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AUTOMORPHISM GROUPS OF HOMOGENEOUS SEMILINEAR ORDERS: NORMAL SUBGROUPS AND COMMUTATORS

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1. **Introduction.** A partially ordered set $(T, \leq)$ is called a *tree* if it is semilinearly ordered, i.e. any two elements have a common lower bound but no two incomparable elements have a common upper bound, and contains an infinite chain and at least two incomparable elements. Let $k \in \mathbb{N}$. We say that a partially ordered set $(T, \leq)$ is *k-homogeneous*, if each isomorphism between two k-element subsets of T extends to an automorphism of $(T, \leq)$, and *weakly k-transitive*, if for any two k-element subchains of T there exists an automorphism of $(T, \leq)$ taking one to the other. In this paper, we study the normal subgroup lattice and the commutator subgroup of the automorphism group $A(T)$ of weakly 2-transitive trees $(T, \leq)$. Such trees T and their automorphism groups $A(T)$ have been studied in [8, 9], to which this paper is a sequel. (For the convenience of the reader, we have included all (we hope) results from [8, 9] needed for the proofs of the present paper, so that it may be read largely independently from [8, 9].) For any lower-directed partial order T, let

$$S(T) := \{f \in A(T) : \exists x \in T, \forall t \in T, t^f \neq t \Rightarrow x < t\}, \text{ and}$$
$$R(T) := \{f \in A(T) : \exists x \in T, \forall t \in T, t < x \Rightarrow t^f = t\}.$$

Then $S(T)$ and $R(T)$ are normal subgroups of $A(T)$, and if T is a weakly 2-transitive tree, then $\{1\} \underset{\neq}{<} S(T) \underset{\neq}{<} R(T) \underset{\neq}{<} A(T)$. In [8] we showed that for any weakly 2-transitive tree T, $S(T)$ is simple and contained in every non-trivial normal subgroup of $A(T)$. Moreover, if T is countable, then the normal subgroup lattice of $A(T)$ contains antichains of size $2^{2^{\aleph_0}}$ as well as chains isomorphic to $(\mathbb{R}, \leq)$ or to $(\omega_1, \leq)$, indicating that the structure of the normal subgroup lattice is quite complicated. Here, we will first show that, at least immediately above $S(T)$, there is some order in the chaos, related to the structure of T.

Countable doubly homogeneous trees are interesting for various reasons. Their automorphism groups have many Jordan sets, and they occur in a recent classification theorem of Adeleke and Neumann [1] of certain Jordan groups. Also, for such trees T the numbers $n_k\big(A(T)\big)$ of orbits of $A(T)$ on unordered k-subsets of T grow comparatively slowly—exponentially, but not faster than exponentially (see [3, 4]). For primitive permutation groups, this phenomenon is rare. Furthermore, they are $\aleph_0$-categorical structures and homogeneous in the sense of Fraïssé [10, 11] over an appropriate finite

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relational language. Finally, they occur as a large class of infinite posets with transitive automorphism groups, see [5]. We note that up to isomorphism there are countably many countable doubly homogeneous trees ([5]), whereas there are continuously many countable weakly 2-transitive trees ([9]).

If $M, N \lhd A(T)$, we say that N is a *cover* of M if $M \underset{\neq}{<} N$ and there is no $L \lhd A(T)$ with $M \underset{\neq}{<} L \underset{\neq}{<} N$. A set $\mathcal{C}$ of covers of M is *complete*, if every normal subgroup $L \lhd A(T)$ with $M \underset{\neq}{<} L$ contains some member of $\mathcal{C}$. Now let $(T^+, \leq)$ be the smallest tree containing T which is a meet-semilattice (up to isomorphism, T^+ always exists and is unique). An element $x \in T^+$ is called a *ramification point* of T, if there are two incomparable elements $a, b \in T$ with $x = \inf\{a, b\}$ in T^+. The set of all ramification points of T is denoted by $\mathrm{ram}(T)$ and we have $T^+ = T \cup \mathrm{ram}(T)$. The elements of $A(T)$ extend naturally to an action on T^+ and hence on $\mathrm{ram}(T)$. If $x \in \mathrm{ram}(T)$, then the maximal lower-directed subsets of $\{t \in T : x < t\}$ are called the *cones* at x. For reasons of simplicity, we will first consider trees T which are *nice*, i.e., for which no cone has a smallest element. Then a cardinal number λ is called *a ramification order* of T if there exists a ramification point $x \in T^+$ which has precisely λ cones. We will show:

THEOREM 1.1. *Let T be a countable weakly 2-transitive nice tree and $n \in \mathbb{N} \cup \{\aleph_0\}$. The following are equivalent:*

(1) $S(T)$ has precisely n covers.

(2) $A(T)$ has precisely n orbits on $\mathrm{ram}(T)$.

(3) $(T, \leq)$ has precisely n different ramification orders.

Moreover, $S(T)$ has only finitely many covers if and only if $S(T)$ has a complete set of covers.

REMARK. By the characterization of countable weakly 2-transitive trees given in [9; Theorem 2.15] we obtain: for each $2 \leq n \in \mathbb{N}$ there are, up to isomorphism, $\aleph_0$ countable weakly 2-transitive nice trees $(T, \leq)$ such that $S(T)$ has precisely n covers. There are $2^{\aleph_0}$ countable weakly 2-transitive nice trees $(T, \leq)$ such that $S(T)$ has $\aleph_0$ covers.

Theorem 1.1 generalizes [8; Proposition 6.3] where the result was shown under the stronger assumption that $(T, \leq)$ be doubly homogeneous (in which case T has only one ramification order). Indeed, we will obtain an explicit construction of the covers of $S(T)$ in Theorem 1.1, and we indicate how to generalize this result to arbitrary countable weakly 2-transitive trees. It seems to be complicated, however, to delete the assumption of countability of T in Theorem 1.1.

In [8], it was also shown for any weakly 2-transitive tree T that $R(T)$ is the largest proper normal subgroup of $A(T)$ if and only if T has *countable coinitiality*, i.e., T contains a countable subset which is unbounded below in T. Here we will prove:

THEOREM 1.2. *Let T be a weakly 2-transitive tree of uncountable coinitiality. Then among the normal subgroups of $A(T)$ not contained in $R(T)$, there is a smallest one N. Furthermore, $R(T) \not\subseteq N$.*

Finally, we investigate the commutator subgroup of $A(T)$. By results of [8], if T is weakly 2-transitive and has countable coinitiality then $A(T) = A(T)'$ and, moreover, each element of $A(T)$ is a product of two commutators.

THEOREM 1.3. *For each uncountable cardinal κ there exist doubly homogeneous trees $(T_1, \leq_1)$ and $(T_2, \leq_2)$ each of cardinality κ and uncountable coinitiality, with $A(T_1) = A(T_1)'$ and $A(T_2) \neq A(T_2)'$.*

Theorem 1.1 will be proved in Section 2 and Theorems 1.2 and 1.3 in Section 3.

We conclude the introduction by stating a result which should have been in § 2 of [9]. It shows the extent to which weakly 2-transitive trees are characterized by their first order theories. The notation is explained in [9]. We omit the proof; it is an easy application of the proof of [9; Proposition 2.6] together with compactness (applied to 2-sorted structures (T, G), where T is a tree and G is a weakly 2-transitive group of automorphisms of T).

PROPOSITION 1.4. *Let $(T, \leq)$, $(T', \leq')$ be weakly 2-transitive trees with types $t(T) = \big(m, (i,j), A\big)$, $t(T') = \big(m', (i',j'), A'\big)$. Then $(T, \leq)$ and $(T', \leq')$ are elementarily equivalent if and only if $m = m'$, $i = i'$, $j = j'$, and either $A = A'$ or A, A' are infinite with $A \triangle A' = \{\infty\}$.*

It follows from Proposition 1.4 that if $(T, \leq)$ is a weakly 2-transitive tree, then either $\mathrm{Th}(T, \leq)$ is ω-categorical, or it has precisely two countable weakly 2-transitive models. Since, by a famous result of Vaught [14], no complete theory in a countable language can have precisely two models of cardinality $\aleph_0$, this shows that weak 2-transitivity is not a first order property for trees.

