

AN ELEMENTARY PROOF OF ABRAMOV'S RESULT ON THE ENTROPY OF A FLOW¹⁾

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§1. Preliminaries

Let T be an automorphism on a probability space (Ω, B, P) . Given a finite partition α of Ω , into disjoint measurable sets A_1, A_2, \dots, A_n , its entropy is

$$(D.1) \quad H(\alpha) = - \sum_{i=1}^n P(A_i) \log P(A_i).$$

Put

$$(D.2) \quad h(\alpha, T) = \lim_{n \rightarrow \infty} \frac{H(\alpha \vee T\alpha \vee \dots \vee T^{n-1}\alpha)^{2)3)}}{n}$$

then, the entropy $h(T)$ of T is defined by

$$(D.3) \quad h(T) = \sup_{\alpha} h(\alpha, T),$$

where the supremum is taken over all finite measurable partitions.

When $\{T_t\}$ is a measurable flow on a Lebesgue space (Ω, B, P) , L.M. Abramov [2] proved the formula

$$(F) \quad h(T_t) = |t| h(T_1) \text{ for every } t \in R.$$

He proved with use of representation of measurable flow and formulas of entropy of derived automorphism and skew product automorphism. Recently, G. Maruyama [4] pointed out that measurability for $\{T_t\}$ can be replaced by the continuity of $\{T_t\}$.

In this paper we will give a simple proof of (F), when $\{T_t\}$ is subject to the continuity assumption

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¹⁾ This paper is chiefly due to Pinsker's work [1].

²⁾ $\alpha \vee \beta$ means the partition formed by the sets $A_i \cap B_j, A_i \in \alpha, B_j \in \beta$.

³⁾ According to [3], the limit exists.

$$(D.4) \quad \lim_{t \rightarrow 0} P(T_t A \Delta A) = 0 \text{ for every } A \in B.$$

In what follow, (Ω, B, P) denotes an arbitrary abstract probability space.

For later use, we will mention the well-known relations for the entropy [3]:

$$(1.1) \quad 0 \leq H(\alpha/\beta),$$

$$(1.2) \quad H(\alpha \vee \beta/\gamma) = H(\alpha/\gamma) + H(\beta/\gamma \vee \alpha),$$

$$(1.3) \quad \text{if } \alpha \geq \beta \pmod{0}$$

$$\text{then } H(\alpha/\gamma) \geq H(\beta/\gamma) \text{ and } H(\gamma/\alpha) \leq H(\gamma/\beta),$$

$$(1.4) \quad \text{if } T \text{ is an automorphism on } \Omega \text{ then } H(\alpha/\beta) = H(T\alpha/T\beta).$$

§2. The theorem.

We need the following lemma.

LEMMA. *Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $\{\alpha'_1, \alpha'_2, \dots, \alpha'_n\}$ be sequences of finite measurable partitions, and ε be an arbitrary positive number.*

If $H(\alpha'_i/\alpha_i) < \varepsilon$ for $i = 1, 2, \dots, n$, then $H(\bigvee_{i=1}^n \alpha_i) - H(\bigvee_{i=1}^n \alpha'_i) > -n\varepsilon$.

Proof. (1.1) (1.2) and (1.3) together imply

$$\begin{aligned} & H(\bigvee_{i=1}^n \alpha_i) - H(\bigvee_{i=1}^n \alpha'_i) \\ &= H((\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^n \alpha'_i)) - H(\bigvee_{i=1}^n \alpha'_i / \bigvee_{i=1}^n \alpha_i) \\ &= H((\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^n \alpha'_i)) + H(\bigvee_{i=1}^n \alpha_i / \bigvee_{i=1}^n \alpha'_i) \\ &\geq -H(\bigvee_{i=1}^n \alpha'_i / \bigvee_{i=1}^n \alpha_i) \\ &\geq -\sum_{k=1}^n H(\alpha'_k / (\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^{k-1} \alpha'_i)) \\ &\geq -\sum_{k=1}^n H(\alpha'_k / \alpha_k) > -n\varepsilon. \end{aligned}$$

PROPOSITION. *If $\{T_t\}$ is a continuous flow on a probability space (Ω, B, P) and if α is a finite measurable partition, then*

$$(2.1) \quad \sup_{t \neq 0} \frac{1}{|t|} h(\alpha, T_t) = \lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t).$$

Proof. Since

$$\lim_{t \rightarrow 0} H(T_t \alpha / \alpha) = \lim_{t \rightarrow 0} (H(\alpha \vee T_t \alpha) - H(\alpha)) = 0$$

by (D.1) and (D.4), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(2.2) \quad H(T_t \alpha / \alpha) < \varepsilon \quad \text{for } |t| < \delta.$$

On the other hand, if T and t' are arbitrary positive numbers, there exists an integer n such that $T - t' \leq (n - 1)t' < T$. For any t satisfying $0 < t < \min(\delta, t')$, there exists an integer m such that $(m - 1)t \leq T < mt$.

