissible representation of G and $H_{\pi,K}$ the subspace of K-finite vectors in H_π. The space of intertwining operators

$$\operatorname{Hom}_{(\mathfrak{g},K)}(H_{\pi,K}, C^\infty \operatorname{Ind}_R^G(\eta))$$

of $(\mathfrak{g}, K)$-modules is called the space of *Siegel–Whittaker functionals for* π.

Before we discuss our spherical functions, we recall some facts on universal enveloping algebras. Let $U(\mathfrak{g}_\mathbb{C})$ and $U(\mathfrak{a}_\mathbb{C})$ be the universal enveloping algebras of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}$, the complexification of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. Then the decomposition

$$U(\mathfrak{g}_\mathbb{C}) = U(\mathfrak{a}_\mathbb{C}) \oplus (\mathfrak{n} U(\mathfrak{g}_\mathbb{C}) + U(\mathfrak{g}_\mathbb{C})\mathfrak{k})$$

holds [HC58]. Let $p : U(\mathfrak{g}_\mathbb{C}) \to U(\mathfrak{a}_\mathbb{C})$ be the projection corresponding to the above decomposition and γ the automorphism of $U(\mathfrak{a}_\mathbb{C})$ which takes $H \in \mathfrak{a}_\mathbb{C}$ to $H + \rho(H) \in \mathfrak{a}_\mathbb{C}$. We denote by $W = W(\mathfrak{g}, \mathfrak{a})$ the Weyl group of $\mathfrak{g}$ relative to $\mathfrak{a}$ and $U(\mathfrak{a}_\mathbb{C})^W$ the W-invariant elements in $U(\mathfrak{a}_\mathbb{C})$. Note that $U(\mathfrak{a}_\mathbb{C})$ can be identified with the symmetric algebra $S(\mathfrak{a}_\mathbb{C})$ and $U(\mathfrak{a}_\mathbb{C})^W = S(\mathfrak{a}_\mathbb{C})^W$ is isomorphic to the polynomial ring of two variables over $\mathbb{C}$ in our case. Set

$$U(\mathfrak{g}_\mathbb{C})^K = \{X \in U(\mathfrak{g}_\mathbb{C}) \mid \operatorname{Ad}(k)X = X,\ \forall k \in K\}.$$

Then we have the following theorem.

Theorem 3.3 (Harish-Chandra [HC58]). *The following sequence is exact:*

$$0 \longrightarrow U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C})\mathfrak{k} \longrightarrow U(\mathfrak{g}_\mathbb{C})^K \xrightarrow{\gamma \circ p} U(\mathfrak{a}_\mathbb{C})^W \longrightarrow 0.$$

If we denote by $\boldsymbol{D}(G/K)$ the algebra of G-invariant differential operators on G/K, then

$$\boldsymbol{D}(G/K) \cong S(\mathfrak{a}_\mathbb{C})^W \cong U(\mathfrak{g}_\mathbb{C})^K / U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$$

by Theorem 3.3.

For $\nu \in \mathfrak{a}_\mathbb{C}^*$, define an algebra homomorphism $c_\nu : U(\mathfrak{g}_\mathbb{C})^K \to \mathbb{C}$ by

$$c_\nu(z) = \nu(\gamma \circ p(z)),$$

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for $z \in U(\mathfrak{g}_C)^K$. Then Theorem 3.3 implies that c_ν is trivial on $U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$. Moreover the restriction of c_ν to the center $Z(\mathfrak{g}_\mathbb{C})$ of $U(\mathfrak{g}_\mathbb{C})$ coincides with the infinitesimal character of the class one principal series representation π_ν.

Now we define spherical functions for π_ν. Let v_0 be the unique (up to constant multiple) K-fixed vector in $H_{\pi_\nu,K}$ (§ 3.1). Then, for $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta))$, the V_χ-valued function $\Phi(v_0)$ on G is contained in the space

$$C_\eta^\infty(R\backslash G/K) := \{f : G \to V_\chi, C^\infty \mid f(rgk) = \eta(r)f(g),\ \forall (r,g,k) \in R \times G \times K\}.$$

Moreover it satisfies

$$z\Phi(v_0) = c_\nu(z)\Phi(v_0), \quad \forall z \in U(\mathfrak{g}_\mathbb{C})^K.$$

We call $\Phi(v_0)$ a *Siegel–Whittaker function of type* $(\pi_\nu; \chi, \xi)$ and define

$$\mathrm{SW}(\pi_\nu; \eta) = \mathrm{SW}(\pi_\nu; \chi, \xi) := \{\Phi(v_0) \mid \Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta))\}.$$

Then we have

$$\mathrm{SW}(\pi_\nu; \eta) = \{f \in C_\eta^\infty(R\backslash G/K) \mid zf = c_\nu(z)f, \quad \forall z \in U(\mathfrak{g}_\mathbb{C})^K\}$$

and

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta)) \cong \mathrm{SW}(\pi_\nu; \eta).$$

We consider the system of differential equations

$$zf = c_\nu(z)f, \quad \forall z \in U(\mathfrak{g}_\mathbb{C})^K.$$

Since f is right K-invariant and c_ν is trivial on $U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$, the system is equivalent to

$$zf = c_\nu(z)f, \quad \forall z \in U(\mathfrak{g}_\mathbb{C})^K / U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C})\mathfrak{k}.$$

Because of the isomorphism $U(\mathfrak{a}_\mathbb{C})^W \cong \mathbb{C}[A_1^2 + A_2^2, A_1^2 A_2^2]$ and Theorem 3.3, we can identify $U(\mathfrak{g}_\mathbb{C})^K / U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$ with $\mathbb{C}[C_2, C_4]$. Here C_2 and C_4 are generators of $Z(\mathfrak{g}_\mathbb{C})$ with degrees two and four, respectively. Thus we have

$$\mathrm{SW}(\pi_\nu; \eta) = \{f \in C^\infty(R\backslash G/K) \mid zf = c_\nu(z)f,\ \forall z \in \mathbb{C}[C_2, C_4]\}.$$

Note that Nakajima [Nak82] obtained an explicit formula for the system of generators of $\boldsymbol{D}(G/K)$.

4. Invariant differential operators

In this section we give explicit forms of C_2 and C_4. We recall the method of calculating a system of generators of $Z(\mathfrak{g}_\mathbb{C})$ which is classically known (cf. [Bou75]). Let $S(\mathfrak{g}_\mathbb{C})$ be the symmetric algebra over $\mathfrak{g}_\mathbb{C}$ and $I(\mathfrak{g}_\mathbb{C}) = \{P \in S(\mathfrak{g}_\mathbb{C}) \mid \mathrm{ad}X(P) = 0, \forall X \in \mathfrak{g}_\mathbb{C}\}$. Then the symmetrizer map $\lambda : I(\mathfrak{g}_\mathbb{C}) \to Z(\mathfrak{g}_\mathbb{C})$ defined by $\lambda(X_1 \cdots X_n) = (1/n!) \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$ gives an isomorphism as $\mathbb{C}$-vector spaces.

The generators of $I(\mathfrak{g}_\mathbb{C})$ are given as follows. Let $M = (M_{ij})_{1 \leqslant i,j \leqslant 3}$ be the matrix of size $q+2$ with

$$M_{11} = \begin{pmatrix} 0 & (Z_1 - Z_2 + Z_{-1} + Z_{-2})/2 \\ (Z_1 + Z_2 - Z_{-1} - Z_{-2})/2 & 0 \end{pmatrix},$$
$$M_{22} = (K_{i,j})_{1 \leqslant i,j \leqslant q-2},$$

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$$M_{33} = \begin{pmatrix} 0 & (Z_1 - Z_2 - Z_{-1} + Z_{-2})/2 \\ (-Z_1 + Z_2 + Z_{-1} - Z_{-2})/2 & 0 \end{pmatrix},$$

$$M_{13} = {}^{\mathrm{t}}M_{31} = \begin{pmatrix} (-Z_1 + Z_2 - Z_{-1} + Z_{-2})/2 & A_1 \\ A_2 & (-Z_1 - Z_2 - Z_{-1} - Z_{-2})/2 \end{pmatrix},$$

$$M_{12} = {}^{\mathrm{t}}M_{21} = \begin{pmatrix} (X_1 + X_{-1})/2 & \cdots & (X_{q-2} + X_{-(q-2)})/2 \\ (Y_1 + Y_{-1})/2 & \cdots & (Y_{q-2} + Y_{-(q-2)})/2 \end{pmatrix},$$

$$M_{32} = -{}^{\mathrm{t}}M_{23} = \begin{pmatrix} (Y_1 - Y_{-1})/2 & \cdots & (Y_{(q-2)} - Y_{-(q-2)})/2 \\ (X_1 - X_{-1})/2 & \cdots & (X_{q-2} - X_{-(q-2)})/2 \end{pmatrix}.$$

Then the set $\{\mathrm{Tr}(M^i)\ (1 \leqslant i \leqslant (q+1)/2)\}$ (respectively $\{\mathrm{Tr}(M^i)\ (1 \leqslant i \leqslant q/2),\ \det(M)\}$) is a system of generators of $I(\mathfrak{g}_{\mathbb{C}})$ for odd (respectively even) q. For our purpose we only compute the explicit form of $\mathrm{Tr}(M^2)$ and $\mathrm{Tr}(M^4)$:

$$\mathrm{Tr}(M^2) = A_1^2 + A_2^2 + \sum_i (X_i X_{-i} + Y_i Y_{-i}) + 2(Z_1 Z_{-1} + Z_2 Z_{-2}) - \sum_{i<j} K_{i,j}^2$$

and

$$\begin{aligned}
\mathrm{Tr}(M^4) = {}& 2(A_1^4 + A_2^4) + 8(A_1^2 + A_2^2)(Z_1 Z_{-1} + Z_2 Z_{-2}) \\
& + 8A_1 A_2 (Z_1 Z_{-1} - Z_2 Z_{-2}) + 4(Z_1^2 Z_{-1}^2 + Z_2^2 Z_{-2}^2) + 24 Z_1 Z_2 Z_{-1} Z_{-2} \\
& + 4\sum_i (A_1^2 X_i X_{-i} + A_2^2 Y_i Y_{-i}) - 4\sum_i (A_1 + A_2)(Z_1 Y_i X_{-i} + Z_{-1} X_i Y_{-i}) \\
& - 4\sum_i (A_1 - A_2)(Z_2 X_{-i} Y_{-i} + Z_{-2} X_i Y_i) \\
& + 2\sum_i (X_i X_{-i} + Y_i Y_{-i})^2 + \sum_{i<j} (X_i X_{-j} + X_j X_{-i} + Y_i Y_{-j} + Y_j Y_{-i})^2 \\
& - 4\sum_i (-Z_1 Z_2 X_{-i}^2 - Z_{-1} Z_{-2} X_i^2 + Z_1 Z_{-2} Y_i^2 + Z_2 Z_{-1} Y_{-i}^2) \\
& + 4\sum_i (Z_1 Z_{-1} + Z_2 Z_{-2})(X_i X_{-i} + Y_i Y_{-i}) \\
& + 2\sum_{i<j} K_{i,j}^4 + 4\sum_{i<j,\,l\neq i,j} K_{i,l}^2 K_{j,l}^2 + 4\sum_{i<j,\,l<m} K_{i,l} K_{j,l} K_{i,m} K_{j,m} \\
& - 4\sum_{i<j} (X_i X_{-i} + X_j X_{-j} + Y_i Y_{-i} + Y_j Y_{-j}) K_{i,j}^2 \\
& - 4\sum_{i<j,\,l\neq i,j} (X_i X_{-j} + X_j X_{-i} + Y_i Y_{-j} + Y_j Y_{-i}) K_{i,l} K_{j,l} \\
& + 4\sum_{i<j} \{A_1 (X_i X_{-j} - X_j X_{-i}) + A_2 (Y_i Y_{-j} - Y_j Y_{-i})\} K_{i,j} \\
& - 4\sum_{i\neq j} (Z_1 Y_i X_{-j} + Z_2 X_{-j} Y_{-i} + Z_{-1} X_i Y_{-j} + Z_{-2} X_i Y_j) K_{i,j}.
\end{aligned}$$

Here indices run from 1 to $q-2$. Let us define the elements of $Z(\mathfrak{g}_{\mathbb{C}})$ by

$$C_2 = \lambda(\mathrm{Tr}(M^2)), \quad C_4 = \lambda(\tfrac{1}{2}\mathrm{Tr}(M^2)^2 - \mathrm{Tr}(M^4)),$$

which are degrees two and four, respectively. Then we obtain the following proposition.