2. **Covers of $S(T)$.** In this section we wish to prove Theorem 1.1. We will also give an analogous result for countable weakly 2-transitive trees which are *not* nice. Let us first fix our notation and recall from [5, 8, 9] basic geometric properties of weakly 2-transitive trees.

Let $(P, \leq)$ be a poset. For $a, b \in P$ we write $a \parallel b$ if a and b are incomparable. If $A, B \subseteq P$, we write $A < B (A \parallel B)$ if $a < b$ $(a \parallel b)$ for all $a \in A$ and $b \in B$, respectively. We let $a < B$ abbreviate $\{a\} < B$ for any $a \in P$. We say that $(P, \leq)$ is *Dedekind-complete*, if each non-empty subset of P which is bounded above in P has a supremum in P, or equivalently if each non-empty subset of P which is bounded below in P has an infimum in P. It is easy to see that a tree $(T, \leq)$ is Dedekind-complete if and only if each maximal chain in T is Dedekind-complete and T is a meet-semilattice. We first note that each tree $(T, \leq)$ has a unique Dedekind-completion.

PROPOSITION 2.1 (cf.[12, 5]). *Let $(T, \leq)$ be a tree. Then there is a Dedekind-complete tree $(\bar{T}, \leq')$ with the following properties:*

(i) $T \subseteq \bar{T}$, and $\leq', \leq$ coincide on T;

(ii) for each $x \in \bar{T}$, $x = \sup\{t \in T : t \leq x\}$ in $(\bar{T}, \leq')$ and there are $a, b \in T$ with $a \leq x \leq b$.

Furthermore, given trees T_1, T_2 and corresponding Dedekind-complete trees $\bar{T}_1, \bar{T}_2$ satisfying (i) and (ii), each isomorphism $f: T_1 \longrightarrow T_2$ extends to a unique isomorphism $\bar{f}: \bar{T}_1 \longrightarrow \bar{T}_2$.

We will always denote by $(\bar{T}, \leq)$ the Dedekind-completion of a tree $(T, \leq)$. Note that every $f \in A(T)$ extends to a unique automorphism of $\bar{T}$, also denoted by f. Note that the set of *ramification points* of T is

$$\mathrm{ram}(T) := \{ a \in \bar{T} : a = \inf\{ b, c\} \text{ for some } b, c \in T \text{ with } b \parallel c \}.$$

Clearly, if T is weakly 2-transitive, then either $T \subseteq \mathrm{ram}(T)$ or $T \cap \mathrm{ram}(T) = \emptyset$. We recall that $T^+ = T \cup \mathrm{ram}(T)$, and if $a \in \mathrm{ram}(T)$, the maximal lower directed subsets of $\{ x \in T : a < x\}$ are called the *cones* at a. Let $C(a)$ be the set of all cones at a.

If A is a subset (usually a chain) of a tree T, we put

$$\begin{aligned} A^{\mathrm{cl}} &:= \{ x \in T : \exists a, b \in A,\ a \leq x \leq b \text{ or } (a \leq x \text{ and } x \parallel b)\} \\ &= \{ x \in T : a \leq x \text{ for some } a \in A\} \setminus \{ x \in T : A \leq x\}, \end{aligned}$$

the *closure* of A in T. If $a, b \in T$ with $a < b$, we put $\langle a, b\rangle := \{ a, b\}^{\mathrm{cl}}$ and call $\langle a, b\rangle$ an *interval* of T. Provided T has no maximal elements, a will be the smallest element of $\langle a, b\rangle$ and b the unique maximal element of $\langle a, b\rangle$, so any isomorphism $f : \langle a, b\rangle \longrightarrow \langle c, d\rangle$ will take a to c and b to d. Now we have

PROPOSITION 2.2 ([8; 3.3, 3.2(a)]). *Let $(T, \leq)$ be a tree. Then the following are equivalent:*

(1) $(T, \leq)$ is weakly k-transitive for some $k \geq 2$.
(2) $(T, \leq)$ is weakly k-transitive for each $k \in \mathbb{N}$.
(3) $(T, \leq)$ has no maximal or minimal elements, and whenever $a, b, c, d \in T$ with $a < b$ and $c < d$, then $\langle a, b\rangle \cong \langle c, d\rangle$.

Moreover, in this case for any $x, y \in T$ with $x < y$ there is $a \in T$ with $x < a < y$, and, furthermore, if $z \in \mathrm{ram}(T)$, there is $f \in A(T)$ such that $x < z^f < y$.

Because of Proposition 2.2, the transitivity condition we will usually use is weak 2-transitivity. Next we recall two main results from [8].

THEOREM 2.3 ([8; THEOREM 1.1]). *Let T be a weakly 2-transitive tree. Then $S(T)$ is simple and is contained in every non-trivial normal subgroup of $A(T)$. Also, each $f \in S(T)$ is a commutator in $S(T)$, and if $f \in S(T)$ and $1 \neq g \in A(T)$, then there are $h_1, \ldots, h_4 \in S(T)$ such that $f = (g^{-1})^{h_1} \cdot g^{h_2} \cdot (g^{-1})^{h_3} \cdot g^{h_4}$.*

If T is a tree, then the *coinitiality* of T is defined to be

$$\mathrm{coi}(T) := \min\{ |A| : A \subseteq T, \text{ for all } x \in T \text{ there is } a \in A \text{ with } a \leq x\}.$$

THEOREM 2.4 ([8; THEOREMS 1.3(a), 5.2]). *Let T be a weakly 2-transitive tree. Then the following are equivalent:*

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(1) $R(T)$ contains every proper normal subgroup of $A(T)$.
(2) $A(T)/R(T)$ is simple.
(3) T has countable coinitiality.

Moreover, in this case each $f \in A(T)$ is a product of two commutators in $A(T)$.

Now we turn to the description of the covers of $S(T)$. First we examine cones in some more detail.

PROPOSITION 2.5. *Let T be a weakly 2-transitive tree.*
(a) Any cone in T with no smallest element is itself a weakly 2-transitive tree.
(b) Any two cones in T with no smallest elements and with countable coinitiality are isomorphic.

PROOF. (a) Let C be a cone in T. By Proposition 2.2, $(C, \leq)$ is a tree. Let $a, b, c, d \in C$ with $a < b$ and $c < d$. Choose $e \in C$ with $e < \{a, c\}$. Again by Proposition 2.2, there is $f \in A(T)$ with $\{e, a, b\}^f = \{e, c, d\}$. Let $g: C \longrightarrow C$ coincide with f on $\{x \in C : e \leq x\}$ and fix the rest of C pointwise. Clearly $g \in A(C)$ and $\{a, b\}^g = \{c, d\}$.

(b) If A, B are two such cones in T, choose two sequences $\{a_i : i \in \mathbb{N}\} \subseteq A$, $\{b_i : i \in \mathbb{N}\} \subseteq B$ which are unbounded below in A and B, respectively, such that $a_{i+1} < a_i$ and $b_{i+1} < b_i$ for each $i \in \mathbb{N}$. There are $f, f_i \in A(T)$ such that $a_1^f = b_1$ and $\{a_{i+1}, a_i\}^{f_i} = \{b_{i+1}, b_i\}$ for each $i \in \mathbb{N}$. Now let $g: A \longrightarrow B$ coincide on $\{x \in T : a_1 \leq x\}$ with f and on each interval $\langle a_{i+1}, a_i\rangle$ with f_i $(i \in \mathbb{N})$. Then g is the required isomorphism.

Let T be a tree. If $a \in \mathrm{ram}(T)$, we say that a is a *special ramification point* of T if a has a cone with a smallest element. Let $\mathrm{ram}_s(T)$ denote the set of all special ramification points of T. Note that if T is weakly 2-transitive, then $T \cap \mathrm{ram}_s(T) = \emptyset$ by Proposition 2.2. Then T is *nice*, if $\mathrm{ram}_s(T) = \emptyset$; that is, no cone in T has a smallest element. (The reason for formulating this condition is illustrated by Proposition 2.5). For $2 \leq n \in \mathbb{N}$ let

$$\mathrm{ram}_n(T) := \{a \in \mathrm{ram}(T) \setminus \mathrm{ram}_s(T) : |C(a)| = n\} \text{ and}$$
$$\mathrm{ram}_\infty(T) := \{a \in \mathrm{ram}(T) \setminus \mathrm{ram}_s(T) : C(a) \text{ is infinite}\}.$$

Put $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$. Next we construct covers of $S(T)$.

