Proceedings of the Edinburgh Mathematical Society (2014) **57**, 393–404
DOI:10.1017/S0013091513000473

THE COMPLEXITY OF THE LIE MODULE

KARIN ERDMANN[1], KAY JIN LIM[2] AND KAI MENG TAN[2]

[1] *Mathematical Institute, University of Oxford, Oxford OX1 3LB, UK* (erdmann@maths.ox.ac.uk)
[2] *Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076, Singapore* (limkj@ntu.edu.sg; tankm@nus.edu.sg)

(Received 3 October 2011)

Abstract We show that the complexity of the Lie module $\mathrm{Lie}(n)$ in characteristic p is bounded above by m, where p^m is the largest p-power dividing n, and, if n is not a p-power, is equal to the maximum of the complexities of $\mathrm{Lie}(p^i)$ for $1 \leqslant i \leqslant m$.

Keywords: Lie module; p-modular representation; symmetric group; wreath product; complexity

2010 *Mathematics subject classification:* Primary 20C30

1. Introduction

The Lie module of the symmetric group $\mathfrak{S}_n$ appears in many contexts; in particular, it is closely related to the free Lie algebra. Here, we take it to be the left ideal of $\mathbb{F}\mathfrak{S}_n$ generated by the Dynkin–Specht–Wever element

$$\omega_n = (1 - d_2)(1 - d_3) \cdots (1 - d_n),$$

where d_i is the i-cycle $(i, i-1, \ldots, 1)$ and we compose the elements of $\mathfrak{S}_n$ from right to left. We write that $\mathrm{Lie}(n) = \mathbb{F}\mathfrak{S}_n\omega_n$ for this module, and we assume that $\mathbb{F}$ is an algebraically closed field of characteristic p.

One motivation comes from the work of Selick and Wu [**13**]. They reduce the problem of finding natural homotopy decompositions of the loop suspension of a p-torsion suspension to an algebraic question, and in this context it is important to know a maximal projective submodule of $\mathrm{Lie}(n)$ when the field has characteristic p. The Lie module also occurs naturally as the homology of configuration spaces, and in other contexts. Moreover, the representation theory of symmetric groups over prime characteristic is difficult and many basic questions are open; naturally occurring representations are therefore of interest and may give new understanding.

In this paper we study homological invariants. More precisely, we provide upper bounds for the complexity of $\mathrm{Lie}(n)$. The complexity of a module may be defined to be the rate of growth of dimensions in its minimal projective resolution. A module for the group algebra of a finite group has a cohomological variety defined via group cohomology, now known as the support variety. Its dimension equals the complexity of the module. The computation of this variety can be reduced to the case of maximal elementary abelian p-subgroups. A module for an elementary abelian p-group also has a rank variety that, in principle, is very explicit, and the support variety is isomorphic to the rank variety. Details and references may be found in [**2**, Chapter 5]. Our results are obtained via this route, that is, we study the action of maximal elementary abelian p-groups on $\mathrm{Lie}(n)$.

A main result of [**5**] provides a decomposition theorem for the homogeneous parts $L^n(V)$ of the free Lie algebra on a vector space V over $\mathbb{F}$. It shows that its module structure, for arbitrary n, can be reduced to the cases when $n = p^m$ for some $m \geqslant 1$. We make use of this theorem. By work in [**10**], it may be transferred to the context of symmetric groups. Our main results (Theorems 3.2 and 3.3) show that the complexity of $\mathrm{Lie}(n)$ is bounded above by m, where p^m is the largest p-power dividing n, and, if n is not a p-power, is equal to the maximum of the complexities of $\mathrm{Lie}(p^i)$ with $1 \leqslant i \leqslant m$. We conjecture that our upper bound is in fact an equality, and show that this conjecture is equivalent to the assertion that the complexity of $\mathrm{Lie}(p^m)$ as an $\mathbb{F}\mathcal{E}_m$-module is m, where $\mathcal{E}_m$ is a regular elementary abelian subgroup of $\mathfrak{S}_{p^m}$ of order p^m.