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PROPOSITION 4.1. *C_2 and C_4 are of the form*

$$\begin{aligned}C_2 &= A_1^2 + A_2^2 - qA_1 - (q-2)A_2 + 2(Z_1Z_{-1} + Z_2Z_{-2}) \\ &\quad + \sum_i (X_iX_{-i} + Y_iY_{-i}) - \sum_{i<j} K_{i,j}^2\end{aligned}$$

and

$$\begin{aligned}C_4 = {} & 4A_1^2A_2^2 - 8A_1A_2(Z_1Z_{-1} - Z_2Z_{-2}) + 4(Z_1^2Z_{-1}^2 + Z_2^2Z_{-2}^2) - 8Z_1Z_2Z_{-1}Z_{-2} \\
& - \sum_{i\neq j}(X_{-i}^2X_j^2 + Y_{-i}^2Y_j^2) + 2\sum_{i<j}(X_iX_jX_{-i}X_{-j} + Y_iY_jY_{-i}Y_{-j}) \\
& - 2\sum_{i\neq j}(Y_iY_{-j}X_iX_{-j} + Y_iY_{-j}X_jX_{-i}) + 4\sum_{i\neq j} Y_jY_{-j}X_iX_{-i} \\
& + 4\sum_i (A_1^2Y_iY_{-i} + A_2^2X_iX_{-i}) + 4\sum_i (Z_1Z_{-1} + Z_2Z_{-2})(X_iX_{-i} + Y_iY_{-i}) \\
& + 4\sum_i (Z_{-1}Z_{-2}X_i^2 + Z_1Z_2X_{-i}^2 - Z_1Z_{-2}Y_i^2 - Z_{-1}Z_2Y_{-i}^2) \\
& + 4\sum_i (A_1 - A_2)(Z_{-2}Y_iX_i + Z_2Y_{-i}X_{-i}) \\
& + 4\sum_i (A_1 + A_2)(Z_{-1}Y_{-i}X_i + Z_1Y_iX_{-i}) \\
& - 4\sum_{i<j}(A_1^2 + A_2^2)K_{i,j}^2 - 8\sum_{i<j}(Z_1Z_{-1} + Z_2Z_{-2})K_{i,j}^2 \\
& - 4\sum_{i\neq j}(A_1X_iX_{-j} + A_2Y_iY_{-j})K_{i,j} - 4\sum_{i<j, l\neq i,j}(X_lX_{-l} + Y_lY_{-l})K_{i,j}^2 \\
& + 4\sum_{i<j, l\neq i,j}(X_iX_{-j} + X_jX_{-i} + Y_iY_{-j} + Y_jY_{-i})K_{i,l}K_{j,l} \\
& - 4\sum_{i\neq j}(Z_1Y_jX_{-i} + Z_2X_{-i}Y_{-j} + Z_{-1}X_jY_{-i} + Z_{-2}X_jY_i)K_{i,j} \\
& + 4\sum_{i<j<l<m}(K_{i,j}^2K_{l,m}^2 + K_{i,l}^2K_{j,m}^2 + K_{i,m}^2K_{j,l}^2) \\
& - 8\sum_{i<j<l<m}(K_{i,l}K_{j,l}K_{i,m}K_{j,m} + K_{i,j}K_{l,j}K_{i,m}K_{l,m} + K_{i,j}K_{j,m}K_{i,l}K_{m,l}) \\
& - 4(q-2)A_1^2A_2 - 4qA_1A_2^2 - 4(q-1)A_1Z_1Z_{-1} \\
& - 4(q-1)A_1Z_2Z_{-2} - 4(q-5)A_2Z_1Z_{-1} - 4(q+1)A_2Z_2Z_{-2} \\
& - 2(q-1)\sum_i A_1X_iX_{-i} - 4(q-2)\sum_i A_1Y_iY_{-i} - 4(q-3)\sum_i A_2X_iX_{-i} \\
& - 2(q-3)\sum_i A_2Y_iY_{-i} + 4q\sum_{i<j} A_1K_{i,j}^2 + 4(q-2)\sum_{i<j} A_2K_{i,j}^2 \\
& + \sum_{i<j}\{2(q-5)(X_iX_{-j} + Y_iY_{-j}) + 2(q-3)(X_jX_{-i} + Y_jY_{-i})\}K_{i,j} \\
& + 2(q-3)\sum_i (Z_1Y_iX_{-i} + Z_2Y_{-i}X_{-i} - Z_{-1}Y_{-i}X_i - Z_{-2}Y_iX_i) \\
& + 4\sum_{i<j<l<m}(K_{i,j}K_{i,l}K_{j,l} + K_{i,j}K_{m,i}K_{j,m} + K_{i,l}K_{i,m}K_{l,m} + K_{j,l}K_{m,j}K_{m,l})\end{aligned}$$

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$$-\frac{1}{3}(q^2-49q+96)A_1^2-\frac{1}{3}(q^2+11q-36)A_2^2+4(q-2)^2A_1A_2$$
$$-\frac{2}{3}(q-1)(q-12)(Z_1Z_{-1}+Z_2Z_{-2})+\frac{1}{3}(q^2-25q+192)\sum_{i<j}K_{i,j}^2$$
$$-\frac{1}{3}(q^2-37q+108)\sum_i X_iX_{-i}-\frac{1}{3}(q-4)(q-9)\sum_i Y_iY_{-i}$$
$$+\frac{1}{3}(q^3-37q^2+156q-168)A_1+\frac{1}{3}(q-2)(q-4)(q-9)A_2.$$

Proof. By using $X_{-i}X_i=X_iX_{-i}-2A_1$, $Y_{-i}Y_i=Y_iY_{-i}-2A_2$, $Z_{-1}Z_1=Z_1Z_{-1}-A_1+A_2$ and $Z_{-2}Z_2=Z_2Z_{-2}-A_1-A_2$, we have the formula for C_2. As for C_4, we need a more tedious calculation. First we can check that $\frac{1}{2}\mathrm{Tr}(M^2)^2-\mathrm{Tr}(M^4)$ is the sum of the following 15 terms by the formulas for $\mathrm{Tr}(M^2)$ and $\mathrm{Tr}(M^4)$:

$$T_1=4A_1^2A_2^2-8A_1A_2(Z_1Z_{-1}-Z_2Z_{-2})+4(Z_1^2Z_{-1}^2+Z_2^2Z_{-2}^2)-8Z_1Z_2Z_{-1}Z_{-2},$$
$$T_2=-\sum_{i\neq j}(X_{-i}^2X_j^2+Y_{-i}^2Y_j^2),$$
$$T_3=2\sum_{i<j}(X_iX_jX_{-i}X_{-j}+Y_iY_jY_{-i}Y_{-j}),$$
$$T_4=-2\sum_{i\neq j}(Y_iY_{-j}X_iX_{-j}+Y_iY_{-j}X_jX_{-i}),$$
$$T_5=4\sum_{i\neq j}Y_jY_{-j}X_iX_{-i},$$
$$T_6=4\sum_i(A_1^2Y_iY_{-i}+A_2^2X_iX_{-i}),$$
$$T_7=4\sum_i(Z_1Z_{-1}+Z_2Z_{-2})(X_iX_{-i}+Y_iY_{-i}),$$
$$T_8=4\sum_i(Z_{-1}Z_{-2}X_i^2+Z_1Z_2X_{-i}^2-Z_1Z_{-2}Y_i^2-Z_{-1}Z_2Y_{-i}^2),$$
$$T_9=4\sum_i\{(A_1-A_2)(Z_{-2}Y_iX_i+Z_2Y_{-i}X_{-i})+(A_1+A_2)(Z_{-1}Y_{-i}X_i+Z_1Y_iX_{-i})\},$$
$$T_{10}=-4\sum_{i<j}(A_1^2+A_2^2)K_{i,j}^2-8\sum_{i<j}(Z_1Z_{-1}+Z_2Z_{-2})K_{i,j}^2,$$
$$T_{11}=-4\sum_{i<j,l\neq i,j}(X_lX_{-l}+Y_lY_{-l})K_{i,j}^2,$$
$$T_{12}=-4\sum_{i\neq j}(A_1X_iX_{-j}+A_2Y_iY_{-j})K_{i,j},$$
$$T_{13}=4\sum_{i<j,l\neq i,j}(X_iX_{-j}+X_jX_{-i}+Y_iY_{-j}+Y_jY_{-i})K_{i,l}K_{j,l},$$
$$T_{14}=-4\sum_{i\neq j}(Z_1Y_jX_{-i}+Z_2X_{-i}Y_{-j}+Z_{-1}X_jY_{-i}+Z_{-2}X_jY_i)K_{i,j},$$
$$T_{15}=4\sum_{i<j<l<m}(K_{i,j}^2K_{l,m}^2+K_{i,l}^2K_{j,m}^2+K_{i,m}^2K_{j,l}^2),$$
$$T_{16}=-8\sum_{i<j<l<m}(K_{i,l}K_{j,l}K_{i,m}K_{j,m}+K_{i,j}K_{l,j}K_{i,m}K_{l,m}+K_{i,j}K_{m,j}K_{i,l}K_{m,l}).$$

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We write down the image of the symmetrizer map of each term:

$$\begin{aligned}
\lambda(T_1) &= T_1 - 8A_1A_2^2 - 4A_1(Z_1Z_{-1} + Z_2Z_{-2}) + 12A_2(Z_1Z_{-1} - Z_2Z_{-2}) \\
&\quad - \frac{2}{3}A_1^2 + \frac{10}{3}A_2^2 + \frac{20}{3}(Z_1Z_{-1} + Z_2Z_{-2}) + \frac{4}{3}A_1, \\
\lambda(T_2) &= T_2 - 4\sum_{i\neq j}(X_iX_{-j} + Y_iY_{-j})K_{i,j} + 24\sum_{i<j}K_{i,j}^2 \\
&\quad + \frac{4}{3}(q-3)\sum_i(X_iX_{-i} + Y_iY_{-i}) - \frac{10}{3}(q-2)(q-3)(A_1 + A_2), \\
\lambda(T_3) &= T_3 - 2(q-3)\sum_i(A_1X_iX_{-i} + A_2Y_iY_{-i}) \\
&\quad + 2\sum_{i\neq j}(X_iX_{-j} + Y_iY_{-j})K_{i,j} + \frac{4}{3}(q-3)\sum_i(X_iX_{-i} + Y_iY_{-i}) \\
&\quad + (q-2)(q-3)(A_1^2 + A_2^2) - 4\sum_{i<j}K_{i,j}^2 + \frac{2}{3}(q-2)(q-3)(A_1 + A_2), \\
\lambda(T_4) &= T_4 - 2(q-3)\sum_i(Z_1Y_iX_{-i} + Z_2Y_{-i}X_{-i} - Z_{-1}Y_{-i}X_i - Z_{-2}Y_iX_i) \\
&\quad + 2(q-2)(q-3)(Z_1Z_{-1} + Z_2Z_{-2}) + \frac{16}{3}(q-3)\sum_i X_iX_{-i} \\
&\quad - \frac{8}{3}(q-3)\sum_i Y_iY_{-i} - \frac{22}{3}(q-2)(q-3)A_1 + \frac{8}{3}(q-2)(q-3)A_2, \\
\lambda(T_5) &= T_5 - 4(q-3)\sum_i(A_1Y_iY_{-i} + A_2X_iX_{-i}) + 4(q-2)(q-3)A_1A_2, \\
\lambda(T_6) &= T_6 - 4(q-2)A_1^2A_2 - 4(q-2)A_1A_2^2, \\
\lambda(T_7) &= T_7 - 4(q-2)(A_1 + A_2)(Z_1Z_{-1} + Z_2Z_{-2}) - 4\sum_i A_1(X_iX_{-i} + Y_iY_{-i}) \\
&\quad + \frac{2}{3}\sum_i(X_iX_{-i} + Y_iY_{-i}) - \frac{8}{3}(q-2)(Z_1Z_{-1} + Z_2Z_{-2}) \\
&\quad + 4(q-2)A_1^2 + 4(q-2)A_1A_2 + 2(q-2)A_1 - \frac{2}{3}(q-2)A_2 \\
\lambda(T_8) &= T_8 + \frac{16}{3}(q-2)(Z_1Z_{-1} + Z_2Z_{-2}) - \frac{4}{3}\sum_i(X_iX_{-i} + Y_iY_{-i}) \\
&\quad - 4(q-2)A_1 + \frac{4}{3}(q-2)A_2, \\
\lambda(T_9) &= T_9 + \frac{8}{3}(q-2)(Z_1Z_{-1} + Z_2Z_{-2}) - \frac{4}{3}\sum_i(X_iX_{-i} + Y_iY_{-i}) \\
&\quad + \frac{32}{3}(q-2)A_1^2 - \frac{16}{3}(q-2)A_2^2 - \frac{4}{3}(q-2)A_1 + \frac{4}{3}(q-2)A_2, \\
\lambda(T_{10}) &= T_{10} + 8\sum_{i<j}A_1K_{i,j}^2, \\
\lambda(T_{11}) &= T_{11} + 4(q-4)\sum_{i<j}(A_1 + A_2)K_{i,j}^2,
\end{aligned}$$