DEFINITION 2.6. Let T be a countable weakly 2-transitive tree and let $2 \leq n \in \mathbb{N}_\infty$ such that $\mathrm{ram}_n(T) \neq \emptyset$. Fix a maximal chain C in T^+. For each $c \in C \cap \mathrm{ram}(T)$, put

$$T_c := \bigcup\{D : D \text{ is a cone at } c \text{ with } C \cap D = \emptyset\}.$$

Now let $A \subseteq C \cap \mathrm{ram}_n(T)$ be unbounded below in C of order type ω^* (the reverse of ω). For each $a \in A$, choose a cone D_a at a disjoint to C; by Proposition 2.5(a), D_a is a weakly 2-transitive tree. Choose any element $1 \neq g_a \in S(D_a)$. Now define $g \in A(T)$ so

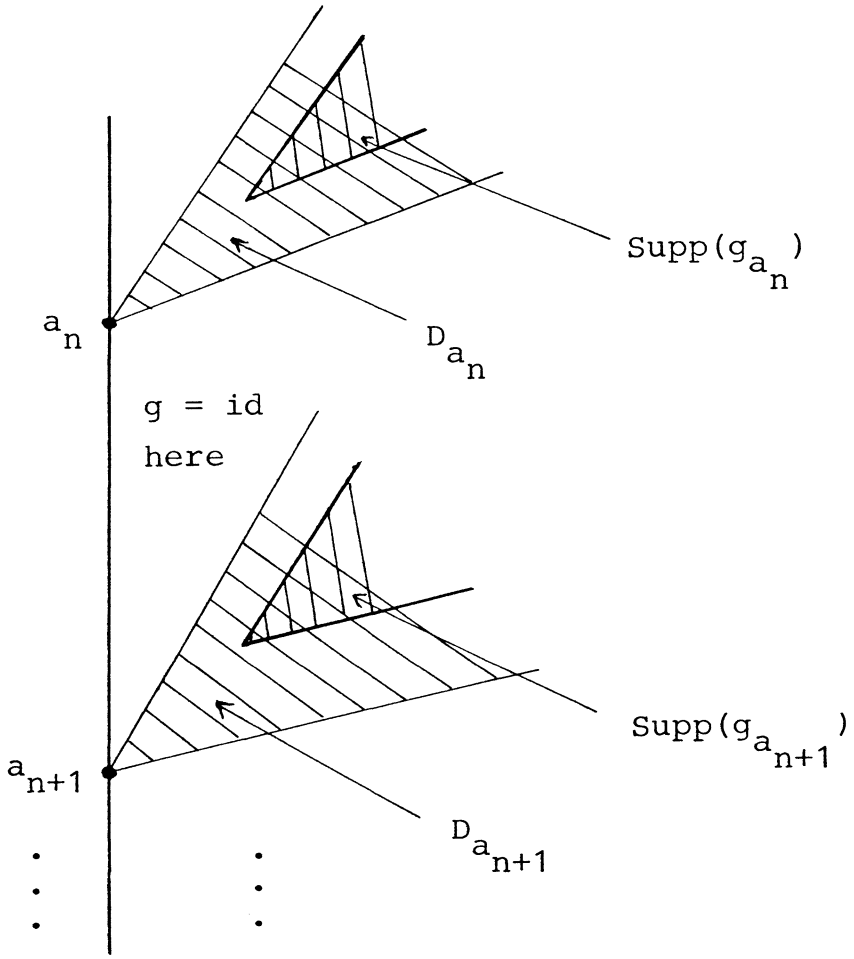

FIGURE 1

that g coincides on D_a with g_a for each $a \in A$, and g is the identity everywhere else (see Fig. 1 below). Put $M_n(T) := \langle g \rangle^{A(T)}$, the normal subgroup of $A(T)$ generated by g.

Clearly, by Theorem 2.3, we have $S(T) \underset{\neq}{<} M_n(T) \leq R(T)$, and we next wish to show that $M_n(T)$ is a cover of $S(T)$. The following auxiliary result shows that $A(T)$ acts weakly 2-transitively on $\mathrm{ram}_n(T)$ and is a consequence of [9; Propositions 2.3, 2.7]. We include a straightforward argument for it.

LEMMA 2.7. *Let T be a countable weakly 2-transitive tree, and let $a, b, c, d \in \mathrm{ram}_n(T)$ with $a < b, c < d$ for some $2 \leq n \in \mathbb{N}_\infty$. Then there is $f \in A(T)$ with $a^f = c$ and $b^f = d$.*

PROOF. We first show that for any $a, b \in \mathrm{ram}_n(T)$ there is $h \in A(T)$ with $a^h = b$. Note that $a = \sup\{x \in T : x < a\}$ in $(T, \leq)$; similarly for b. Hence there are sequences

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$\{a_i : i \in \mathbb{N}\}$, $\{b_i : i \in \mathbb{N}\}$ in T such that $a_1 = b_1$, $a_i < a_{i+1}$, $b_i < b_{i+1}$ for each $i \in \mathbb{N}$, and $a = \sup\{a_i : i \in \mathbb{N}\}$, $b = \sup\{b_i : i \in \mathbb{N}\}$ in $\bar{T}$. Since a, b have the same number of cones all of which are isomorphic, there exists $h: T \longrightarrow T$ which maps $\{x \in T : a < x\}$ isomorphically onto $\{y \in T : b < y\}$ and each interval $\langle a_i, a_{i+1}\rangle$ isomorphically onto $\langle b_i, b_{i+1}\rangle$ $(i \in \mathbb{N})$ and fixes $\{x \in T : a_1 \not\leq x\}$ pointwise. Then $h \in A(T)$ and $a^h = b$.

Now, to prove the claim of the lemma, it suffices to consider the case where $b = d$ and, say, $a < c$. By the above, choose $h \in A(T)$ with $a^h = c$. Then $c < c^h$. It could happen that b and c^h belong to different cones at c, i.e., that $c = \inf\{b, c^h\}$. In this case, by Proposition 2.5(b), there exists $h' \in A(T)$ which fixes $\{x \in T : c \not\leq x\}$ pointwise and maps c^h into the cone at c containing b; then consider $g' = hh'$ in place of h. We may therefore assume that $c < \inf\{b, c^h\}$. Choose $y, z \in T$ with $c < y < z < \{b, c^h\}$ and put $x = y^{h^{-1}}$; then $a < x < c$. There is $k \in A(T)$ with $\{x, z\}^k = \{y, z\}$. Now let $f: T \longrightarrow T$ coincide on $\{t \in T : x \not\leq t\}$ with h, on $\langle x, z\rangle$ with k, and on $\{t \in T : z \leq t\}$ with the identity. Then $f \in A(T)$, $a^f = c$, and $b^f = b = d$.

We will also use

LEMMA 2.8 ([8; LEMMA 4.1]). *Let T be a weakly 2-transitive tree, let C be a maximal chain in T, and let $c \in C, f \in A(T)$. Then there is $h \in S(T)$ such that $C^{fh} = C$ and fh fixes $\{x \in T : c \leq x\}$ pointwise.*

The characterization of the covers of $S(T)$ will follow from the following two main lemmas.

LEMMA 2.9. *Under the assumptions of Definition 2.6, let $B \subseteq C \cap \mathrm{ram}_n(T)$ be unbounded below in C and assume that $h \in A(T)$ fixes C pointwise and acts non-trivially on T_b for each $b \in B$. Then $M_n(T) \leq \langle h\rangle^{A(T)}$.*

PROOF. We may assume that B has order type ω^* (otherwise select an appropriate subset). Next we claim that we can assume without loss of generality that $A = B$. Indeed, enumerate $A = \{a_i : i \in \mathbb{N}\}$, $B = \{b_i : i \in \mathbb{N}\}$ such that $a_{i+1} < a_i$, $b_{i+1} < b_i$ for each $i \in \mathbb{N}$. By Lemma 2.7, there are $f, f_i \in A(T)$ such that $b_1^f = a_1$ and $\{b_{i+1}, b_i\}^{f_i} = \{a_{i+1}, a_i\}$ for each $i \in \mathbb{N}$. Let $f^1: T^+ \longrightarrow T^+$ coincide on $\{x \in T^+ : b_1 \leq x\}$ with f and on each interval $\langle b_{i+1}, b_i\rangle$ of T^+ with f_i $(i \in \mathbb{N})$; then $f^* = f^1|_T \in A(T)$ and $B^{f^*} = A$. Since $\langle h\rangle^{A(T)} = \langle h^{f^*}\rangle^{A(T)}$, this proves our claim.