Computer calculations in [**6**] suggest that the problem of determining the module structure of $\mathrm{Lie}(p^n)$ explicitly is very hard, but understanding its rank variety, and complexity, may help. These computations can be used to obtain the complexities of $\mathrm{Lie}(8)$ and $\mathrm{Lie}(9)$ in characteristic 2 and 3, respectively, and they provide some evidence in support of our conjecture.

The paper has the following structure. We give a summary of the background theory in § 2 and prove some preliminary results. These include a result (Proposition 2.7) on the complexity of certain modules for some wreath products in general. We prove the main results in § 3, and we conclude the paper with some examples in § 4.

2. Preliminaries

In this section, we provide the necessary background theory and prove some preliminary results.

Throughout, $\mathbb{F}$ denotes an algebraically closed field of characteristic p.

2.1. Complexities and cohomological varieties of modules

Let G be a finite group. Denote by V_G the affine variety defined by the maximum ideal spectrum of the cohomology ring $H^\bullet(G,\mathbb{F}) = \mathrm{Ext}^{\cdot}_{\mathbb{F}G}(\mathbb{F},\mathbb{F})$. Given a finitely generated $\mathbb{F}G$-module M, its cohomological variety $V_G(M)$ is defined to be the subvariety of V_G consisting of maximal ideals of $H^\bullet(G,\mathbb{F})$ containing the annihilator of $\mathrm{Ext}^*_{\mathbb{F}G}(M,M)$ (thus, $V_G(\mathbb{F}) = V_G$). The complexity of M, denoted by $c_G(M)$, is equal to the (Krull) dimension of $V_G(M)$.

Let H be a subgroup of G. We write $V_H(M)$ and $c_H(M)$ for the cohomological variety and complexity, respectively, of M as an $\mathbb{F}H$-module.

We collect together some results relating to complexities and varieties of modules that we require.

Theorem 2.1 (see [2, Theorem 5.1.1]). *Let G be a finite group, and let M be a finitely generated $\mathbb{F}G$-module.*

(1) $c_G(M) = 0$ *if and only if M is projective.*

(2) $c_G(M) = \max_E\{c_E(M)\}$, *where E runs over representatives of conjugacy classes of maximal elementary abelian p-subgroups of G.*

(3) *If H is a subgroup of G, then* $c_G(\mathrm{Ind}_H^G(M)) = c_H(M)$.

(4) *If N is another finitely generated $\mathbb{F}G$-module, then* $V_G(M \otimes_{\mathbb{F}} N) = V_G(M) \cap V_G(N)$ *and* $c_G(M \oplus N) = \max\{c_G(M), c_G(N)\}$.

2.2. Rank varieties of modules

Let E be an elementary abelian p-group, that is, E is isomorphic to $(C_p)^k$, and assume that E has generators $g_1, g_2, \dots, g_k$. Let M be a finitely generated $\mathbb{F}E$-module. For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{F}^k$, with $\alpha \neq 0$, let

$$u_\alpha = 1 + \sum_{i=1}^{k} \alpha_i(g_i - 1) \in \mathbb{F}E.$$

Then $(u_\alpha)^p = 1$. Write $\langle u_\alpha \rangle$ for the cyclic group of order p generated by u_α. Then the group algebra $\mathbb{F}\langle u_\alpha \rangle$ is a subalgebra of $\mathbb{F}E$.

Let M be a finitely generated $\mathbb{F}E$-module. The rank variety $V_E^{\#}(M)$ of M is defined as

$$V_E^{\#}(M) = \{\alpha \in \mathbb{F}^k \mid \alpha \neq 0,\ M \text{ is non-projective as an } \mathbb{F}\langle u_\alpha \rangle\text{-module}\} \cup \{0\}.$$

This is an affine subvariety of $\mathbb{F}^k$ and is independent of the choice and order of the generators (in the sense that two varieties obtained using different choices of generators are isomorphic). More importantly, we have the following.