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$$\lambda(T_{12}) = T_{12} + 8\sum_{i<j}(A_1+A_2)K_{i,j}^2 + \frac{2}{3}(q-3)\sum_i (X_iX_{-i}+Y_iY_{-i}) + \frac{16}{3}\sum_{i<j}K_{i,j}^2$$
$$- \frac{4}{3}(q-2)(q-3)(A_1^2+A_2^2) - \frac{2}{3}(q-2)(q-3)(A_1+A_2),$$

$$\lambda(T_{13}) = T_{13} + 2(q-4)\sum_{i<j}(X_iX_{-j}+X_jX_{-i}+Y_iY_{-j}+Y_jY_{-i})K_{i,j}$$
$$- \frac{1}{3}(q-3)(q-4)\sum_i (X_iX_{-i}+Y_iY_{-i}) - \frac{16}{3}(q-4)\sum_{i<j}K_{i,j}^2$$
$$+ \frac{1}{3}(q-2)(q-3)(q-4)(A_1+A_2),$$

$$\lambda(T_{14}) = T_{14} - \frac{8}{3}(q-2)(q-3)(Z_1Z_{-1}+Z_2Z_{-2}) + \frac{4}{3}(q-3)\sum_i (X_iX_{-i}+Y_iY_{-i})$$
$$+ \frac{32}{3}\sum_{i<j}K_{i,j}^2 + \frac{4}{3}(q-2)(q-3)A_1 - \frac{4}{3}(q-2)(q-3)A_2,$$

$$\lambda(T_{15}) = T_{15},$$

$$\lambda(T_{16}) = T_{16} + \frac{1}{3}(q-4)(q-5)\sum_{i<j}K_{i,j}^2$$
$$+ 4\sum_{i<j<l<m}(K_{i,j}K_{i,l}K_{j,l} + K_{i,j}K_{m,i}K_{j,m} + K_{i,l}K_{i,m}K_{l,m} + K_{j,l}K_{m,j}K_{m,l}).$$

Thus we have the explicit formula for C_4. □

5. Irreducible representations of $SO(\xi)$

In this section we recall the irreducible representations of $SO(\xi) \cong SO(q-1)$. As is well known, the equivalent classes of the irreducible finite-dimensional representations of $SO(q-1)$ are parametrized by the set of dominant integral weights,

$$\begin{cases} \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0\} & \text{if } q-1 = 2m+1, \\ \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{m-1} \geqslant |\lambda_m|\} & \text{if } q-1 = 2m. \end{cases}$$

We denote by $(\chi_\lambda, V_\lambda)$ the irreducible finite-dimensional representation of $SO(q-1)$ with highest weight λ. To write down the infinitesimal actions of $\mathfrak{so}(q-1)$, we use the Gel'fand–Zetlin basis [GZ50, VK92].

Let $m_i = (m_{1,i}, m_{2,i}, \dots, m_{[i/2],i})$ $(2 \leqslant i \leqslant q-1)$ be vectors which satisfy the following conditions of Gel'fand–Zetlin patterns:

i) $m_{j,i}$ are all integers,

ii) $m_{q-1} = \lambda$,

iii) $m_{1,2k+1} \geqslant m_{1,2k} \geqslant m_{2,2k+1} \geqslant m_{2,2k} \geqslant \cdots \geqslant m_{k,2k+1} \geqslant m_{k,2k} \geqslant -m_{k,2k+1}$,

iv) $m_{1,2k} \geqslant m_{1,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \cdots \geqslant m_{k-1,2k} \geqslant m_{k-1,2k-1} \geqslant |m_{k,2k}|$.

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For the above m_i, we can define a Gel'fand–Zetlin pattern:

$$M = (m_{q-1},\dots,m_2) = \begin{pmatrix} m_{1,q-1}\ m_{2,q-1}\ \cdots\ m_{[(q-1)/2],q-1} \\ m_{2,q-2}\ \cdots \\ \cdots \\ m_{1,5}\ m_{2,5} \\ m_{1,4}\ m_{2,4} \\ m_{1,3} \\ m_{1,2} \end{pmatrix}.$$

Then the set of diagrams

$$GZ(\lambda) := \{M \mid m_i (2 \leqslant i \leqslant q-1) \text{ satisfy the conditions i–iv}\}$$

parametrize a basis of χ_λ and we call the basis $\{v(M) \mid M \in GZ(\lambda)\}$ the *Gel'fand–Zetlin basis*. The infinitesimal actions of the basis of $\mathfrak{so}(q-1)$ can be described as follows. For $F_{i,j} = E_{i,j} - E_{j,i}$ $(1 \leqslant i < j \leqslant q-1)$,

$$d\chi_\lambda(F_{2p,2p+1})v(M) = \sum_{k=1}^{p} A_{2p}^k(M)v(M_{2p,k}^+) - \sum_{k=1}^{p} A_{2p}^k(M_{2p,k}^-)v(M_{2p,k}^-),$$

$$\begin{aligned} d\chi_\lambda(F_{2p+1,2p+2})v(M) &= \sum_{k=1}^{p} B_{2p+1}^k(M)v(M_{2p+1,k}^+) \\ &\quad - \sum_{k=1}^{p} B_{2p+1}^k(M_{2p+1,k}^-)v(M_{2p+1,k}^-) + \sqrt{-1}C_{2p}(M)v(M). \end{aligned}$$

By using commuting relations of $F_{i,j}$, we can find explicit formulas for general $F_{i,j}$. Here $M_{i,j}^+$ (respectively $M_{i,j}^-$) means $m_{i,j}$ is replaced by $m_{i,j}+1$ (respectively $m_{i,j}-1$) and the others remain the same, and

$$A_{2p}^k(M) = \frac{1}{2}\left|\frac{\prod_{r=1}^{p-1}((l_{r,2p-1}-\frac{1}{2})^2 - (l_{k,2p}+\frac{1}{2})^2)\prod_{r=1}^{p}((l_{r,2p+1}-\frac{1}{2})^2 - (l_{k,2p}+\frac{1}{2})^2)}{\prod_{r=1,r\neq k}^{p}(l_{r,2p}^2 - l_{k,2p}^2)(l_{r,2p}^2 - (l_{k,2p}+1)^2)}\right|^{1/2},$$

$$B_{2p+1}^k(M) = \left|\frac{\prod_{r=1}^{p}(l_{r,2p}^2 - l_{k,2p+1}^2)\prod_{r=1}^{p+1}(l_{r,2p+2}^2 - l_{k,2p+1}^2)}{l_{k,2p+1}^2(4l_{k,2p+1}^2 - 1)\prod_{r=1,r\neq k}^{p}(l_{r,2p+1}^2 - l_{k,2p+1}^2)(l_{k,2p+1}^2 - (l_{r,2p+1}-1)^2)}\right|^{1/2},$$

$$C_{2p}(M) = \frac{\prod_{r=1}^{p} l_{r,2p}\prod_{r=1}^{p+1} l_{r,2p+2}}{\prod_{r=1}^{p} l_{r,2p+1}(l_{r,2p+1}-1)},$$

with $l_{k,2p} = m_{k,2p} + p - k$, $l_{k,2p+1} = m_{k,2p+1} + p - k + 1$. The following lemma will be used later.

LEMMA 5.1. *Under the above notation, put*

$$S_q = \sum_{i=1}^{q-2} d\chi_\lambda(F_{i,q-1}^2).$$

Then S_q acts on V_λ as scalar multiplication. More precisely

$$S_q v(M) = \Bigl(-\sum_{i=1}^{[(q-1)/2]}\{m_{i,q-1}^2 + (q-2i-1)m_{i,q-1}\} + \sum_{i=1}^{[(q-2)/2]}\{m_{i,q-2}^2 + (q-2i-2)m_{i,q-2}\}\Bigr)v(M).$$

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Proof. We first notice that the action of $F_{i,j} \in \mathfrak{so}(q-1)$ and $F_{i,j} \in \mathfrak{so}(q'-1)$ $(q' \leqslant q)$ on $\mathfrak{so}(q-1)$ is compatible by construction of the Gel'fand–Zetlin basis. (For instance, the action of $F_{1,2}$ is always of the form $d\chi_\lambda(F_{1,2})v(M) = \sqrt{-1}m_{1,2}v(M)$ independent of q.) Then we may regard an element defined by $I_p = \sum_{i=1}^{p-1} F_{i,p-1}^2 \in U(\mathfrak{so}(q-1))$ $(3 \leqslant p \leqslant q)$ as also an element in $U(\mathfrak{so}(q'-1))$. With this identification, we can write

$$(S_3 + \cdots + S_q)v(M) = d\chi_\lambda(I_3 + \cdots + I_q)v(M).$$

Since $I_3 + \cdots + I_q$ is a Casimir element of $U(\mathfrak{so}(q-1))$, it acts on V_λ as scalar multiplication:

$$d\chi_\lambda(I_3 + \cdots + I_q)v(M) = \{(\lambda, \lambda) + (\lambda, 2\rho)\}v(M),$$

where λ is the highest weight and ρ is the half sum of positive roots:

$$\lambda = (m_{1,q-1}, \ldots, m_{[(q-1)/2],q-1}),$$
$$\rho = \begin{cases} (m-1, m-2, \ldots, 0) & \text{if } q-1 = 2m, \\ (m-\frac{1}{2}, m-\frac{3}{2}, \ldots, \frac{1}{2}) & \text{if } q-1 = 2m+1. \end{cases}$$

Then if we write the scalar c_q, we have

$$c_q = (\lambda, \lambda) + (\lambda, 2\rho) = -\sum_{i=1}^{[(q-1)/2]} \{m_{i,q-1}^2 + (q-2i-1)m_{i,q-1}\}.$$

Since $S_q v(M) = (c_q - c_{q-1})v(M)$, the claim follows. □

6. Radial parts of differential operators

6.1 Radial parts, compatibility conditions

Let f be an element in $C_\eta^\infty(R\backslash G/K)$. Then f is determined by its restriction to A because of the decomposition $G = RAK$. To regard f as a function on A, we recall the notion of *radial part*.