Let $a \in A$. Then $h_a := h|_{T_a} \in A(T_a)\backslash\{1\}$. Clearly there is $k_a \in A(T_a)$ which fixes setwise each cone at a contained in T_a such that $h_a(h_a^{-1})^{k_a} \neq \mathrm{id}$. Let $k \in A(T)$ coincide with k_a on T_a for each $a \in A$, and fix the rest of T pointwise. Put $h^* := h(h^{-1})^k \in A(T)$. Then, for each $a \in A$, h^* leaves each cone at a contained in T_a setwise invariant, and acts on at least one of them, say C_a, non-trivially. Everywhere else h^* is the identity. Since all cones at a are isomorphic, we can assume $C_a = D_a$. Hence by Theorem 2.3 applied to each tree $C_a (a \in A)$ there are $h_1, \ldots, h_4 \in A(T)$ such that

$$g = (h^{*-1})^{h_1} \cdot (h^*)^{h_2} \cdot (h^{*-1})^{h_3} \cdot (h^*)^{h_4} \in \langle h\rangle^{A(T)}.$$

LEMMA 2.10. *Let T be a countable weakly 2-transitive tree, and let C be a maximal chain in T^+. As before, for each $c \in C$ we let $T_c := \bigcup\{D : D$ is cone at c with $C \cap D = \emptyset\}$. Assume that $A \subseteq C \cap (\mathrm{ram}(T) \setminus \mathrm{ram}_s(T))$ is unbounded below in C, but $A \cap \mathrm{ram}_n(T)$ is bounded below in C for each $2 \leq n \in \mathbb{N}_\infty$. Let $h \in A(T)$ fix C pointwise and act non-trivially on T_a for each $a \in A$. Then $\langle h \rangle^{A(T)}$ is not a cover of $S(T)$.*

PROOF. Let $I := \{n \in \mathbb{N}_\infty : n \geq 2, A \cap \mathrm{ram}_n(T)\} \neq \emptyset$. By our assumptions on A, we can split $I = J \cup K$ into two disjoint infinite subsets such that both $A \cap \bigcup_{n \in J} \mathrm{ram}_n(T)$ and $A \cap \bigcup_{n \in K} \mathrm{ram}_n(T)$ are unbounded below in C. Whenever $j \in J$ and $a \in A \cap \mathrm{ram}_j(T)$, choose a cone D_a at a disjoint to C and any $1 \neq f_a \in S(D_a)$. Now define $f \in A(T)$ so that f coincides on D_a with f_a for each $a \in A \cap \mathrm{ram}_j(T)$ ($j \in J$), and f is the identity everywhere else. Then clearly $S(T) \underset{\neq}{<} \langle f \rangle^{A(T)}$, and $f \in \langle h \rangle^{A(T)}$ follows as in the argument for Lemma 2.9. It only remains to show that $h \notin \langle f \rangle^{A(T)}$.

Indeed, otherwise there are $f_i \in \{f, f^{-1}\}$ and $k_i \in A(T)$ ($i = 1, \ldots, m$; $m \in \mathbb{N}$) such that $h = \prod_{i=1}^m f_i^{k_i}$. Choose any $y \in C$ and then $x \in C$ with $x < y^{k_i}$ for all $i = 1, \ldots, m$. Then $x^{k_i^{-1}} < y$, hence $x^{k_i^{-1}} \in C$ for all $i = 1, \ldots, m$. By our assumptions on K, there are $n \in K$ and $a \in A \cap \mathrm{ram}_n(T)$ with $a < x$. For each $i \in \{1, \ldots, m\}$, we have $a_i = a^{k_i^{-1}} \in C \cap \mathrm{ram}_n(T)$ and thus f_i fixes T_{a_i} pointwise. Hence h fixes T_a pointwise, a contradiction.

Now we can describe the covers of $S(T)$:

THEOREM 2.11. *Let T be a countable weakly 2-transitive nice tree.*

(a) The covers of $S(T)$ are precisely the groups $M_n(T)$ ($2 \leq n \in \mathbb{N}_\infty$ with $\mathrm{ram}_n(T) \neq \emptyset$), as constructed in (2.6).

(b) $S(T)$ has a complete set of covers if and only if $S(T)$ has only finitely many covers.

PROOF. Let C be a maximal chain in T^+. For each $c \in C$, as before let $T_c := \bigcup\{D : D$ is a cone at c with $C \cap D = \emptyset\}$.

(a) We first show that the groups $M_n(T)$ are covers of $S(T)$. Let $f \in A(T)$ with $S(T) \underset{\neq}{<} \langle f \rangle^{A(T)} \leq M_n(T)$. By Lemma 2.8 and Theorem 2.3, we may assume that f fixes C pointwise. There exists a subset $B \subseteq C \cap \mathrm{ram}(T)$ which is unbounded below in C such that f acts non-trivially on T_b for each $b \in B$. As $f \in M_n(T)$, there exists $x \in C$ such that each $b \in B$ with $b < x$ belongs to $\mathrm{ram}_n(T)$. Hence $\langle f \rangle^{A(T)} = M_n(T)$ by Lemma 2.9.

Next we show that each cover H of $S(T)$ is one of the groups $M_n(T)$. Indeed, $H \leq R(T)$ by Theorem 2.4, and hence by Lemma 2.8 and Theorem 2.3 there is $h \in H \setminus S(T)$ which fixes C pointwise. Let $A := \{c \in C \cap \mathrm{ram}(T) : h$ acts non-trivially on $T_c\}$. Then A is unbounded below in C. By Lemma 2.10, $A \cap \mathrm{ram}_n(T)$ is unbounded below in C for some $2 \leq n \in \mathbb{N}_\infty$. But then $M_n(T) \leq \langle h \rangle^{A(T)} \leq H$ by Lemma 2.9 and thus $H = M_n(T)$.

(b) Let $I := \{n \in \mathbb{N}_\infty : n \geq 2, \mathrm{ram}_n(T) \neq \emptyset\}$. By (a), $S(T)$ has precisely $|I|$ covers. Now assume I is finite and $h \in A(T) \setminus S(T)$. We claim that $M_n(T) \leq \langle h \rangle^{A(T)}$ for some $n \in I$. By Theorem 2.4, we may assume $h \in R(T)$ and hence, by Theorem 2.3 and Lemma 2.8, that h fixes C pointwise. Let $A = \{c \in C \cap \mathrm{ram}(T) : h$ acts non-trivially on

$T_c\}$. Then A is unbounded below in C. As I is finite, $A \cap \mathrm{ram}_n(T)$ is unbounded below in C for some $n \in I$. Then $M_n(T) \leq \langle h \rangle^{A(T)}$ by Lemma 2.9.

Finally, assume that I is infinite. Choose a subset $A \subseteq C \cap \mathrm{ram}(T)$ which is unbounded below in C such that $|A \cap \mathrm{ram}_n(T)| = 1$ for each $n \in I$. Define $f \in R(T) \setminus S(T)$ such that f fixes C pointwise and acts non-trivially on $T_c\big(c \in C \cap \mathrm{ram}(T)\big)$ iff $c \in A$. The argument at the end of the proof of Lemma 2.10 shows that $M_n(T) \not\leq \langle f \rangle^{A(T)}$ for each $n \in I$. Hence, by (a), the set of covers of $S(T)$ is not complete in this case. The result follows.

Now we can give the

PROOF OF THEOREM 1.1. (1)$\leftrightarrow$(3): Immediate by Theorem 2.11(a). (2)$\leftrightarrow$(3): By Lemma 2.7, the orbits of $A(T)$ on $\mathrm{ram}(T)$ are the sets $\mathrm{ram}_n(T)$, if non-empty, for $2 \leq n \in \mathbb{N}_\infty$.