Theorem 2.2 (see [2, Theorem 5.8.3]). *Let $E = (C_p)^k$, and let M be a finitely generated $\mathbb{F}E$-module. Then $V_E(M)$ and $V_E^{\#}(M)$ are isomorphic as affine varieties. In particular, $c_E(M) \leqslant k$.*

Lemma 2.3. *Let $E = E_1 \times E_2$ be an elementary abelian p-group. Suppose that M is a finitely generated $\mathbb{F}E$-module such that M is projective as an $\mathbb{F}E_1$-module. Then $c_{E_1 \times E_2}(M) \leqslant s$, where $E_2 \cong (C_p)^s$.*

Proof. Let $E_1 \cong (C_p)^r$, and choose generators $g_1, \dots, g_{r+s}$ for E such that $g_1, \dots, g_r \in E_1$ and $g_{r+1}, \dots, g_{r+s} \in E_2$. Embedding $\mathbb{F}^r$ into $\mathbb{F}^{r+s}$ in the obvious way, we have that

$$\mathbb{F}^r \cap V_E^{\#}(M) = V_{E_1}^{\#}(M).$$

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

Thus,

$$0 = c_{E_1}(M) = \dim(\mathbb{F}^r \cap V_E^{\#}(M)) \geqslant r + c_E(M) - (r+s)$$

by [**9**, Chapter I, Proposition 7.1], so that $c_E(M) \leqslant s$. □

2.3. Symmetric groups

Let $n \in \mathbb{Z}^+$. Denote by $\mathfrak{S}_n$ the symmetric group on n letters. We identify $\mathfrak{S}_n$ with the permutation group on $\{1, 2, \ldots, n\}$, and we compose the elements in $\mathfrak{S}_n$ from right to left. For $m \in \mathbb{Z}^+$, with $m \leqslant n$, we view $\mathfrak{S}_m$ as the subgroup of $\mathfrak{S}_n$ fixing $\{m+1, m+2, \ldots, n\}$ pointwise.

Let $r, s \in \mathbb{Z}^+$. For $1 \leqslant i \leqslant s$ and $\sigma \in \mathfrak{S}_r$, write $\sigma[i] \in \mathfrak{S}_{rs}$ for the permutation sending $(i-1)r + j$ to $(i-1)r + \sigma(j)$, for each $1 \leqslant j \leqslant r$, and fixing everything else pointwise. Also, let

$$\Delta_s \sigma = \prod_{i=1}^{s} \sigma[i].$$

If H is a subgroup of $\mathfrak{S}_r$, let $H[i] = \{\sigma[i] \mid \sigma \in H\}$. For $\tau \in \mathfrak{S}_s$, write $\tau^{[r]} \in \mathfrak{S}_{rs}$ for the permutation sending $(i-1)r + j$ to $(\tau(i)-1)r + j$ for each $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant r$. If K is a subgroup of $\mathfrak{S}_s$, let $K^{[r]} = \{\tau^{[r]} \mid \tau \in K\}$.

Let $a_p = (1, 2, \ldots, p) \in \mathfrak{S}_p$. For $r \in \mathbb{Z}^+$, let

$$\mathcal{E}_r = \langle \Delta_{p^{r-1}} a_p, \Delta_{p^{r-2}} a_p^{[p]}, \ldots, a_p^{[p^{r-1}]} \rangle \subseteq \mathfrak{S}_{p^r}.$$

This is an elementary abelian p-subgroup of $\mathfrak{S}_{p^r}$ isomorphic to $(C_p)^r$. These $\mathcal{E}_r$ are the building blocks of distinguished representatives of the conjugacy classes of maximal elementary abelian p-subgroups of $\mathfrak{S}_n$.