Let $C_\eta^\infty(A, V_\chi)$ be the space of V_χ-valued C^∞-functions on A satisfying the condition

$$\phi(a) = \eta(m)\phi(a), \quad \forall m \in R \cap M. \tag{6.1}$$

Here $M = Z_K(A)$, the centralizer of A in K (§ 2). Then we can see that the restriction map $\mathrm{res}|_A : C_\eta^\infty(R\backslash G/K) \to C_\eta^\infty(A, V_\chi)$ is a linear injection. For any linear map $D : C_\eta^\infty(R\backslash G/K) \to C_\eta^\infty(R\backslash G/K)$, there exists a linear map

$$R(D) : C_\eta^\infty(A, V_\chi) \to C_\eta^\infty(A, V_\chi)$$

satisfying $R(D) \circ \mathrm{res}|_A = \mathrm{res}|_A \circ D$ and $R(D)$ is said to be the *radial part* of D. We also call $\mathrm{res}|_A(f)$ the *radial part* of f for each $f \in C_\eta^\infty(R\backslash G/K)$ and denote it by $f(a) = (\mathrm{res}|_A(f))(a)$ for short. According to § 4, $f(a)$ can be expressed as

$$f(a) = \sum_{M \in GZ(\lambda)} f_M(a)v(M).$$

By the following lemma, the compatibility condition (6.1) implies that almost all f_M but one vanish. More precisely, we have the following lemma.

Lemma 6.1. *We assume $\xi = \xi_0$ and let χ_λ be the irreducible finite-dimensional representation of $SO(\xi_0) \cong SO(q-1)$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_{[(q-1)/2]})$. If $\lambda = (\lambda_1, 0, \ldots, 0)$, then any*

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non-zero element $\phi \in C^\infty_\eta(A, V_{\chi_\lambda})$ *is of the form*

$$\phi(a) = \phi_{M_0}(a)v(M_0), \quad M_0 := \begin{pmatrix} \lambda_1 \; 0 \ldots 0 \\ 0 \ldots 0 \\ \cdots \\ 0 \end{pmatrix}.$$

Otherwise, $C^\infty_\eta(A, V_{\chi_\lambda}) = \{0\}$.

Proof. Since $R \cap M = M \cong SO(q-2)$ (cf. Remark 3.1), condition (6.1) implies that

$$d\chi_\lambda(K_{i,j})v(M) = 0$$

for all $1 \leqslant i < j \leqslant q-2$. Firstly if $q = 4$,

$$d\chi_\lambda(K_{1,2})v(M) = \sqrt{-1}m_{1,2}v(M) = 0$$

means $m_{1,2} = 0$. Secondly if $q = 5$, the equalities

$$\begin{aligned} d\chi_\lambda(K_{1,2})v(M) &= \sqrt{-1}m_{1,2}v(M) = 0, \\ d\chi_\lambda(K_{2,3})v(M) &= \frac{1}{2}\sqrt{(m_{1,3} - m_{1,2})(m_{1,3} + m_{1,2} + 1)}v(M^+_{1,2}) \\ &\quad - \frac{1}{2}\sqrt{(m_{1,3} - m_{1,2} + 1)(m_{1,3} + m_{1,2})}v(M^-_{1,2}) = 0 \end{aligned}$$

and the conditions for the Gel'fand–Zetlin patterns

$$m_{1,4} \geqslant m_{1,3} \geqslant |m_{2,4}|, \quad m_{1,3} \geqslant m_{1,2} \geqslant -m_{1,3}, \quad m_{1,4} \geqslant |m_{2,4}|$$

imply $m_{1,2} = m_{1,3} = m_{2,4} = 0$ and $m_{1,4} \geqslant 0$. Since the action of $d\chi_\lambda(K_{i,j})$ and the conditions for the Gel'fand–Zetlin patterns are 'the same' for any q, we can prove $m_{i,j} = 0$ except for the component $m_{1,q-1}$ in Gel'fand–Zetlin pattern M by induction on q for the general case. □

6.2 Radial parts of differential operators

To write down the differential equations which characterize Siegel–Whittaker functions, we calculate the radial parts of C_2 and C_4. For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, put

$$(Xf)(g) = \frac{d}{dt}(f(g\exp(tX)))\Big|_{t=0}$$

and we extend it to the action of $U(\mathfrak{g}_\mathbb{C})$ in the usual manner.

Proposition 6.2. *Let* $a = \exp(\log a_1 A_1 + \log a_2 A_2) \in A$. *If we use the symbol* $\partial_{a_i} = a_i(\partial/\partial a_i)$ $(i = 1, 2)$, *then for* $\phi \in C^\infty_\eta(A, V_\chi)$,

$$\begin{aligned} (R(A_i)\phi)(a) &= \partial_{a_i}\phi(a) \quad (i = 1, 2), \\ (R(X_i)\phi)(a) &= 2\pi\sqrt{-1}\xi_{i+1}a_1\phi(a) \quad (1 \leqslant i \leqslant q-2), \\ (R(Z_1)\phi)(a) &= \pi\sqrt{-1}(\xi_1 - \xi_q)a_1a_2^{-1}\phi(a), \\ (R(Z_2)\phi)(a) &= \pi\sqrt{-1}(\xi_1 + \xi_q)a_1a_2\phi(a). \end{aligned}$$

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Proof. We only prove the formula $(R(Z_1)\phi)(a)$

$$\begin{aligned}(R(Z_1)\phi)(a) &= \frac{d}{dt}\phi(\exp t(aZ_1a^{-1})\cdot a)|_{t=0}\\ &= \frac{d}{dt}\phi(n_s(\tfrac{1}{2}a_1a_2^{-1}t, 0, \ldots, 0, -\tfrac{1}{2}a_1a_2^{-1}t)\cdot a)|_{t=0}\\ &= \frac{d}{dt}\exp(\pi\sqrt{-1}a_1a_2^{-1}(\xi_1-\xi_q)t)|_{t=0}\phi(a)\\ &= \pi\sqrt{-1}(\xi_1-\xi_q)a_1a_2^{-1}\phi(a).\end{aligned}$$

We can verify the others in the same way. □

Proposition 6.3. *We assume $\xi=\xi_0$ for the unitary character of N_s. Then*

$$(R(Y_i^2)\phi)(a) = \frac{2a_2^2}{a_2^2-1}(\partial_{a_2}\phi)(a) + \frac{4a_2^2}{(a_2^2-1)^2}(d\chi(F_{i,q-1}^2))\phi(a) \quad (1\leqslant i\leqslant q-2).$$

Here $F_{i,j}=E_{i+2,j+2}-E_{j+2,i+2}$ $(1\leqslant i<j\leqslant q-1)$ are generators of $\mathfrak{so}(\xi_0)$.

Proof. We can easily find

$$Y_i = -t(a_2)F_{i,q-1} + s(a_2)^{-1}(\mathrm{Ad}(a^{-1})F_{i,q-1}),$$

with $t(a)=1+c(a)/s(a)=2a/(a-a^{-1})$. Then

$$\begin{aligned}Y_i^2 &= t(a_2)^2F_{i,q-1}^2 + s(a_2)^{-2}(\mathrm{Ad}(a^{-1})F_{i,q-1})^2\\ &\quad - s(a_2)^{-1}t(a_2)\left\{F_{i,q-1}(\mathrm{Ad}(a^{-1})F_{i,q-1}) + (\mathrm{Ad}(a^{-1})F_{i,q-1})F_{i,q-1}\right\}\\ &= t(a_2)A_2 + s(a_2)^{-2}(\mathrm{Ad}(a^{-1})F_{i,q-1})^2 + t(a_2)^2F_{i,q-1}^2 - 2s(a_2)^{-1}t(a_2)(\mathrm{Ad}(a^{-1})F_{i,q-1})F_{i,q-1}.\end{aligned}$$

By using

$$(R(\mathrm{Ad}(a^{-1})Y)\phi)(a) = \frac{d}{dt}\phi(\exp(tY)a)|_{t=0} = \frac{d}{dt}\chi(\exp(tY))|_{t=0}\phi(a) = d\chi(Y)\phi(a)$$

for $Y\in\mathfrak{so}(\xi)$ and Proposition 6.2, we have the assertion. □

Now we write down the radial parts of C_2 and C_4.

Proposition 6.4. *We assume $\xi=\xi_0$ for the unitary character of N_s. Under the same notation as in Proposition 6.2, the radial parts of C_2 and C_4 are given as follows:*

$$(R(C_2)\phi)(a) = \left[\partial_{a_1}^2 + \partial_{a_2}^2 - q\partial_{a_1} + (q-2)\frac{a_2^2+1}{a_2^2-1}\partial_{a_2} - 2\pi^2a_1^2(a_2^2+a_2^{-2}) + \frac{4a_2^2}{(a_2^2-1)^2}S_q\right]\phi(a)$$

and

$$\begin{aligned}(R(C_4)\phi)(a) = \Bigg[&4\partial_{a_1}^2\partial_{a_2}^2 + 4(q-2)\frac{a_2^2+1}{a_2^2-1}\partial_{a_1}^2\partial_{a_2} - 4q\partial_{a_1}\partial_{a_2}^2 - \frac{1}{3}(q-1)\{q\partial_{a_1}^2 + (q-12)\partial_{a_2}^2\}\\ &+ \left\{-8\pi^2a_1^2(a_2^2-a_2^{-2}) - 4q(q-2)\frac{a_2^2+1}{a_2^2-1}\right\}\partial_{a_1}\partial_{a_2}\\ &+ \left\{-4q\pi^2a_1^2(a_2^2+a_2^{-2}) - 8(q-2)\pi^2a_1^2 + \frac{1}{3}q^2(q-1)\right\}\partial_{a_1}\\ &+ \left\{4(q-2)\pi^2a_1^2(a_2^2-a_2^{-2}) - \frac{1}{3}(q-1)(q-2)(q-12)\frac{a_2^2+1}{a_2^2-1}\right\}\partial_{a_2}\\ &+ 4\pi^4a_1^4(a_2^2-a_2^{-2})^2 + \frac{2}{3}q(7q-19)\pi^2a_1^2(a_2^2+a_2^{-2}) + 8\pi^2(q-2)(q-3)a_1^2\\ &+ \frac{4a_2^2}{(a_2^2-1)^2}\left\{4\partial_{a_1}^2 - 4q\partial_{a_1} - 4\pi^2a_1^2(a_2-a_2^{-1})^2 - \frac{1}{3}(q-1)(q-12)\right\}S_q\Bigg]\phi(a).\end{aligned}$$

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Proof. By Proposition 4.1,

$$\begin{aligned}C_2 &= A_1^2 + A_2^2 - qA_1 - (q-2)A_2 + 2\{Z_1^2 - Z_1(Z_1 - Z_{-1}) + Z_2^2 - Z_2(Z_2 - Z_{-2})\}\\ &\quad + \sum_i \{X_i^2 - X_i(X_i - X_{-i}) + Y_i^2 - Y_i(Y_i - Y_{-i})\} - \sum_{i<j} K_{i,j}^2.\end{aligned}$$

Since $f \in C_\eta^\infty(R\backslash G/K)$ is annihilated by $U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$ and $\mathfrak{g}_{e_1}U(\mathfrak{g}_\mathbb{C})$ (because of the assumption on ξ and Proposition 6.2), we have

$$C_2 \equiv A_1^2 + A_2^2 - qA_1 - (q-2)A_2 + \sum_i Y_i^2 + 2(Z_1^2 + Z_2^2).$$