Consequently, there exists a subsequence $\{m_k\}_{k=1}^{n-1}$ of $1, 2, \dots, m$ such that $|kt' - m_k t| < \delta$ for $k = 1, 2, \dots, n - 1$.

From (2.2) one obtains

$$(2.3) \quad H(T_{t'}^k \alpha / T_t^{m_k} \alpha) = H(T_{kt' - m_k t} \alpha / \alpha) < \varepsilon.$$

From Lemma and (2.3) follows

$$(2.4) \quad H\left(\bigvee_{k=0}^{m-1} T_t^k \alpha\right) \geq H\left(\bigvee_{k=0}^{n-1} T_{m_k t} \alpha\right) > H\left(\bigvee_{k=0}^{n-1} T_{t'}^k \alpha\right) - n\varepsilon.$$

Then, since $\lim_{T \rightarrow \infty} T/m = t$ and $\lim_{T \rightarrow \infty} T/n = t'$, on making $T \rightarrow \infty$ in (2.4) divided by T and in view of (D.2) we have the relation

$$\frac{1}{t} h(\alpha, T_t) > \frac{1}{t'} h(\alpha, T_{t'}) - \frac{\varepsilon}{t'}.$$

Since ε is arbitrary and since

$$h(\alpha, T_t) = h(\alpha, T_{-t}) \text{ for all } t \in R,$$

$$(2.5) \quad \lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) \geq \frac{1}{|t'|} h(\alpha, T_{t'})$$

holds. We obtain from (2.5),

$$\lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) \geq \sup_{t' \neq 0} \frac{1}{|t'|} h(\alpha, T_{t'}).$$

Thus, there exists $\lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t)$

and

$$\lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) = \sup_{t \neq 0} \frac{1}{|t|} h(\alpha, T_t).$$

THEOREM. *If $\{T_t\}$ is a continuous flow, then*

$$h(T_t) = |t|h(T_1) \text{ for any real } t.$$

Proof. Define now $h(\{T_t\}) = \sup_{\substack{t \neq 0 \\ \alpha}} \frac{1}{|t|} h(\alpha, T_t)$, where the supremum is taken over all finite measurable partitions and all non-zero numbers.

On the one hand, from (D.3)

$$(2.6) \quad h(\{T_t\}) = \sup_{t \neq 0} \frac{1}{|t|} h(T_t) \geq \overline{\lim}_{t \rightarrow 0} \frac{1}{|t|} h(T_t);$$

On the other hand, from (2.1)

$$(2.7) \quad \begin{aligned} \lim_{t \rightarrow 0} \frac{1}{|t|} h(T_t) &\geq \lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) \\ &= \lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) \text{ for all } \alpha. \end{aligned}$$

According to (2.1),

$$(2.8) \quad h(\{T_t\}) = \sup_{\alpha} \left(\lim_{t \rightarrow 0} \frac{1}{|t|} h(\alpha, T_t) \right).$$

Combining (2.6), (2.7) and (2.8), we obtain

$$(2.9) \quad \lim_{t \rightarrow 0} \frac{1}{|t|} h(T_t) = h(\{T_t\}).$$

Finally by the formula $h(T^k) = |k|h(T)$ in [3], which holds for any automorphism T and integer k , one gets for any real $t (\neq 0)$

$$\frac{1}{|t|} h(T_t) = \frac{1}{\left| \frac{t}{2^n} \right|} h\left(T_{\frac{t}{2^n}}\right) \quad n = 1, 2, \dots, ;$$

so that from (2.9),

$$\frac{1}{|t|} h(T_t) = \lim_{n \rightarrow \infty} \frac{1}{\left| \frac{t}{2^n} \right|} h\left(T_{\frac{t}{2^n}}\right) = h(\{T_t\})$$

for any real $t (\neq 0)$.

This proves the theorem.

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