Theorem 2.4 (see [1, Chapter VI, Theorem 1.3]). *Let $n \in \mathbb{Z}^+$, and let $k = \lfloor n/p \rfloor$. Every maximal elementary abelian p-subgroup of $\mathfrak{S}_n$ is conjugate to one of the following form:*

$$\prod_{j=1}^{m} \mathcal{E}_{r_j}[s_j/p^{r_j}],$$

where $(r_1, r_2, \ldots, r_m)$ is a decreasing sequence of positive integers such that

$$\sum_{i=1}^{m} p^{r_i} = pk \quad \text{and} \quad s_j = \sum_{i=1}^{j} p^{r_i}.$$

Remark 2.5. The support of each factor $\mathcal{E}_{r_j}[s_j/p^{r_j}]$ is $\{s_{j-1}+1, s_{j-1}+2, \ldots, s_j\}$, so that these factors have disjoint support.

2.4. Wreath products

Let G be a finite group, and let $n \in \mathbb{Z}^+$. The wreath product $G \wr \mathfrak{S}_n$ has the underlying set $\{(g_1, \ldots, g_n)\sigma \mid g_1, \ldots, g_n \in G,\ \sigma \in \mathfrak{S}_n\}$, and it is the group with group composition defined by

$$((g_1, \ldots, g_n)\sigma) \cdot ((g_1', \ldots, g_n')\tau) = (g_1 g'_{\sigma^{-1}(1)}, \ldots, g_n g'_{\sigma^{-1}(n)})(\sigma\tau).$$

We identify $\mathfrak{S}_n$ with the subgroup $\{(1, \ldots, 1)\sigma \mid \sigma \in \mathfrak{S}_n\}$ of $G \wr \mathfrak{S}_n$.

Let M be a finitely generated (non-zero) left $\mathbb{F}G$-module. Then $M^{\otimes n}$ admits a natural left $\mathbb{F}(G \wr \mathfrak{S}_n)$-action via

$$((g_1, \ldots, g_n)\sigma) \cdot (m_1 \otimes \cdots \otimes m_n) = (g_1 m_{\sigma^{-1}(1)}) \otimes \cdots \otimes (g_n m_{\sigma^{-1}(n)}),$$
$$g_1, \ldots, g_n \in G,\ \sigma \in \mathfrak{S}_n,\ m_1, \ldots, m_n \in M.$$

Suppose that

$$M = \bigoplus_{i \in I} M(i)$$

is a decomposition of M as $\mathbb{F}G$-modules. For each $\boldsymbol{i} = (i_1, \ldots, i_n) \in I^n$, write $M(\boldsymbol{i})$ for the subset $M(i_1) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} M(i_n)$ of $M^{\otimes n}$, so that

$$M^{\otimes n} = \bigoplus_{\boldsymbol{i} \in I^n} M(\boldsymbol{i}).$$

For each $A \subseteq I^n$, write $M(A)$ for $\bigoplus_{\boldsymbol{i} \in A} M(\boldsymbol{i})$ (thus, $M(I^n) = M^{\otimes n}$). The set I^n admits a natural left action of $\mathfrak{S}_n$ via place permutation, i.e. $\sigma \cdot (i_1, \ldots, i_n) = (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(n)})$. We note that the $\mathbb{F}(G \wr \mathfrak{S}_n)$-action on $M^{\otimes n}$ satisfies that $\sigma(M(\boldsymbol{i})) = M(\sigma \cdot \boldsymbol{i})$ for all $\sigma \in \mathfrak{S}_n$. Thus, if $\mathcal{O}$ is an $\mathfrak{S}_n$-orbit of I^n, then $M(\mathcal{O})$ is an $\mathbb{F}(G \wr \mathfrak{S}_n)$-submodule of $M^{\otimes n}$, and

$$M^{\otimes n} = \bigoplus_{\mathcal{O}} M(\mathcal{O}),$$

where $\mathcal{O}$ runs over all $\mathfrak{S}_n$-orbits of I^n.