In the same way as above, we arrive at

$$\begin{aligned}C_4 &\equiv 4A_1^2A_2^2 - 8(Z_1^2 - Z_2^2)A_1A_2 + 4(Z_1^4 + Z_2^4 - 2Z_1^2Z_2^2) - 4(q-2)A_1^2A_2 - 4qA_1A_2^2\\ &\quad + 4qZ_1^2A_1 + 4(q-2)Z_1^2A_2 + 4qZ_2^2A_1 - 12(q-2)Z_2^2A_2 + 8(q-2)Z_1Z_2(A_1 + A_2)\\ &\quad - \tfrac{1}{3}q(q-1)A_1^2 - \tfrac{1}{3}(q-1)(q-12)A_2^2 + 4(q-2)qA_1A_2\\ &\quad - \tfrac{2}{3}q(7q-19)(Z_1^2 + Z_2^2) - 8(q-2)(q-3)Z_1Z_2\\ &\quad + \tfrac{1}{3}q^2(q-1)A_1 + \tfrac{1}{3}(q-1)(q-2)(q-12)A_2\\ &\quad + \{4A_1^2 - 4qA_1 + 4Z_1^2 + 4Z_2^2 - 8Z_1Z_2 - \tfrac{1}{3}(q-1)(q-12)\}\sum_i Y_i^2.\end{aligned}$$

By using Propositions 6.2 and 6.3, we obtain the formulas. □

6.3 Eigenvalues of C_2 and C_4

To find differential equations for Siegel–Whittaker functions, let us compute the eigenvalues of C_2 and C_4, that is, $c_\nu(C_2)$ and $c_\nu(C_4)$, where c_ν is the infinitesimal character of the class one principal series representation π_ν (§ 3.1). Since c_ν is trivial on $\mathfrak{n}U(\mathfrak{g}_\mathbb{C})$ and $U(\mathfrak{g}_\mathbb{C})\mathfrak{k}$, we have

$$c_\nu(C_2) = c_\nu(A_1^2 + A_2^2 - qA_1 - (q-2)A_2)$$

and

$$\begin{aligned}c_\nu(C_4) &= c_\nu(4A_1^2A_2^2 - 4(q-2)A_1^2A_2 - 4qA_1A_2^2 - \tfrac{1}{3}q(q-1)A_1^2 - \tfrac{1}{3}(q-1)(q-12)A_2^2\\ &\quad + 4q(q-2)A_1A_2 + \tfrac{1}{3}q^2(q-1)A_1 + \tfrac{1}{3}(q-1)(q-2)(q-12)A_2)\end{aligned}$$

by Proposition 4.1 and the calculation in the proof of Proposition 6.4. In view of $c_\nu(A_1) = \nu_1 + q/2$ and $c_\nu(A_2) = \nu_2 + (q-2)/2$, we get our next lemma.

LEMMA 6.5.

$$c_\nu(C_2) = \nu_1^2 + \nu_2^2 - \frac{q^2}{2} + q - 1,$$

$$c_\nu(C_4) = 4\nu_1^2\nu_2^2 - \frac{1}{3}(4q^2 - 13q + 12)(\nu_1^2 + \nu_2^2) + \frac{1}{12}(5q^4 - 30q^3 + 80q^2 - 100q + 48).$$

7. System of partial differential equations and explicit formulas for Siegel–Whittaker functions

7.1 System of partial differential equations

Summing up the results of previous sections we obtain a system of differential equations for Siegel–Whittaker functions.

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Theorem 7.1. *We assume $\xi = \xi_0$ and $\lambda = (\lambda_1, 0, \dots, 0)$ for the highest weight of χ_λ. Let $f(a) = \sum_{M \in GZ(\lambda)} f_M(a) v(M) = f_{M_0}(a) v(M_0)$ be the radial part of the Siegel–Whittaker function of type $(\pi_\nu; \xi_0, \chi_\lambda)$. Further, we introduce new variables given by $y = (y_1, y_2) = (\pi a_1 a_2^{-1}, \pi a_1 a_2)$. Then $f_{M_0}(y)$ satisfies the following differential equations:*

$$\left[D_1^{(q)} - \frac{2\lambda_1(\lambda_1 + q - 3) y_1 y_2}{(y_1 - y_2)^2}\right] f_{M_0}(y) = \frac{1}{2}\left(\nu_1^2 + \nu_2^2 - \frac{q^2}{2} + q - 1\right) f_{M_0}(y), \tag{7.1}$$

$$\begin{aligned}\left[D_2^{(q)} - \frac{4\lambda_1(\lambda_1 + q - 3) y_1 y_2}{(y_1 - y_2)^2} D_3^{(q)}\right] & f_{M_0}(y) \\ &= \left(\nu_1 + \frac{q-2}{2}\right)\left(\nu_2 + \frac{q-2}{2}\right)\left(-\nu_1 + \frac{q-2}{2}\right)\left(-\nu_2 + \frac{q-2}{2}\right) f_{M_0}(y). \end{aligned}\tag{7.2}$$

Here

$$\begin{aligned}
D_1^{(q)} &= y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + (q-2)\frac{y_1 y_2}{y_1 - y_2}\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right) - (y_1^2 + y_2^2), \\
D_2^{(q)} &= y_1^4 \frac{\partial^4}{\partial y_1^4} + y_2^4 \frac{\partial^4}{\partial y_2^4} - 2 y_1^2 y_2^2 \frac{\partial^4}{\partial y_1^2 \partial y_2^2} + \left(4 + \frac{2(q-2) y_2}{y_1 - y_2}\right) y_1^3 \frac{\partial^3}{\partial y_1^3} \\
&\quad + \left(4 - \frac{2(q-2) y_1}{y_1 - y_2}\right) y_2^3 \frac{\partial^3}{\partial y_2^3} + \frac{2(q-2) y_1}{y_1 - y_2} y_1^2 y_2 \frac{\partial^3}{\partial y_1^2 \partial y_2} - \frac{2(q-2) y_2}{y_1 - y_2} y_1 y_2^2 \frac{\partial^3}{\partial y_1 \partial y_2^2} \\
&\quad + \left\{-2(y_1^2 - y_2^2) - q(q-3) - 2(q-2)(q-3)\frac{y_2}{y_1 - y_2}\right\} y_1^2 \frac{\partial^2}{\partial y_1^2} \\
&\quad + \left\{2(y_1^2 - y_2^2) - q(q-3) + 2(q-2)(q-3)\frac{y_1}{y_1 - y_2}\right\} y_2^2 \frac{\partial^2}{\partial y_2^2} \\
&\quad + \left\{-4 y_1^2 - 2(q-2) y_2^2 - 2(q-2) y_1 y_2\right\} y_1 \frac{\partial}{\partial y_1} \\
&\quad + \left\{-2(q-2) y_1^2 - 4 y_2^2 - 2(q-2) y_1 y_2\right\} y_2 \frac{\partial}{\partial y_2} \\
&\quad + (y_1^2 - y_2^2)^2 + q(q-3)(y_1^2 + y_2^2) + 2(q-2)(q-3) y_1 y_2,
\end{aligned}$$

and

$$D_3^{(q)} = E_y^2 - q E_y - (y_1 - y_2)^2 + q - 1,$$

with

$$E_y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Proof. We get differential equations for $f_{M_0}(a)$ by using Proposition 6.4, Lemma 6.5 and $S_q v(M_0) = -\lambda_1(\lambda_1 + q - 3) v(M_0)$ (see Lemma 5.1). We remark that (7.2) can be deduced from $R(C_4') f = c_\nu(C_4') f$ with $C_4' = \frac{1}{4} C_4 + \frac{1}{12} q(q-1) C_2$. ◻

The validity of our calculations is supported by [OS95]. If we denote by D_1 and D_2 the differential operators of the left-hand side of (7.1) and (7.2) respectively, we can find that D_1 and D_2 are commutative as differential operators. On the other hand, [OS95] implies that D_2 can be uniquely determined (up to D_1, D_1^2) by the commutativity $D_1 \cdot D_2 = D_2 \cdot D_1$ and the invariance of the action of the Weyl group on (y_1, y_2).

If we put $q = 3$ in the above system, it agrees with the differential equations satisfied by class one principal series Siegel–Whittaker functions on $\mathrm{Sp}(2, \mathbb{R})$ (see [Niw91, Ish02]).

We can show the following in the same way as in [Ish02].

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1) The singular divisors of the system are $y_1 = 0$, $y_2 = 0$ and $y_1 - y_2 = 0$. Since they are not normal crossing, we blow up $\mathbb{C}^2$ at the origin $(y_1, y_2) = (0, 0)$. If we consider the formal power series solutions and compute the characteristic indices, it turns out that the system in Theorem 7.1 is holonomic of rank 8.
2) If we assume ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers, so that π_ν is irreducible, then there exists a four-dimensional space of holomorphic solutions along the singular divisor $y_1 - y_2 = 0$, that is,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\nu,K}, C^\infty \mathrm{Ind}_R^G(\eta)) \leqslant 4.$$

More precisely, let $f(y) = \sum_{m,n \geqslant 0} c_{m,n} y_1^{\tau_1 + m} (y_2/y_1 - 1)^{\tau_2 + n}$ $(c_{0,0} \neq 0)$ be the formal power series solution along $y_1 - y_2 = 0$. Then the characteristic indices are

$$(\tau_1, \tau_2) = (\pm\nu_i + q/2, \lambda_1), (\pm\nu_i + q/2, -\lambda_1 - q + 3).$$

3) If we write the holomorphic solutions along $y_1 - y_2 = 0$ corresponding to $\tau_2 = \lambda_1$ as $f(y) = \sum_{n \geqslant 0} \varphi_n(y_1)(y_2/y_1 - 1)^{\lambda_1 + n}$ $(\varphi_0(y_1) \neq 0)$, the boundary value $\varphi_0(y_1)$ satisfies the fourth-order differential equation:

$$\left[-4y_1^2(\theta - \lambda_1 - q + 3)(\theta - \lambda_1 - q + 2) + \prod_{i=1,2} \left(\theta + \nu_i - \frac{q}{2}\right)\left(\theta - \nu_i - \frac{q}{2}\right)\right]\varphi_0(y_1) = 0.$$

Here $\theta = y_1(d/dy_1)$. The basis of the space of solution is constructed by Meijer's G-functions (see [Erd53, 5.4]). Further, up to a constant multiple, there exists a unique solution which decreases rapidly as $y_1 \to \infty$, which is of the form

$$G_{2,4}^{4,0}\left(y_1^2 \,\middle|\, \begin{matrix} \dfrac{\lambda_1 + q - 1}{2}, & \dfrac{\lambda_1 + q}{2} \\ \dfrac{q}{4} + \dfrac{\nu_1}{2}, \dfrac{q}{4} + \dfrac{\nu_2}{2}, \dfrac{q}{4} - \dfrac{\nu_1}{2}, \dfrac{q}{4} - \dfrac{\nu_2}{2} \end{matrix}\right).$$

We remark that Niwa [Niw91] found an explicit integral representation for the solution mentioned in item 3 in the case of $q = 3$.

7.2 Reductions of the system of differential equations

As a result of the following, we can reduce the problem to the case $q = 3$ (cf. [Ish03, Theorem 3.2]).