Next we wish to consider countable weakly 2-transitive trees T which are not nice and state a result similar to Theorem 1.1 also for them.

A characterization of all countable weakly 2-transitive trees was obtained in [9; §2]. For any poset $(P, \leq)$ and $a, b \in P$, we say that b *covers* a if $a < b$ and there is no $x \in P$ with $a < x < b$. Now let T be a tree and $a \in \mathrm{ram}(T)$. Then a is a special ramification point of T iff a is covered in T^+ by some $b \in T$. We say that T has *special ramification order* (i,j), if T is not nice and for any $a \in \mathrm{ram}_s(T)$, a has precisely i cones with a smallest element and j cones without a smallest element; then always $i \geq 1$ and $i+j \geq 2$. Now let T be weakly 2-transitive and $a, b \in \mathrm{ram}_s(T)$. Choose $c, d \in T$ which cover a, b, respectively. There is $f \in A(T)$ with $cf = d$ and thus $af = b$. This shows that any weakly 2-transitive non-nice tree T has a special ramification order and $\mathrm{ram}_s(T)$ is a single orbit of $A(T)$ on $\mathrm{ram}(T)$. The following is the analogue of Theorem 1.1 for trees which are not nice (for notational simplicity, let $\infty + i = \infty$ for each $i \in \mathbb{N}$).

THEOREM 2.12. *Let T be a countable weakly 2-transitive tree with special ramification order (i,j), and let $n \in \mathbb{N} \cup \{0, \infty\}$. Put $I := \{m \in \mathbb{N}_\infty : \mathrm{ram}_m(T) \neq \emptyset\}$, and let $\bar{n} := n + 1$ (respectively $n + 2; n + 3; n + 4$) if $j = 0$ (respectively $i = j = 1; i = 1, j \geq 2$ or $i \geq 2, j = 1; i, j \geq 2$). Then the following are equivalent:*

(1) $S(T)$ has precisely $\bar{n}$ covers.

(2) $A(T)$ has precisely $n + 1$ orbits on $\mathrm{ram}(T)$.

(3) $|I| = n$.

Moreover, $S(T)$ has only finitely many covers if and only if $S(T)$ has a complete set of covers.

REMARK. The remark after Theorem 1.1 holds with 'not nice' replacing 'nice'.

Here, the equivalence (2)$\leftrightarrow$(3) in Theorem 2.12 follows from Lemma 2.7 and the fact, noted above, that $\mathrm{ram}_s(T)$ is a single orbit of $A(T)$ on $\mathrm{ram}(T)$. (In this context we note, however, that if $j \geq 1$, then in contrast to Lemma 2.7 $A(T)$ has two orbits on the set of all 2-element subchains of $\mathrm{ram}_s(T)$; for if $a, b, c \in \mathrm{ram}_s(T)$ and $g \in A(T)$ with

$a < b < c, a^g = b$ and $c^g = c$, then a is covered in T^+ by some $t \in T$ with $t < c$ if and only if b is.) The rest of the argument for Theorem 2.12 is very similar to the one given for Theorem 1.1. Therefore we only describe how to construct the covers of $S(T)$ in the present case, leaving the actual proof to the reader.

We proceed as in Definition 2.6.

Let T be a countable weakly 2-transitive tree with special ramification order (i,j), let C be a maximal chain in T^+, and let $I := \{n \in \mathbb{N}_\infty : n \geq 2, \mathrm{ram}_n(T) \neq \emptyset\}$. For each $n \in I$, choose a subset $A \subseteq C \cap \mathrm{ram}_n(T)$ as before; the normal subgroups $M_n(T)$ $(n \in I)$ then provide $|I|$ covers of $S(T)$. However, as T is not nice, $S(T)$ has either 1,2,3 or 4 more new covers, depending on the following cases.

CASE 1. (possible iff $i \geq 2$) We can choose $A \subseteq C \cap \mathrm{ram}_s(T)$ with order type ω^* and unbounded below in C such that each $a \in A$ is covered in T^+ by some $t \in C$. For each $a \in A$, let D'_a be a cone at a disjoint to C containing a smallest element, say m_a. If $m_a \notin \mathrm{ram}(T)$ (then $T \cap \mathrm{ram}(T) = \emptyset$), put $D_a = D'_a \setminus \{m_a\}$. If $m_a \in \mathrm{ram}(T)$ (so $T \subseteq \mathrm{ram}(T) \setminus \mathrm{ram}_s(T)$), let D_a be any cone at m_a. Now choose any $1 \neq g_a \in S(D_a)$.

CASE 2. (possible iff $j \geq 1$) Choose $A \subseteq C \cap \mathrm{ram}_s(T)$ as in Case 1. For each $a \in A$, let D_a be a cone at a disjoint to C and with no smallest element, and let $1 \neq g_a \in S(D_a)$.

CASE 3. (possible iff $j \geq 1$) Choose $A \subseteq C \cap \mathrm{ram}_s(T)$ of order type ω^* and unbounded below in C such that no $a \in A$ is covered by any $t \in C$. For each $a \in A$, let D'_a be a cone at a disjoint to C and with a smallest element, say m_a, and proceed as in Case 1.

CASE 4. (possible iff $j \geq 2$) Choose $A \subseteq C \cap \mathrm{ram}_s(T)$ as in Case 3. For each $a \in A$, let D_a be a cone at a disjoint to C and without a smallest element, and let $1 \neq g_a \in S(D_a)$.

Note that at least one of these cases always occurs. Now define $g \in A(T)$ as in Definition 2.6. Thus each of these four cases gives rise to a normal subgroup of $A(T)$, and it can be shown that these groups together with the groups $M_n(T)$ are all the covers of $S(T)$.

Finally, we note that there is also an explicit characterization of all the covers of $S(T)$ for arbitrary weakly 2-transitive trees T with countable coinitiality. As a consequence, there is a technical generalization of the equivalences (1)$\leftrightarrow$(2) of Theorems 1.1 and 2.12 for such trees T, in which condition (2) of Theorem 1.1 is replaced by a condition on the number of orbits of $A(T)$ on $\mathrm{ram}(T)$, the number of orbits of $A(T)$ on the set of 2-element chains in $\mathrm{ram}(T)$, and the numbers of non-isomorphic cones at ramification points. This shows that if T is weakly 2-transitive and has countable coinitiality, then still $S(T)$ has only finitely many covers iff $S(T)$ has a complete set of covers; moreover, if T is doubly homogeneous and $\mathrm{coi}(T) = \aleph_0$, then $S(T)$ has a unique cover. (This latter statement can also be proved directly by following the argument for [8; Proposition 6.3], using [5; Lemma 5.30].) However, corresponding results for trees T with uncountable coinitiality, even if T is assumed to be doubly homogeneous, remain open.

3. **Normal subgroups and commutators in** $A(T)$. In this section, we wish to prove Theorems 1.2 and 1.3. We also derive two results which describe, for weakly 2-transitive trees T with uncountable coinitiality, sufficient conditions which imply $A(T) = A(T)'$, respectively $A(T) \neq A(T)'$. We will use a few results on groups of automorphisms of chains and trees from [8, 9] which we first summarize for the convenience of the reader.

If G is a permutation group on a set X and if $Y \subseteq X$, then G_Y denotes the setwise stabilizer of Y in G, and if $H \leq G_Y$, then H^Y denotes the permutation group induced by H on Y. If G is a group and $H \leq G$, then H^G denotes the least normal subgroup of G containing H. We denote by $\big(\mathcal{N}(G), \subseteq\big)$ the lattice of all normal subgroups of G.

Now let C be a chain with no endpoints. We say that C has *countable coterminality,* if it contains a countable subset which is unbounded above and below in C. We denote by $(\bar{C}, \leq)$ the Dedekind-completion of $(C, \leq)$, and by $(\bar{\bar{C}}, \leq)$ the absolute completion of $(C, \leq)$; that is, $\bar{\bar{C}} = \bar{C} \cup \{-\infty, \infty\}$ where $-\infty < x < \infty$ for all $x \in \bar{C}$. Each automorphism f of C extends uniquely to an automorphism, also denoted by f, of $\bar{\bar{C}}$. For each $f \in A(C)$, let $F(f) := \{x \in \bar{\bar{C}} : xf = x\}$, the fixed point set of f in $\bar{\bar{C}}$. If $G \leq A(C)$, let $\mathcal{F}(G) := \{F(g) : g \in G\}$.