Suppose further that G is a subgroup of a finite group K. Let $N = \mathrm{Ind}_G^K M$ and, for each $i \in I$, let $N(i) = \mathrm{Ind}_G^K M(i)$. Then $N = \bigoplus_{i \in I} N(i)$ and, using notation analogous to that introduced above, we see that

$$N^{\otimes n} = \bigoplus_{\mathcal{O}} N(\mathcal{O}),$$

where $\mathcal{O}$ runs over all $\mathfrak{S}_n$-orbits of I^n, is a decomposition of $N^{\otimes n}$ as $\mathbb{F}(K \wr \mathfrak{S}_n)$-modules. In addition, for each $\mathfrak{S}_n$-orbit $\mathcal{O}$ of I^n, we have the following.

Lemma 2.6.

$$N(\mathcal{O}) \cong \mathrm{Ind}_{G \wr \mathfrak{S}_n}^{K \wr \mathfrak{S}_n} M(\mathcal{O}).$$

Proof. Let T be a set of left coset representatives of G in K. Then T^n is a set of left coset representatives of $G \wr \mathfrak{S}_n$ in $K \wr \mathfrak{S}_n$. The reader may check that the map $(t_1 \otimes v_1) \otimes \cdots \otimes (t_n \otimes v_n) \mapsto (t_1, \ldots, t_n) \otimes (v_1 \otimes \cdots \otimes v_n)$ for $t_1, \ldots, t_n \in T$, $v_1 \in M(i_1)$, $v_2 \in M(i_2)$, …, $v_n \in M(i_n)$, with $(i_1, \ldots, i_n) \in \mathcal{O}$, gives the required isomorphism. $\square$

Proposition 2.7. *Let G be an abelian p'-subgroup of a finite group K. Let M be a non-zero $\mathbb{F}G$-module, and let $N = \mathrm{Ind}_G^K M$. Let $n \in \mathbb{Z}^+$, and let S be an $\mathbb{F}\mathfrak{S}_n$-module, so that S becomes an $\mathbb{F}(K \wr \mathfrak{S}_n)$-module via inflation. Then*

$$c_{K \wr \mathfrak{S}_n}(N^{\otimes n} \otimes_{\mathbb{F}} S) = c_{\mathfrak{S}_n}(S).$$

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

Proof. Since $\mathrm{char}(\mathbb{F}) = p \nmid |G|$, we see that M is completely reducible, so that

$$M = \bigoplus_{i \in I} M(i),$$

where, since G is abelian, each $M(i)$ is one dimensional and $I \neq \varnothing$. Let $N(i) = \mathrm{Ind}_G^K M(i)$ for each $i \in I$, so that $N = \bigoplus_{i \in I} N(i)$. We have, by Lemma 2.6, that

$$N^{\otimes n} = \bigoplus_{\mathcal{O}} N(\mathcal{O}) \cong \bigoplus_{\mathcal{O}} \mathrm{Ind}_{G \wr \mathfrak{S}_n}^{K \wr \mathfrak{S}_n} M(\mathcal{O}),$$

where the sum runs over all $\mathfrak{S}_n$-orbits $\mathcal{O}$ of I^n, so that

$$N^{\otimes n} \otimes_{\mathbb{F}} S \cong \left(\bigoplus_{\mathcal{O}} \mathrm{Ind}_{G \wr \mathfrak{S}_n}^{K \wr \mathfrak{S}_n} M(\mathcal{O}) \right) \otimes_{\mathbb{F}} S \cong \bigoplus_{\mathcal{O}} (\mathrm{Ind}_{G \wr \mathfrak{S}_n}^{K \wr \mathfrak{S}_n} (M(\mathcal{O}) \otimes_{\mathbb{F}} S)).$$