PROPOSITION 7.2. *Under the same assumptions as in Theorem 7.1, if we put*

$$f_{M_0}(y) = \left(\frac{y_1 y_2}{y_1 - y_2}\right)^{(q-3)/2} g(y),$$

then $g(y)$ satisfies the following:

$$\left[D_1^{(3)} - \frac{(2\lambda_1 + q - 3)^2 y_1 y_2}{2(y_1 - y_2)^2}\right] g(y) = \frac{1}{2}\left((\nu_1^2 + \nu_2^2) - \frac{5}{2}\right) g(y), \tag{7.3}$$

$$\left[D_2^{(3)} - \frac{(2\lambda_1 + q - 3)^2 y_1 y_2}{(y_1 - y_2)^2} D_3^{(3)}\right] g(y) = \left(\nu_1 + \frac{1}{2}\right)\left(\nu_2 + \frac{1}{2}\right)\left(-\nu_1 + \frac{1}{2}\right)\left(-\nu_2 + \frac{1}{2}\right) g(y), \tag{7.4}$$

and

$$\begin{aligned} &\Bigg[E_y^4 - 6E_y^3 + \left\{-2(y_1 - y_2)^2 - (\nu_1^2 + \nu_2^2) + \frac{27}{2}\right\} E_y^2 \\ &\quad + \left\{2(y_1 - y_2)^2 + 3(\nu_1^2 + \nu_2^2) - \frac{27}{2}\right\} E_y + (y_1 - y_2)^4 + \left(\nu_1^2 + \nu_2^2 - \frac{5}{2}\right)(y_1 - y_2)^2 \\ &\quad + \left(\nu_1 + \frac{3}{2}\right)\left(\nu_2 + \frac{3}{2}\right)\left(-\nu_1 + \frac{3}{2}\right)\left(-\nu_2 + \frac{3}{2}\right) - 4y_1^2 y_2^2 \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)^2\Bigg] g(y) = 0. \end{aligned} \tag{7.5}$$

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Proof. If we write the multiplier $\Delta = (y_1y_2/(y_1-y_2))^{(q-3)/2}$, then

$$\Delta^{-1}\left(\frac{\partial}{\partial y_1}\right)^k\Delta = \left(\frac{\partial}{\partial y_1} - \frac{q-3}{2}\frac{y_2}{y_1(y_1-y_2)}\right)^k,$$

$$\Delta^{-1}\left(\frac{\partial}{\partial y_2}\right)^k\Delta = \left(\frac{\partial}{\partial y_2} + \frac{q-3}{2}\frac{y_1}{y_2(y_1-y_2)}\right)^k.$$

By using these formulas we get (7.3) and (7.4). Equation (7.5) can be deduced from (7.3) and (7.4) by eliminating the terms $(2\lambda_1+q-3)^2y_1y_2/2(y_1-y_2)^2$. Note that $D_3^{(3)}$ and $y_1y_2/(y_1-y_2)^2$ are commutative as differential operators. □

We notice that Equation (7.5), which does not appear in the previous paper [Ish02], is useful to examine the space of solutions.

Now we introduce new variables $x=(x_1,x_2)=(y_1-y_2,y_1+y_2)$ and put $g(y)=(x_1^2-x_2^2)h(x)$, after [Niw91]. Then the system in Proposition 7.2 is rewritten as follows:

$$\begin{aligned}
&\bigg[(x_1^2+x_2^2)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+4x_1x_2\frac{\partial^2}{\partial x_1\partial x_2}+4\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)+\frac{x_2^2-x_1^2}{x_1}\frac{\partial}{\partial x_1}\\
&\quad-(x_1^2+x_2^2)+\frac{(2\lambda_1+q-3)^2}{4}\frac{x_1^2-x_2^2}{x_1^2}-(\nu_1^2+\nu_2^2)+\frac{1}{2}\bigg]h(x)=0, \qquad (7.6)
\end{aligned}$$

$$\begin{aligned}
&\bigg[x_1^2x_2^2\left(\frac{\partial^4}{\partial x_1^4}+\frac{\partial^4}{\partial x_2^4}\right)+2x_1x_2(x_1^2+x_2^2)\left(\frac{\partial^4}{\partial x_1^3\partial x_2}+\frac{\partial^4}{\partial x_1\partial x_2^3}\right)\\
&\quad+(x_1^4+4x_1^2x_2^2+x_2^4)\frac{\partial^4}{\partial x_1^2\partial x_2^2}+x_1(x_1^2+7x_2^2)\frac{\partial^3}{\partial x_1^3}+8x_2(2x_1^2+x_2^2)\frac{\partial^3}{\partial x_1^2\partial x_2}\\
&\quad+\frac{6x_1^4+17x_1^2x_2^2+x_2^4}{x_1}\frac{\partial^3}{\partial x_1\partial x_2^2}+2x_2(3x_1^2+x_2^2)\frac{\partial^3}{\partial x_2^3}-2(x_1^2x_2^2-2x_1^2-5x_2^2)\frac{\partial^2}{\partial x_1^2}\\
&\quad-\frac{2x_2}{x_1}(x_1^4+x_1^2x_2^2-12x_1^2-2x_2^2)\frac{\partial^2}{\partial x_1\partial x_2}-2(x_1^2x_2^2-3x_1^2-4x_2^2)\frac{\partial^2}{\partial x_2^2}\\
&\quad-\frac{x_1^4+7x_1^2x_2^2-2x_1^2-2x_2^2}{x_1}\frac{\partial}{\partial x_1}-2x_2(3x_1^2+x_2^2-2)\frac{\partial}{\partial x_2}\\
&\quad+(x_1^2x_2^2-2x_1^2-4x_2^2)-\left(\nu_1+\frac{1}{2}\right)\left(\nu_2+\frac{1}{2}\right)\left(-\nu_1+\frac{1}{2}\right)\left(-\nu_2+\frac{1}{2}\right)\\
&\quad+\frac{(2\lambda_1+q-3)^2}{4}\frac{x_1^2-x_2^2}{x_1^2}(E_x^2+E_x-x_1^2)\bigg]h(x)=0, \qquad (7.7)
\end{aligned}$$

$$\begin{aligned}
&\bigg[(E_x+2)^4-6(E_x+2)^3+\left\{-2x_1^2-(\nu_1^2+\nu_2^2)+\frac{27}{2}\right\}(E_x+2)^2\\
&\quad+\left\{2x_1^2+3(\nu_1^2+\nu_2^2)-\frac{27}{2}\right\}(E_x+2)+\left(\nu_1+\frac{3}{2}\right)\left(\nu_2+\frac{3}{2}\right)\left(-\nu_1+\frac{3}{2}\right)\left(-\nu_2+\frac{3}{2}\right)\\
&\quad-(x_1^2-x_2^2)^2\frac{\partial^2}{\partial x_2^2}+4(x_1^2-x_2^2)x_2\frac{\partial}{\partial x_2}+x_1^4+\left(\nu_1^2+\nu_2^2-\frac{1}{2}\right)x_1^2-2x_2^2\bigg]h(x)=0. \qquad (7.8)
\end{aligned}$$

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Here we use the notation

$$E_x = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}.$$

7.3 Explicit formulas for Siegel–Whittaker functions

We have reduced the problem to the case $q=3$. However, Niwa's explicit formula does not work in the case of even q because $\lambda_1+(q-3)/2$ must be integer to apply his formula. Therefore we take another approach to find explicit formulas.

Let us consider the formal solution around $x_1 = y_1 - y_2 = 0$ and put

$$h(x) = \sum_{n\geqslant 0}\varphi_n(x_2)x_1^{\tau+n} \quad (\varphi_0 \neq 0).$$

Then we get $\tau = \pm(\lambda_1+(q-3)/2)$ and the following difference–differential equations for $\varphi_n = \varphi_n(x_2)$ from the differential equations for $h(x)$:

$$\begin{aligned}
&(n+2)(n+2\tau+2)x_2^2\varphi_{n+2} + [x_2^2\varphi_n'' + 4(n+\tau+1)x_2\varphi_n' \\
&\quad + \{-x_2^2 + n^2 + 2(\tau+1)n + 2\tau(\tau+1) - (\nu_1^2+\nu_2^2) + \tfrac{1}{2}\}\varphi_n] + (\varphi_{n-2}'' - \varphi_{n-2}) = 0, \qquad (7.9)\\
&(n+2)(n+2\tau+2)x_2^2(\theta+n+\tau+2)(\theta+n+\tau+3)\varphi_{n+2}\\
&\quad + [x_2^2\{-2(n+\tau+1)\theta - 2n^2 - (4\tau+5)n - (\tau+1)(\tau+4)\}\\
&\quad + (\theta+n+\tau)(\theta+n+\tau+1)\{2(n+\tau+1)\theta + n + \tau(\tau+1)\}\\
&\quad - (\nu_1+\tfrac{1}{2})(\nu_2+\tfrac{1}{2})(-\nu_1+\tfrac{1}{2})(-\nu_2+\tfrac{1}{2})]\varphi_n\\
&\quad + [x_2^2 + \{-2\theta^2 - 2(n+\tau)\theta - n - \tau(\tau+1)\}\\
&\quad + x_2^{-2}\theta(\theta-1)(\theta+n+\tau-1)(\theta+n+\tau-2)]\varphi_{n-2} = 0, \qquad (7.10)\\
&\left\{-x_2^2(\theta+1)(\theta+2) + \prod_{i=1,2}\left(\theta+n+\tau+\frac{5}{2}+\nu_i\right)\left(\theta+n+\tau+\frac{5}{2}-\nu_i\right)\right\}\varphi_{n+2}\\
&\quad + \left\{-4(n+\tau+1)\theta - 2n^2 - (4\tau+6)n - 2\tau^2 - 6\tau + (\nu_1^2+\nu_2^2) - \frac{9}{2}\right\}\varphi_n\\
&\quad + (\varphi_{n-2} - \varphi_{n-2}'') = 0. \qquad (7.11)
\end{aligned}$$

Here we use the notation $\theta = x_2(d/dx_2)$.

As in [Niw91] and [Ish02], let us examine the holomorphic solutions corresponding to $\tau = \lambda_1 + (q-3)/2$. We firstly obtain a fourth-order differential equation for $\varphi_0(x_2)$ by substituting $n=-2$ into (7.11):

$$\left[-x_2^2(\theta+1)(\theta+2) + \prod_{i=1,2}\left(\theta+\lambda_1+\frac{q}{2}-1+\nu_i\right)\left(\theta+\lambda_1+\frac{q}{2}-1-\nu_i\right)\right]\varphi_0(x_2) = 0.$$

The space of solutions is constructed by using Meijer's G-functions. In particular, there exists a unique (up to a constant) solution which decreases rapidly as $x_2 \to \infty$:

$$\varphi_0(x_2) = G^{4,0}_{2,4}\left(\frac{x_2^2}{4}\,\middle|\,\begin{matrix} \frac{1}{2},\ 0 \\ \frac{-\lambda_1+\nu_1+1}{2}-\frac{q}{4},\ \frac{-\lambda_1+\nu_2+1}{2}-\frac{q}{4},\ \frac{-\lambda_1-\nu_1+1}{2}-\frac{q}{4}\ \frac{-\lambda_1-\nu_2+1}{2}-\frac{q}{4}\end{matrix}\right).$$

From now on we deduce an explicit formula of $\varphi_n(x_2)$ $(n\geqslant 1)$ corresponding to the above $\varphi_0(x_2)$. By applying (7.9) we can find φ_n inductively. However, we have to check that such a solution satisfies (7.10) to state the multiplicity-one property of Siegel–Whittaker functions. It is immediately seen

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that $\varphi_n = 0$ for odd n. For even n we consider Mellin transformation of $\varphi_n(x_2)$. If we put

$$M_n(s) = \int_0^\infty \varphi_n(x_2) x_2^{s-1}\, dx_2,$$

then we can easily get the following recurrence relations from (7.9), (7.10) and (7.11):

$$\begin{aligned}
&(n+2)(n+2\tau+2)M_{n+2}(s+2) - M_n(s+2)\\
&\quad + \{s^2 - (4n+4\tau+3)s + n^2 + 2(\tau+1)n + 2\tau(\tau+1) - (\nu_1^2+\nu_2^2) + \tfrac{1}{2}\}M_n(s)\\
&\quad - M_{n-2}(s) + (s-2)(s-1)M_{n-2}(s-2) = 0,
\end{aligned} \tag{7.12}$$

$$\begin{aligned}
&(n+2)(n+2\tau+2)(-s+n+\tau)(-s+n+\tau+1)M_{n+2}(s+2)\\
&\quad + \{2(n+\tau+1)s - 2n^2 - (4\tau+1)n - \tau(\tau+1)\}M_n(s+2)\\
&\quad + [(-s+n+\tau)(-s+n+\tau+1)\{-2(n+\tau+1)s + n + \tau(\tau+1)\}\\
&\quad - (\nu_1+\tfrac{1}{2})(\nu_2+\tfrac{1}{2})(-\nu_1+\tfrac{1}{2})(-\nu_2+\tfrac{1}{2})]M_n(s)\\
&\quad + M_{n-2}(s+2) + \{-2s^2 + 2(n+\tau)s - n - \tau(\tau+1)\}M_{n-2}(s)\\
&\quad + (s-2)(s-1)(-s+n+\tau+1)(-s+n+\tau)M_{n-2}(s-2) = 0,
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
&-(s+1)sM_{n+2}(s+2)\\
&\quad + \prod_{i=1,2}\left(-s+n+\tau+\frac{5}{2}+\nu_i\right)\left(-s+n+\tau+\frac{5}{2}-\nu_i\right)M_{n+2}(s)\\
&\quad + \left\{4(n+\tau+1)s - 2n^2 - (4\tau+6)n - 2\tau^2 - 6\tau + (\nu_1^2+\nu_2^2) - \frac{9}{2}\right\}M_n(s)\\
&\quad + M_{n-2}(s) - (s-2)(s-1)M_{n-2}(s-2) = 0.
\end{aligned} \tag{7.14}$$

By solving the difference equations (7.12), (7.13) and (7.14), we obtain the following proposition.