PROPOSITION 3.1([8; THEOREM 2.3, PROPOSITIONS 2.9, 4.2]). *Let T be a weakly 2-transitive tree, let C be a maximal chain in T, and let $G := A(T)_C^C := \big(A(T)_C\big)^C$.*

(a) *$(C, \leq)$ is doubly homogeneous. The mapping $\Phi : \big(\mathcal{N}(G), \subseteq\big) \longrightarrow \big(\mathcal{N}(A(C)), \subseteq\big)$, defined by $N^\Phi := N^{A(C)}$ for any $N \lhd G$ is a lattice isomorphism, and Φ^{-1} maps each $N^* \lhd A(C)$ onto $N^* \cap G$.*

(b) *Let $A \subseteq \bar{\bar{C}}$. Then $A \in \mathcal{F}(G)$ iff $-\infty, \infty \in A$ and $\bar{\bar{C}} \setminus A$ is a disjoint union of open intervals each with countable coterminality.*

We note that in [8] a more general statement than Proposition 3.1 has been proved: under the assumptions of Proposition 3.2, the group G is, in some precise sense, a "large" subgroup of $A(C)$, and indeed, any such large subgroup of the automorphism group of a doubly homogeneous chain C satisfies the assertions of Proposition 3.1.

Let T be a tree. We call a subset $S \subseteq T$ *convex,* if $a, b \in S$, $x \in T$ and $a < x < b$ imply $x \in S$. A convex subset S of T will be called a *non-trivial orbital* of $g \in A(T)$, if S is an infinite chain, $S^g = S$, and for some (equivalently, all) $s \in S$ the set $\{s^{g^i} : i \in \mathbb{Z}\}$ is unbounded above and below in S. We then say that g has *positive* (respectively *negative*) *parity* on S if $s < s^g$ (respectively, $s^g < s$) for some (equivalently, all) $s \in S$. The next lemma gives a sufficient condition for conjugacy in $A(T)$.

CONJUGATION LEMMA 3.2 ([8; LEMMA 3.5]). *Let T be a weakly 2-transitive tree, let $g, h \in A(T)$ and let $S \subseteq T$ be a non-trivial orbital of g and of h on which g and h have the same parity. Then there exists $f \in A(T)$ which maps S^{cl} onto itself and fixes $T \setminus S^{\mathrm{cl}}$ pointwise such that h and $f^{-1} \cdot g \cdot f$ coincide on S^{cl}.*

Next, we introduce some more notation for chains. Let C be a chain with no endpoints. Then let

$$\mathrm{cof}(C) := \min\{\,|A| : A \subseteq C \text{ is unbounded above in } C\},$$

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the *cofinality* of C, and dually

$$\operatorname{coi}(C) := \min\{\,|A| : A \subseteq C \text{ is unbounded below in } C\},$$

the *coinitiality* of C. If $\operatorname{cof}(C) = \operatorname{coi}(C)$, this is called the *coterminality* of C. A subset A of C is said to be *dense* in C, if for any $a, b \in C$ with $a < b$ there exists $x \in A$ with $a < x < b$. Now let C be dense in itself and $c \in \bar{C}$. Then

$$\operatorname{cof}(c) := \operatorname{cof}(\{x \in \bar{C} : x < c\}) = \min\{\,|A| : A \subseteq \bar{C}, A < c,\ c = \sup A\}$$

is called the *cofinality* of c in $\bar{C}$, and dually

$$\operatorname{coi}(c) := \operatorname{coi}(\{x \in \bar{C} : c < x\})$$

is called the *coinitiality* of c in $\bar{C}$. If $\operatorname{cof}(c) = \operatorname{coi}(c)$, this is called the *coterminality* of c. If $a, b \in \bar{C}$ with $a < b$ and $A \subseteq \bar{C}$, put $[a,b]_A := \{x \in A : a \leq x \leq b\}$ and $(a, \infty)_A := \{x \in A : a < x\}$. The set $(a,b)_A$ is defined analogously.

LEMMA 3.3. *Let C be an infinite chain.*

(a) If C has uncountable coinitiality and $A_i \subseteq \bar{C}\,(i = 1, \ldots, n;\ n \in \mathbb{N})$ are closed and unbounded below in $\bar{C}$, then $\bigcap_{i=1}^{n} A_i$ is also closed and unbounded below in $\bar{C}$.

(b) Let $(C, \leq)$ be doubly homogeneous. Then for each $c \in \bar{C}$ the sets $\{x \in \bar{C} : \operatorname{cof}(x) = \operatorname{cof}(c)\}$ and $\{x \in \bar{C} : \operatorname{coi}(x) = \operatorname{coi}(c)\}$ are dense in $\bar{C}$. In particular, the sets $\{x \in \bar{C} : \operatorname{cof}(x) = \aleph_0\}$ and $\{x \in \bar{C} : \operatorname{coi}(x) = \aleph_0\}$ are dense in $\bar{C}$.

PROOF. Straightforward.

Now we can give the

PROOF OF THEOREM 1.2. Let $\bar{T}$ be the Dedekind-completion of T, C be a maximal chain in T, and $\tilde{C}$ be the maximal chain in $\bar{T}$ containing C. Clearly $\tilde{C}$ is Dedekind-complete, so we will assume that the Dedekind-completion $\bar{C}$ of C satisfies $C \subseteq \bar{C} \subseteq \tilde{C}$. By Lemma 3.3(b) and transfinite induction, there is an inversely well-ordered closed subset Z of $\bar{C}$ which is unbounded below in $\bar{C}$, such that for each $a \in Z$, if $a^* := \max\{z \in Z : z < a\}$, then $\operatorname{coi}(a^*) = \aleph_0$ in $\bar{C}$. By Proposition 3.1(a) and Lemma 3.3(b), for each $a \in Z$, choose $a' \in \bar{C}$ with $a^* < a' < a$ and $\operatorname{cof}(a') = \aleph_0$ in $\bar{C}$. Also let $a_0 := \max Z$. Put $A := \{-\infty, \infty\} \dot\cup \dot\bigcup_{a \in Z} [a', a]_{\bar{C}} \dot\cup (a_0, \infty)_{\bar{C}} \subseteq \bar{\bar{C}}$. Let $G := A(T)_C^C$. Since $\bar{\bar{C}} \setminus A = \dot\bigcup_{a \in Z} (a^*, a')_{\bar{C}}$, we have $A \in \mathcal{F}(G)$ by Proposition 3.1(b). Choose $g \in G$ with $A = F(g)$. The only fixed point of g in $\bar{\bar{C}}$ not contained in a non-trivial interval fixed pointwise by g is $-\infty$. It follows from [2; Lemma 4.7] that $\langle g \rangle^{A(C)}$ is the smallest normal subgroup of $A(C)$ not contained in $R(C)$ (recall the definition of $R(C)$ given in the introduction). Hence, by Proposition 3.1(a), $\langle g \rangle^{G}$ is the smallest normal subgroup of G not contained in $G \cap R(C)$. Let $h' \in A(T)_C$ satisfy $h'|_C = g$. Define $h \in A(T)$ so that h coincides with h' on each interval $(a^*, a')^{\mathrm{cl}}$ $(a \in Z)$ and fixes the rest of T pointwise. Then $C^h = C$, $h|_C = g$, and $h \notin R(T)$. Put $N := \langle h \rangle^{A(T)}$. Thus $N \not\subseteq R(T)$, and we claim that if M is any normal subgroup of $A(T)$ with $M \not\subseteq R(T)$, then $N \leq M$.

Indeed, M_C^C is a normal subgroup of G containing $G \cap R(C)$, as $S(T) \leq M$ by Theorem 2.3. Moreover, by $S(T) \leq M$ and Lemma 2.8, M_C^C strictly contains $G \cap R(C)$. Hence by the above we have $\langle g \rangle^G \leq M_C^C$. Choose $k \in M_C$ such that $g = k|_C$. The non-trivial orbitals S of k contained in C are precisely the sets $(a^*, a')_C$ $(a \in Z)$, and for each such orbital S, k coincides with h on S. Hence k fixes S^{cl} setwise, and by Lemma 3.2 there is an automorphism p_S of S^{cl} such that $k \cdot h = p_S^{-1} \cdot k \cdot p_S$ on S^{cl}. Define $p \in A(T)$ so that p coincides with p_S on S^{cl} for each of these orbitals $S \subseteq C$ of k, and with the identity on the rest of T. Then $h = [k,p]$ on all of T, showing $h \in M$ and thus $N \leq M$.