Thus, $c_{K \wr \mathfrak{S}_n}(N^{\otimes n} \otimes_{\mathbb{F}} S) = \max_{\mathcal{O}}\{c_{G \wr \mathfrak{S}_n}(M(\mathcal{O}) \otimes_{\mathbb{F}} S)\}$ by Theorem 2.1 (3) and (4). Since $p \nmid |G|$, we may pick representatives of the conjugacy classes of maximal elementary abelian p-subgroups of $G \wr \mathfrak{S}_n$ to be subgroups of $\mathfrak{S}_n$. For each such representative E,

$$V_E(M(\mathcal{O}) \otimes_{\mathbb{F}} S) = V_E(M(\mathcal{O})) \cap V_E(S) \subseteq V_E(S)$$

by Theorem 2.1 (4), so $c_E(M(\mathcal{O}) \otimes_{\mathbb{F}} S) \leqslant c_E(S) \leqslant c_{\mathfrak{S}_n}(S)$. Thus,

$$c_{G \wr \mathfrak{S}_n}(M(\mathcal{O}) \otimes_{\mathbb{F}} S) = \max_E \{c_E(M(\mathcal{O}) \otimes_{\mathbb{F}} S)\} \leqslant c_{\mathfrak{S}_n}(S)$$

for all $\mathfrak{S}_n$-orbits $\mathcal{O}$ of I^n. On the other hand, if $i \in I$, then $\mathcal{O}_i = \{(i, \ldots, i)\}$ is a singleton $\mathfrak{S}_n$-orbit of I^n. Consequently, $M(\mathcal{O}_i) = (M(i))^{\otimes n}$ is one dimensional, on which $\mathfrak{S}_n$ acts trivially. Thus,

$$V_E(M(\mathcal{O}_i) \otimes_{\mathbb{F}} S) = V_E(M(\mathcal{O}_i)) \cap V_E(S) = V_E \cap V_E(S) = V_E(S).$$

This implies that $c_{G \wr \mathfrak{S}_n}(M(\mathcal{O}_i) \otimes_{\mathbb{F}} S) = c_{\mathfrak{S}_n}(S)$, and hence

$$c_{K \wr \mathfrak{S}_n}(N^{\otimes n} \otimes_{\mathbb{F}} S) = \max_{\mathcal{O}} \{c_{G \wr \mathfrak{S}_n}(M(\mathcal{O}) \otimes_{\mathbb{F}} S)\} = c_{\mathfrak{S}_n}(S).$$

□

2.5. Lie module

Denote by $\mathrm{Lie}(n)$ the Lie module for the symmetric group $\mathfrak{S}_n$. This is the left ideal of $\mathbb{F}\mathfrak{S}_n$ generated by the Dynkin–Specht–Wever element

$$\omega_n = (1 - d_2)(1 - d_3) \cdots (1 - d_n),$$

where d_i is the descending i-cycle $(i, i-1, \ldots, 1)$ of $\mathfrak{S}_n$. (Recall that we compose the elements of $\mathfrak{S}_n$ from right to left.)

The following lemma about $\mathrm{Lie}(n)$ is well known, but we are unable to find an appropriate reference in the existing literature.

Lemma 2.8. *As an $\mathbb{F}\mathfrak{S}_{n-1}$-module,* $\mathrm{Lie}(n)$ *is free of rank* 1*.*

Proof. It is well known that $(\omega_n)^2 = n\omega_n$, and $\dim_{\mathbb{F}}(\mathrm{Lie}(n)) = (n-1)!$ (see, for example, [**12**, Theorem 8.16], and [**11**, Theorem 5.11] with $n_1 = n_2 = \cdots = 1$). We first claim that $\omega_n = -\omega_{r-1}d_r\omega_n$ whenever $2 \leqslant r \leqslant n$ (note that $\omega_1 = 1$ by definition). To prove this, we have that

$$\omega_r = \omega_{r-1}(1-d_r) = \omega_{r-1} - \omega_{r-1}d_r,$$
$$\omega_s\omega_n = \omega_s\omega_s(1-d_{s+1})\cdots(1-d_n) = s\omega_s(1-d_{s+1})\cdots(1-d_n) = s\omega_n$$

for all $1 \leqslant s \leqslant n$. Thus,

$$(1+\omega_{r-1}d_r)\omega_n = (1+\omega_{r-1}-\omega_r)\omega_n = \omega_n + (r-1)\omega_n - r\omega_n = 0,$$

proving the claim.