Proposition 7.3. *The Mellin transformation $M_n(s)$ of $\varphi_n(x_2)$ is given by*

$$M_{2k}(s) = P_k(s)Q_k(s), \quad M_{2k+1}(s) = 0.$$

Here

$$P_k(s) = \frac{2^{-2k}}{\sqrt{\pi}k!}\Gamma\begin{bmatrix} s,\ \lambda_1 + \dfrac{q-1}{2} \\ k+\lambda_1+\dfrac{q-1}{2}\end{bmatrix}\prod_{i=1,2}\Gamma\begin{bmatrix}\dfrac{s}{2}-k+a_i, \dfrac{s}{2}-k+b_i \\ \dfrac{s}{2}-k, \dfrac{s+1}{2}-k\end{bmatrix},$$

$$Q_k(s) = Q_k^1(s)Q_k^2(s), \quad Q_k^i(s) = {}_3F_2\left(\left.\begin{matrix}-k,\ \dfrac{s}{2}-k+a_i,\ \dfrac{s}{2}-k+b_i \\ \dfrac{s}{2}-k,\ \dfrac{s+1}{2}-k\end{matrix}\ \right|\ 1\right),$$

with

$$a_i = -\frac{\tau}{2} - \frac{1}{4} + \frac{\nu_i}{2} = \frac{-\lambda_1+\nu_i+1}{2} - \frac{q}{4}, \quad b_i = -\frac{\tau}{2} - \frac{1}{4} - \frac{\nu_i}{2} = \frac{-\lambda_1-\nu_i+1}{2} - \frac{q}{4}.$$

Here we use the notation

$$\Gamma\begin{bmatrix}a_1,\dots,a_n \\ b_1,\dots,b_m\end{bmatrix} = \prod_{i=1}^n \Gamma(a_i)\Big/\prod_{i=1}^m\Gamma(b_i)$$

and

$${}_3F_2\left(\left.\begin{matrix}a_1,\ a_2,\ a_3 \\ b_1,\ b_2\end{matrix}\ \right|\ z\right) = \sum_{n=0}^\infty \Gamma\begin{bmatrix}a_1+n, & a_2+n, & a_3+n, & b_1, & b_2 \\ a_1, & a_2, & a_3, & b_1+n, & b_2+n\end{bmatrix}\frac{z^n}{n!},$$

the generalized hypergeometric function.

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Proof. We use induction on k. By using the formula

$$\int_0^\infty G^{4,0}_{2,4}\left(x \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix}\right) x^{s-1}\,dx = \Gamma\begin{bmatrix} b_1+s,\ b_2+s,\ b_3+s,\ b_4+s \\ a_1+s,\ a_2+s \end{bmatrix},$$

we have

$$M_0(s) = 2^{s-1}\Gamma\begin{bmatrix} \dfrac{s}{2}+a_1,\ \dfrac{s}{2}+a_2,\ \dfrac{s}{2}+b_1,\ \dfrac{s}{2}+b_2 \\ \dfrac{s}{2},\ \dfrac{s+1}{2} \end{bmatrix}.$$

Then the claim for $n=0$ is obvious from $\Gamma(s) = 2^{s-1}\pi^{-1/2}\Gamma(s/2)\Gamma((s+1)/2)$.

Since $M_{2k}(s)$ is uniquely determined by (7.12), we have to check that $P_k(s)Q_k(s)$ satisfies (7.12) and (7.13) to prove the proposition. First we prepare two sublemmas on the generalized hypergeometric functions. We notice that every ${}_3F_2(1)$ in our calculation is a finite sum of fractional gamma functions.

LEMMA 7.4. *Let k be a positive integer. Then*

$${}_3F_2\left(\begin{matrix} -k,\ a_1,\ a_2 \\ b_1,\ b_2 \end{matrix}\,\middle|\, 1\right) - {}_3F_2\left(\begin{matrix} -(k-1),\ a_1,\ a_2 \\ b_1,\ b_2 \end{matrix}\,\middle|\, 1\right) = -\frac{a_1a_2}{b_1b_2}\,{}_3F_2\left(\begin{matrix} -(k-1),\ a_1+1,\ a_2+1 \\ b_1+1,\ b_2+1 \end{matrix}\,\middle|\, 1\right).$$

LEMMA 7.5. *Let k be a positive integer and $Q_k^i(s)$ be as in Proposition 7.3. Then*

$$s(s+1)Q_k^i(s+2) - 4k(k+\tau)Q_{k-1}^i(s) - (s-2k)(s-2k+1)Q_k^i(s) = 0.$$

The proof of Lemma 7.4 is immediate from the definition and the proof of Lemma 7.5 can be obtained by induction on k and Lemma 7.4.

Now we return to the proof of Proposition 7.3. We begin with (7.12). If we substitute $n=2k$ and $M_{2k}(s) = P_k(s)Q_k(s)$ into the left-hand side of (7.12), it is equal to $P_k(s)$ times

$$\begin{aligned}
&4(k+1)(k+\tau+1)\frac{P_{k+1}(s+2)}{P_k(s)}Q_{k+1}(s+2) - \frac{P_k(s+2)}{P_k(s)}Q_k(s+2) \\
&\quad + \left\{-s(s+1) - 4k(k+\tau) + \sum_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)\right\}Q_k(s) \\
&\quad + (s-1)(s-2)\frac{P_{k-1}(s-2)}{P_k(s)}Q_{k-1}(s-2) - \frac{P_{k-1}(s)}{P_k(s)}Q_{k-1}(s) \\
&= s(s+1)\left[Q_{k+1}(s+2) - Q_k(s) - \frac{\prod_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)}{(s-2k)^2(s-2k+1)^2}Q_k(s+2)\right] \\
&\quad + 4k(k+\tau)\left[Q_{k-1}(s-2) - Q_k(s) - \frac{\prod_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)}{(s-2k)^2(s-2k+1)^2}Q_{k-1}(s)\right] \\
&\quad + \sum_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)Q_k(s).
\end{aligned}$$

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Now we apply Lemma 7.4 to the last terms of the first and second lines. Then the above is equal to

$$
\begin{aligned}
&s(s+1)[Q_k^1(s)(Q_{k+1}^2(s+2)-Q_k^2(s))+Q_k^2(s)(Q_{k+1}^1(s+2)-Q_k^1(s))]\\
&\quad+4k(k+\tau)[Q_k^1(s)(Q_{k-1}^2(s-2)-Q_k^2(s))+Q_k^2(s)(Q_{k-1}^1(s-2)-Q_k^1(s))]\\
&\quad+\sum_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)Q_k^1(s)Q_k^2(s)\\
&=Q_k^1(s)[s(s+1)(Q_{k+1}^2(s+2)-Q_k^2(s))+4k(k+\tau)(Q_{k-1}^2(s-2)-Q_k^2(s))\\
&\quad+(s-2k+2a_2)(s-2k+2b_2)Q_k^2(s)]+Q_k^2(s)[s(s+1)(Q_{k+1}^1(s+2)-Q_k^1(s))\\
&\quad+4k(k+\tau)(Q_{k-1}^1(s-2)-Q_k^1(s))+(s-2k+2a_1)(s-2k+2b_1)Q_k^1(s)].
\end{aligned}
$$

Substituting $Q_{k+1}^i(s+2)-Q_k^i(s)$ and $Q_{k-1}^i(s-2)-Q_k^i(s)$ into the right-hand side of Lemma 7.4:

$$
\begin{aligned}
Q_{k+1}^i(s+2)-Q_k^i(s)&=-\frac{(s-2k+2a_i)(s-2k+2b_i)}{(s-2k)(s-2k+1)}Q_k^i(s+2),\\
Q_{k-1}^i(s-2)-Q_k^i(s)&=\frac{(s-2k+2a_i)(s-2k+2b_i)}{(s-2k)(s-2k+1)}Q_{k-1}^i(s).
\end{aligned}
$$

We can verify that the above is equal to zero from Lemma 7.5. Thus we have proved that the recurrence relation (7.12) implies that $M_{2k}(s)=cP_k(s)Q_k(s)$ with some constant c.

Next we check the compatibility, that is, $M_{2k}(s)=P_k(s)Q_k(s)$ satisfies (7.13) and (7.14). If we compute the difference of (7.12) and (7.14), we get

$$
\begin{aligned}
&\{4(k+1)(k+\tau+1)-s(s+1)\}M_{2k+2}(s+2)\\
&\quad+\prod_{i=1,2}\left(-s+2k+\tau+\frac{5}{2}+\nu_i\right)\left(-s+2k+\tau+\frac{5}{2}-\nu_i\right)M_{2k+2}(s)\\
&\quad-M_{2k}(s+2)+\{s(s+1)-4k^2-4(\tau+2)k-4(\tau+1)\}M_{2k}(s)=0. \qquad (7.15)
\end{aligned}
$$

Then we check (7.15) to prove (7.14). In the same way as in (7.12), the left-hand side of (7.15) is $P_k(s)$ times

$$
\begin{aligned}
&\left(\frac{-s^2(s+1)^2}{4(k+1)(k+\tau+1)}+s(s+1)\right)Q_{k+1}(s+2)+\frac{(s-2k-1)^2(s-2k-2)^2}{4(k+1)(k+\tau+1)}Q_{k+1}(s)\\
&\quad+\{s(s+1)-4(k+1)(k+\tau+1)\}Q_k(s)\\
&\quad-\frac{s(s+1)\prod_{i=1,2}(s-2k+2a_i)(s-2k+2b_i)}{(s-2k)^2(s-2k+1)^2}Q_k(s+2).
\end{aligned}
$$

We rewrite the last term as $-s(s+1)(Q_{k+1}^1(s+2)-Q_k^1(s))(Q_{k+1}^2(s+2)-Q_k^2(s))$ by using Lemma 7.4. Then the above expression becomes

$$
\begin{aligned}
&\left(\frac{-s^2(s+1)^2}{4(k+1)(k+\tau+1)}+s(s+1)\right)Q_{k+1}^1(s+2)Q_{k+1}^2(s+2)\\
&\quad+\frac{(s-2k-1)^2(s-2k-2)^2}{4(k+1)(k+\tau+1)}Q_{k+1}^1(s)Q_{k+1}^2(s)-4(k+1)(k+\tau+1)Q_k^1(s)Q_k^2(s)\\
&\quad+s(s+1)(Q_k^1(s)Q_{k+1}^2(s+2)+Q_k^2(s)Q_{k+1}^1(s+2))\\
&=\frac{1}{4(k+1)(k+\tau+1)}\Bigg[(s-2k-1)^2(s-2k-2)^2Q_{k+1}^1(s)Q_{k+1}^2(s)\\
&\quad-\prod_{i=1,2}\{s(s+1)Q_{k+1}^i(s+2)-4(k+1)(k+\tau+1)Q_k^i(s)\}\Bigg].
\end{aligned}
$$

This is equal to zero by means of Lemma 7.5.