It remains to show that $R(T) \not\subseteq N$. We define $f \in R(T)$ as follows. For each $c \in \bar{C} \cap \mathrm{ram}(T)$, let $T_c := \bigcup (D : D$ is a cone at c with $C \cap D = \emptyset\}$, and choose (as in Section 2) an automorphism $f_c \neq \mathrm{id}$ of T_c. Let f coincide with f_c on T_c for each $c \in \bar{C} \cap \mathrm{ram}(T)$ and fix C pointwise. Then $f \in R(T)$, and we claim $f \notin N$.

Suppose $f \in N$. Then there are $k_i \in A(T)$, $h_i \in \{h, h^{-1}\}$ $(i = 1, \ldots, n;\ n \in \mathbb{N})$ with $f = \prod_{i=1}^n h_i^{k_i}$. Choose $c \in C$ such that $x^{k_i}, x^{k_i^{-1}} \in C$ for each $x \in C$ with $x \leq c$, and each $i = 1, \ldots, n$. Let $B_i := \{x \in \bar{C} : x \leq c, x^{k_i} = x\}$ $(i = 1, \ldots, n)$. Since C has uncountable coinitiality, it follows that each B_i is closed and unbounded below in $\bar{C}$. Hence by Lemma 3.3(a) there is $a \in Z \cap \bigcap_{i=1}^n B_i$. Now choose $b \in C$ with $a' < b < a$ so that $\{a'^{k_i}, a'^{k_i^{-1}}\} < b$ for each $i = 1, \ldots, n$. Then k_i, k_i^{-1} map $[b,a]_{\bar{C}}$ into $[a',a]_{\bar{C}}$, and consequently map $\langle b,a \rangle$ into $\langle a',a \rangle$ $(i = 1, \ldots, n)$. Hence, since h fixes $\langle a',a \rangle$ pointwise, f fixes $\langle b,a \rangle$ pointwise. However, by Proposition 2.2 there is $d \in \mathrm{ram}(T)$ with $b < d < a$. Then, by construction, f is not the identity on $T_d \subseteq \langle b,a \rangle$, a contradiction. This shows $f \in R(T) \setminus N$.

Next we consider Theorem 1.3. Before constructing the actual trees, we first wish to exhibit conditions which are sufficient to imply that $A(T) \neq A(T)'$, respectively, that $A(T) = A(T)'$. For this, we will only consider nice trees T. If then C is a maximal chain in T and $\tilde{C}$ is the maximal chain in $\bar{T}$ containing C, then $\tilde{C}$ is Dedekind-complete and contains C as a dense subset. Hence we may (and will) regard $\tilde{C}$ as the Dedekind-completion of C, i.e., $\bar{C} = \tilde{C}$. If C is any chain with uncountable coinitiality (cofinality) and $Z \subseteq \bar{C}$, we say that Z is *stationary below (above)* in $\bar{C}$ if Z intersects each closed subset A of $\bar{C}$ which is unbounded below (above) in $\bar{C}$, respectively. If T is a nice tree with uncountable coinitiality, a subset Z of $\bar{T}$ is *stationary below* in $\bar{T}$ if $Z \cap \bar{C}$ is stationary below in $\bar{C}$ for each maximal chain C in T. In view of the following result, we remark here that there are weakly 2-transitive nice trees T such that the cones at a point $z \in \mathrm{ram}(T)$ are not all isomorphic. For ramification order 2 such trees are constructed in [7; Theorems 3 and 3.2]. Similar constructions of trees with larger ramification order are possible using [9; Theorem 2.14].

THEOREM 3.4. *Let T be a weakly 2-transitive nice tree of uncountable coinitiality, and suppose that the set*

$$Z := \{\, z \in \mathrm{ram}(T) : z \textit{ has only finitely many but at least three isomorphic cones} \,\}$$

is stationary below in $\bar{T}$. Then $A(T) \neq A(T)'$, and $A(T)/A(T)'$ contains an involution.

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PROOF. Let C be any maximal chain in T. For each $c \in \bar{C} \cap \operatorname{ram}(T)$, let S_c be the set of cones at c which are disjoint from C, and put $T_c = \cup S_c$. Thus, for each $c \in Z$, S_c has finitely many, but at least two, pairwise isomorphic cones. Therefore, for any $c \in \bar{C} \cap \operatorname{ram}(T)$ there is an automorphism f_c of $(T_c, \leq)$ such that f_c^2 fixes T_c pointwise and, if $c \in Z$, f_c acts as a transposition on S_c. Now define $f \in A(T)$ so that f fixes C pointwise and coincides with f_c on T_c for each $c \in \bar{C} \cap \operatorname{ram}(T)$. Then $f \in R(T)$ and $f^2 = 1$, and we claim that $f \notin A(T)'$.

Suppose for a contradiction that there are $g_i, h_i \in A(T)$ $(i = 1, \ldots, n;\ n \in \mathbb{N})$ such that $f = \prod_{i=1}^n [g_i, h_i]$. Choose $d \in C$ such that $x^{g_i}, x^{g_i^{-1}}, x^{h_i}, x^{h_i^{-1}} \in C$ for each $x \leq d$, and each $i = 1, \ldots, n$. For $i = 1, \ldots, n$, let A_i (respectively B_i) be the set of all fixed points of g_i (respectively h_i) in $\{x \in \bar{C} : x \leq d\}$. Clearly, since $\bar{T}$ has uncountable coinitiality, by Proposition 3.1(b) A_i and B_i are closed and unbounded below in $\{x \in \bar{C} : x \leq d\}$, and hence so is $A := \bigcap_{i=1}^n (A_i \cap B_i)$ by Lemma 3.3. By assumption, there are $a, b \in A \cap Z$ with $a < b \leq d$. Since a, b are fixed by each of g_i, h_i, f, we obtain $T_a^f = T_a^{g_i} = T_a^{h_i} = T_a$ for $i = 1, \ldots, n$. Hence the elements f, g_i, h_i permute the set S_a. This is a contradiction, since f induces an odd permutation on S_a whilst each $[g_i, h_i]$ induces an even permutation. Hence $f \notin A(T)'$.

If T is a nice tree, κ a cardinal, and $z \in \bar{T}$ has cofinality (coinitiality) κ in $\bar{T}$ containing z, we say that z has *cofinality (coinitiality)* κ in $\bar{T}$, respectively. If these are equal, they are called the *coterminality* $\operatorname{cot}(z)$ of z in T.

THEOREM 3.5. *Let T be a weakly 2-transitive nice tree each of whose ramification points has countable coinitiality in $\bar{T}$ and precisely two cones. Then $A(T) = A(T)'$, and moreover each $f \in A(T)$ is a product of two commutators in $A(T)$.*

PROOF. By Theorem 2.4 we may assume that T has uncountable coinitiality. Choose $f \in A(T)$. Let C be a maximal chain in T. Since C has uncountable coinitiality, there is a sequence $(a_n : n \in \mathbb{N})$ of fixed points of f in $\bar{C}$, with $a_{n+1} < a_n$ for each $n \in \mathbb{N}$. Let $a := \inf\{a_n : n \in \mathbb{N}\}$. Then $a \in \bar{C}$ and $a^f = a < a_1 = a_1^f$.

For each $c \in \operatorname{ram}(T)$ with $c \leq a$ and $c^f = c$, let T_c be the cone at c not containing a_1. Then $T_c^f = T_c$ and T_c is a tree with countable coinitiality and weakly 2-transitive automorphism group. Hence by Theorem 2.4 there are automorphisms $g_{c,i}$ $(i = 1, 2, 3, 4)$ of T_c such that $f|_{T_c} = [g_{c,1}, g_{c,2}] \cdot [g_{c,3}, g_{c,4}]$.