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Finally (7.13) follows from (7.12) and (7.14). Because, if we denote the left-hand sides of (7.12), (7.13) and (7.14) by $R_k^1(s), R_k^2(s)$ and $R_k^3(s)$ respectively, then we have

$$R_{k-2}^1(s) + R_{k-2}^3(s) + R_k^2(s) - (k+\tau-s)(k+\tau-s+1)R_k^1(s) = 0.$$

Thus we complete the proof of Proposition 7.3. □

By virtue of Proposition 7.3, we give an integral representation of the Siegel–Whittaker function. We first apply the formula

$$\begin{aligned} {}_3F_2\left(\begin{matrix} -k,\ c_1,\ c_2 \\ d_1,\ d_2 \end{matrix}\middle| 1\right) &= \Gamma\begin{bmatrix} d_1,\ d_2 \\ c_1, c_2, d_1-c_1, d_2-c_2 \end{bmatrix} \\ &\quad \times \int_0^1\int_0^1 t^{c_1}u^{c_2}(1-t)^{d_1-c_1}(1-u)^{d_2-c_2}(1-tu)^k\, d^*t\, d^*u, \end{aligned}$$

for $\mathrm{Re}(d_i) > \mathrm{Re}(c_i) > 0$ $(i=1,2)$ (see [Sla66, p. 108]). Here

$$d^*t = \frac{dt}{t(1-t)}.$$

Then we have

$$\begin{aligned} M_{2k}(s) &= \frac{2^{-2k}}{\sqrt{\pi}k!}\Gamma\begin{bmatrix} s,\ \lambda_1+\dfrac{q-1}{2} \\ k+\lambda_1+\dfrac{q-1}{2},\ -a_1,\ -a_2,\ -b_1+\dfrac{1}{2},\ -b_2+\dfrac{1}{2} \end{bmatrix} \\ &\quad \times \prod_{i=1,2}\left(\int_0^1\int_0^1 t_i^{s/2-k+a_i}u_i^{s/2-k+b_i}(1-t_i)^{-a_i}(1-u_i)^{-b_i+1/2}(1-t_iu_i)^k\, d^*t_i\, d^*u_i\right). \end{aligned}$$

By the Mellin inversion formula and

$$\frac{1}{2\pi\sqrt{-1}}\int_{\gamma-\sqrt{-1}\infty}^{\gamma+\sqrt{-1}\infty}\left(\frac{x_2}{\sqrt{t_1t_2u_1u_2}}\right)^{-s}\Gamma(s)\, ds = \exp\left(-\frac{x_2}{\sqrt{t_1t_2u_1u_2}}\right)$$

for $\mathrm{Re}(s) > 0$, we have

$$\begin{aligned} \varphi_{2k}(x_2) &= \frac{2^{-2k}}{\sqrt{\pi}k!}\Gamma\begin{bmatrix} \lambda_1+\dfrac{q-1}{2} \\ k+\lambda_1+\dfrac{q-1}{2},\ -a_1,\ -a_2,\ -b_1+\dfrac{1}{2},\ -b_2+\dfrac{1}{2} \end{bmatrix} \\ &\quad \times \int_0^1\int_0^1\int_0^1\int_0^1 \left(\frac{(1-t_1u_1)(1-t_2u_2)}{t_1t_2u_1u_2}\right)^k \exp\left(-\frac{x_2}{\sqrt{t_1t_2u_1u_2}}\right) \\ &\quad \times \prod_{i=1,2}(t_i^{a_i}u_i^{b_i}(1-t_i)^{-a_i}(1-u_i)^{-b_i+1/2}\, d^*t_i\, d^*u_i) \end{aligned}$$

for $-\lambda_1 - q/2 < \mathrm{Re}(\nu_i) < \lambda_1 + (q-2)/2$ $(i=1,2)$. Thus we obtain

$$\begin{aligned} h(x) &= \sum_{k\geqslant 0}\varphi_{2k}(x_2)x_1^{\lambda_1+(q-3)/2+2k} \\ &= \frac{2^{\lambda_1+(q-3)/2}}{\sqrt{\pi}}\Gamma\begin{bmatrix} \lambda_1+\dfrac{q-1}{2} \\ -a_1,\ -a_2,\ -b_1+\dfrac{1}{2},\ -b_2+\dfrac{1}{2} \end{bmatrix} I(x), \end{aligned}$$

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where

$$I(x) = \int_0^1\int_0^1\int_0^1\int_0^1 I_{\lambda_1+(q-3)/2}\left(x_1\sqrt{\frac{(1-t_1u_1)(1-t_2u_2)}{t_1t_2u_1u_2}}\right)\exp\left(-\frac{x_2}{\sqrt{t_1t_2u_1u_2}}\right)$$
$$\times\prod_{i=1,2}\Big(t_i^{\nu_i/2-1/4}u_i^{-\nu_i/2-1/4}(1-t_i)^{(\lambda_1-\nu_i-1)/2+q/4}(1-u_i)^{(\lambda_1+\nu_i)/2+q/4}$$
$$\times(1-t_iu_i)^{-\lambda_1/2-(q-3)/4}\,d^*t_i\,d^*u_i\Big).$$

Here

$$I_\nu(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2n+\nu}}{n!\,\Gamma(\nu+n+1)},$$

the modified Bessel function of the first kind. To perform more integration, we make the change of variables $t_i \mapsto (1-u_i)t_i + u_i$, $u_i \mapsto u_i/((1-u_i)t_i+u_i)$ for $i=1,2$. Then we have

$$I(x) = \int_0^1\int_0^1\int_0^1\int_0^1 I_{\lambda_1+(q-3)/2}\left(x_1\sqrt{\left(1-\frac{1}{u_1}\right)\left(1-\frac{1}{u_2}\right)}\right)\exp\left(-\frac{x_2}{\sqrt{u_1u_2}}\right)$$
$$\times\prod_{i=1,2}\left[t_i^{(\lambda_1+\nu_i)/2+q/4}(1-t_i)^{(\lambda_1-\nu_i-1)/2+q/4}\left(t_i+\frac{u_i}{1-u_i}\right)^{(-\lambda_1+\nu_i)/2-q/4}\right.$$
$$\left.\times\left(\frac{1-u_i}{u_i}\right)^{\nu_i/2+1/4}\,d^*t_i\,d^*u_i\right].$$

After the integrations with respect to t_1 and t_2 by means of the formula

$$\int_0^1 x^\nu(1-x)^\mu(x+a)^\lambda\,d^*x = a^\lambda\Gamma\begin{bmatrix}\nu,\ \mu\\ \nu+\mu\end{bmatrix}{}_2F_1\left(\begin{matrix}\nu,\ -\lambda\\ \nu+\mu\end{matrix}\,\middle|\,-\frac{1}{a}\right)$$

for $\mathrm{Re}(\nu)>0$, $\mathrm{Re}(\mu)>0$ and $|\arg 1/a|<\pi$ (see [GR00, p. 315, 3.197(8)]), we obtain

$$I(x) = \prod_{i=1,2}\Gamma\begin{bmatrix}\dfrac{\lambda_1+\nu_i}{2}+\dfrac{q}{4},\ \dfrac{\lambda_1-\nu_i-1}{2}+\dfrac{q}{4}\\ \lambda_1+\dfrac{q-1}{2}\end{bmatrix}$$
$$\times\int_0^1\int_0^1 I_{\lambda_1+(q-3)/2}\left(x_1\sqrt{\left(1-\frac{1}{u_1}\right)\left(1-\frac{1}{u_2}\right)}\right)\exp\left(-\frac{x_2}{\sqrt{u_1u_2}}\right)$$
$$\times\prod_{i=1,2}\left[\left(\frac{1-u_i}{u_i}\right)^{\lambda_1/2+(q+1)/4}{}_2F_1\left(\begin{matrix}\dfrac{\lambda_1+\nu_i}{2}+\dfrac{q}{4},\ \dfrac{\lambda_1-\nu_i}{2}+\dfrac{q}{4}\\ \lambda_1+\dfrac{q-1}{2}\end{matrix}\,\middle|\,1-\frac{1}{u_i}\right)d^*u_i\right]$$

for $-\lambda_1-q/2<\mathrm{Re}(\nu_i)<\lambda_1+(q-2)/2$ $(i=1,2)$.

Remark 7.6. This condition for the parameter of class one principal series covers the unitary case, that is, $\mathrm{Re}(\nu_1)=\mathrm{Re}(\nu_2)=0$.

8. Main result

We summarize the results of the previous sections and state the main theorem. Let

$$\mathrm{SW}(\pi_\nu;\chi,\xi)^{\mathrm{rap}} = \{f\in\mathrm{SW}(\pi_\nu;\chi,\xi)\mid f(a)\text{ decreases rapidly as }a_1a_2, a_1/a_2\to\infty\}.$$

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Moreover, let $C^\infty \operatorname{Ind}_R^G(\eta)^{\mathrm{rap}}$ be the subspace of $C^\infty \operatorname{Ind}_R^G(\eta)$ with the same property as above. Then we can see that $C^\infty \operatorname{Ind}_R^G(\eta)^{\mathrm{rap}}$ is a $(\mathfrak{g}, K)$-submodule of $C^\infty \operatorname{Ind}_R^G(\eta)$ (cf. Proposition 6.2) and

$$\operatorname{Hom}_{(\mathfrak{g},K)}(H_{\pi_\nu,K}, C^\infty \operatorname{Ind}_R^G(\eta)^{\mathrm{rap}}) \cong \mathrm{SW}(\pi_\nu; \chi, \xi)^{\mathrm{rap}}.$$

Thus we arrive at our main theorem in this paper.

THEOREM 8.1. *We assume $\nu_1, \nu_2, \nu_1 \pm \nu_2$ are not integers for the class one principal representation π_ν and $\xi = \xi_0$ for the unitary character of N_s. Let χ_λ be the irreducible finite-dimensional representation of $SO(\xi)(\cong SO(q-1))$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_{[(q-1)/2]})$.*

1) *If λ is not of the form $(\lambda_1, 0, \ldots, 0)$, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_{\nu,K}, C^\infty \operatorname{Ind}_R^G(\eta)) = \dim_{\mathbb{C}} \mathrm{SW}(\pi_\nu; \chi_\lambda, \xi_0) = 0.$$

2) *If $\lambda = (\lambda_1, 0, \ldots, 0)$, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_{\nu,K}, C^\infty \operatorname{Ind}_R^G(\eta)^{\mathrm{rap}}) = \dim_{\mathbb{C}} \mathrm{SW}(\pi_\nu; \chi_\lambda, \xi_0)^{\mathrm{rap}} = 1,$$

and let $f(a) = f_{M_0}(a)v_{M_0}$ be the radial part of $f \in \mathrm{SW}(\pi_\nu; \chi_\lambda, \xi_0)^{\mathrm{rap}}$. Then $f(a)$ is of the form

$$\begin{aligned} f(a) &= c a_1^{(q+1)/2}(a_2^{-1} - a_2)^{(3-q)/2} \\ &\times \int_0^1\int_0^1 I_{\lambda_1+(q-3)/2}\left(\pi a_1(a_2^{-1} - a_2)\sqrt{\left(1 - \frac{1}{u_1}\right)\left(1 - \frac{1}{u_2}\right)}\right) \exp\left(-\frac{\pi a_1(a_2^{-1} + a_2)}{\sqrt{u_1 u_2}}\right) \\ &\times \prod_{i=1,2}\left[\left(\frac{1-u_i}{u_i}\right)^{\lambda_1/2+(q+1)/4} {}_2F_1\left(\left.\begin{matrix}\dfrac{\lambda_1+\nu_i}{2} + \dfrac{q}{4}, \dfrac{\lambda_1-\nu_i}{2} + \dfrac{q}{4} \\ \lambda_1 + \dfrac{q-1}{2}\end{matrix}\right| 1 - \frac{1}{u_i}\right) \frac{du_i}{u_i(1-u_i)}\right] \end{aligned}$$

for $-\lambda_1 - q/2 < \mathrm{Re}(\nu_i) < \lambda_1 + (q-2)/2$ $(i = 1, 2)$ with some constant c.

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