If $a \in \operatorname{ram}(T)$, let T^* be the cone at a containing a_1, and if $a \notin \operatorname{ram}(T)$ let $T^* := \{x \in T : a < x\}$. In either case T^* is a weakly 2-transitive tree with countable coinitiality, and $T^{*f} = T^*$. Hence again there are automorphisms g_i^* $(i = 1, 2, 3, 4)$ of T^* with $f|_{T^*} = [g_1^*, g_2^*] \cdot [g_3^*, g_4^*]$. Finally, let S be any non-trivial orbital of f in T with $S < a$. By applying Lemma 3.2 to f and f^2, we obtain an automorphism g_S of S^{cl} with $f|_{S^{\mathrm{cl}}} = [f|_{S^{\mathrm{cl}}}, g_S]$ on S^{cl}. Put $g_{S,1} := f|_{S^{\mathrm{cl}}}$, $g_{S,2} := g_S$, $g_{S,3} := g_{S,4} :=$ identity on S^{cl}.

Now define $g_i \in A(T)$ by patching together the $g_{c,i}$, the g_i^* and the $g_{S,i}$ in the obvious way $(i = 1, \ldots, 4)$. Then $f = [g_1, g_2] \cdot [g_3, g_4]$. (Figure 2 below illustrates the action of f).

For Dedekind complete trees, these results can be combined as follows.

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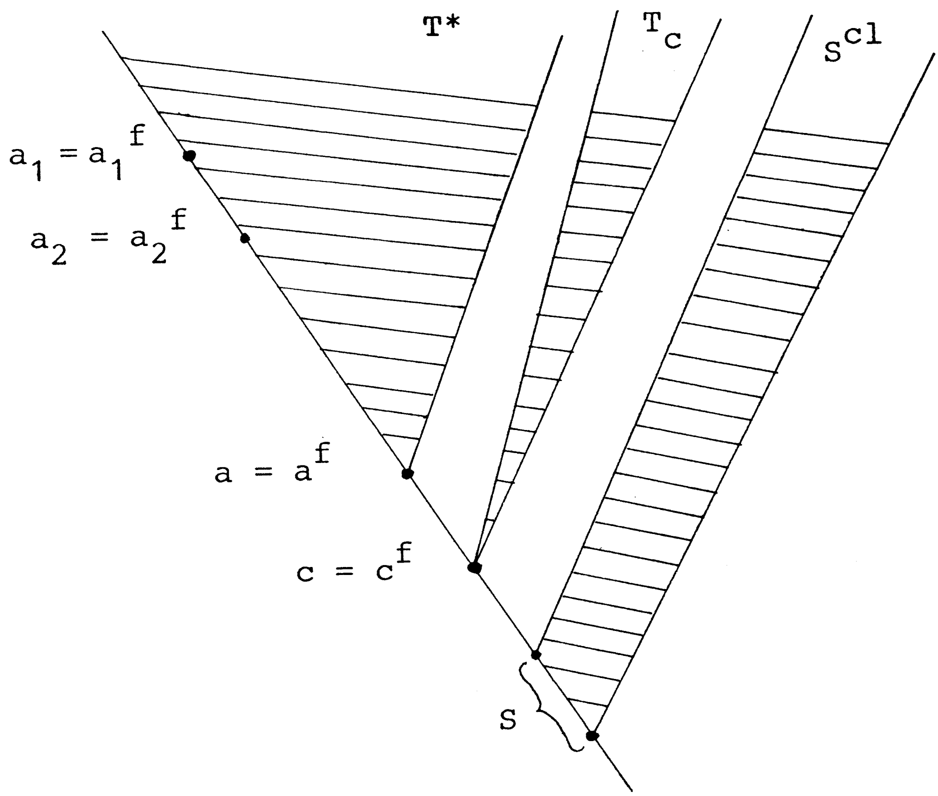

FIGURE 2

COROLLARY 3.6. *Let T be a weakly 2-transitive Dedekind-complete tree of finite ramification order. Then $A(T) = A(T)'$ if and only if either T has countable coinitiality or T has ramification order 2.*

PROOF. This is an easy consequence of Theorems 2.4, 3.4, and 3.5.

Examples of trees satisfying the assumptions of Corollary 3.6 are the trees $T_n(C, C)$ with $2 \leq n \in \mathbb{N}$, where C is a Dedekind-complete chain with transitive automorphism group [5, p. 59].

Next we wish to prove Theorem 1.3. It is easy to obtain trees T as required if $\kappa = \aleph_1$. For instance, let $\mathbb{L}$ be a rational long line constructed as follows. Let $(0, 1)$ be the open unit interval in $\mathbb{Q}$, and put $\mathbb{L} = \omega_1 \overrightarrow{\times} (0, 1)$, ordered lexicographically. Let $(C, \leq) = (\mathbb{L}, \leq)^*$, the reverse of $\mathbb{L}$. Thus C has cardinality and coinitiality $\aleph_1$, and each element of $\bar{C}$ has countable coterminality. Then the trees $T_n = T_n(C, C)$ with $2 \leq n \in \mathbb{N}$, as defined in [5; pp. 57–60], are doubly homogeneous and meet-semilattices (hence they are nice trees by Proposition 2.2), have cardinality and coinitiality $\aleph_1$, each ramification point of $\bar{T}_n$ has precisely n cones and each element of $\bar{T}_n$ has countable coterminality; now apply Theorems 3.4 and 3.5. To obtain Theorem 1.3 in its full generality we will use

LEMMA 3.7. *Let λ be any regular uncountable cardinal and $\kappa \geq \lambda$. Then there exists a doubly homogeneous chain $(C, \leq)$ of cardinality κ and coinitiality λ such that C is stationary below in $\bar{C}$ and each element of C has countable coterminality.*

PROOF. As shown in [6; Theorem 3.2], there exists a doubly homogeneous chain S of cardinality κ and cofinality λ such that there is a subset $R \subseteq \bar{S} \backslash S$ with $|R| = \kappa$, R is stationary above in $\bar{S}$, each $x \in R$ having countable coterminality in $\bar{S}$. Choose $x \in R$, and let U be the $A(S)$-orbit of x in $\bar{S}$. Then $R \subseteq U$, U is dense in $\bar{S}$, and $(U, \leq)$ is doubly homogeneous (cf. McCleary [13; p. 418]). As $|S| = \kappa$, we can choose a dense subset V of $\bar{S}$ with $R \subseteq V \subseteq U$ and $|V| = \kappa$. By the downward Löwenheim-Skolem Theorem there is a chain W such that $V \subseteq W \subseteq U$, $|W| = \kappa$, and $(W, \leq)$ is doubly homogeneous. In particular, $\bar{W} = \bar{U} = \bar{S}$ and hence W is stationary above in $\bar{W}$ and, as $x \in W$, each element of W has countable coterminality. Now let $(C, \leq)$ be the chain W with the reverse ordering.

THEOREM 3.8. *Let λ be any regular uncountable cardinal, $\kappa \geq \lambda$, and μ be an arbitrary cardinal with $2 \leq \mu \leq \kappa$. Then there exists a doubly homogeneous tree $(T, \leq)$ of cardinality κ with the following properties:*

(1) $(T, \leq)$ is a meet-semilattice and has coinitiality λ.

(2) T is stationary below in $\bar{T}$.

(3) Each element of T has countable coterminality in $\bar{T}$ and has precisely μ cones.

PROOF. Choose a doubly homogeneous chain $(C, \leq)$ with all the properties listed in Lemma 3.7, and put $T := T_\mu(C, C)$ (cf. [5; pp. 57–60]). The result follows.

Now the *proof of Theorem 1.3* is immediate by Theorems 3.8, 3.4 and 3.5, observing that by Proposition 2.2 any doubly homogeneous tree which is a meet-semilattice is nice.

Finally, we note that by using [6; Theorem 3.2] in its full generality for Lemma 3.7, it is possible to produce for Theorem 3.8, and hence also for Theorem 1.3, 2^κ trees T, each of cardinality κ and with all the required properties, with pairwise non-isomorphic Dedekind-completions, provided that κ is regular; if κ is singular, there are at least $2^{<\kappa} = \sum_{\lambda<\kappa} 2^\lambda$ such trees T.

The following problem remains open:

QUESTION. *If T is a weakly 2-transitive tree, must every non-identity element of $A(T)/A(T)'$ have order 2?*

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