Now, if $\rho \in \mathfrak{S}_n$ such that $\rho(1) \neq 1$, say $\rho(r) = 1$, then $\rho\omega_n = -\rho\omega_{r-1}d_r\omega_n$, and $-\rho\omega_{r-1}d_r \in \mathbb{F}\mathfrak{S}_{n,1}$, where $\mathfrak{S}_{n,1} = \{\sigma \in \mathfrak{S}_n \mid \sigma(1) = 1\}$, so that $\rho\omega_n \in \mathbb{F}\mathfrak{S}_{n,1}\omega_n$ for all $\rho \in \mathfrak{S}_n$. Thus, the (obviously linear) map $\psi\colon \mathbb{F}\mathfrak{S}_{n,1} \to \mathrm{Lie}(n)$ defined by $x \mapsto x\omega_n$ is surjective, and hence bijective by dimension count. Define $\phi\colon \mathbb{F}\mathfrak{S}_{n-1} \to \mathrm{Lie}(n)$ by $y \to y(1,n)\omega_n$. Then ϕ is clearly an $\mathbb{F}\mathfrak{S}_{n-1}$-module homomorphism. In addition, it is injective since ψ is, and is therefore bijective by dimension count. $\square$

2.6. Tensor powers and Lie powers

Let $n, r, s \in \mathbb{Z}^+$. Let V be a finite-dimensional vector space over $\mathbb{F}$. If V is a left module for the Schur algebra $S_{\mathbb{F}}(n,r)$, then the tensor power $V^{\otimes s}$ is naturally a left $S_{\mathbb{F}}(n,rs)$-module. In addition, $V^{\otimes s}$ admits a commuting right action of $\mathfrak{S}_s$ by place permutation. The Lie power $L^s(V)$ of V may be defined as $(V^{\otimes s})\omega_s$, where ω_s is the Dynkin–Specht–Wever element mentioned in the last subsection; this is a left $S_{\mathbb{F}}(n,rs)$-submodule of $V^{\otimes s}$.

If $\dim_{\mathbb{F}}(V) = n$, then V is naturally a left $S_{\mathbb{F}}(n,1)$-module. Therefore, $V^{\otimes s}$ is a $(S_{\mathbb{F}}(n,s), \mathbb{F}\mathfrak{S}_s)$-bimodule, while $L^s(V)$ is a left $S_{\mathbb{F}}(n,s)$-submodule of $V^{\otimes s}$. When $n \geqslant s$, the Schur functor f_s sends $V^{\otimes s}$ to the $(\mathbb{F}\mathfrak{S}_s, \mathbb{F}\mathfrak{S}_s)$-bimodule $\mathbb{F}\mathfrak{S}_s$, and it sends $L^s(V)$ to the left $\mathbb{F}\mathfrak{S}_s$-module $\mathrm{Lie}(s)$. The effect of the Schur functor f_{rs} on $V^{\otimes s}$ and $L^s(V)$, when V is a left $S_{\mathbb{F}}(n,r)$-module and $n \geqslant rs$, is described in detail in [**10**]. In this paper, we need the latter result.

Theorem 2.9 (see [10, Corollary 3]). *Let $n, r, s \in \mathbb{Z}^+$, with $n \geqslant rs$. Let V be an $S_{\mathbb{F}}(n,r)$-module. Then*

$$f_{rs}L^s(V) \cong \mathrm{Ind}_{\mathfrak{S}_r \wr \mathfrak{S}_s}^{\mathfrak{S}_{rs}}((f_r(V))^{\otimes s} \otimes_{\mathbb{F}} \mathrm{Lie}(s)),$$

where $\mathfrak{S}_r \wr \mathfrak{S}_s$ is identified with the subgroup $(\prod_{i=1}^s \mathfrak{S}_r[i])\mathfrak{S}_s^{[r]}$ of $\mathfrak{S}_{rs}$, and it acts on $(f_r(V))^{\otimes s}$ and $\mathrm{Lie}(s)$ via

$$((\sigma_1,\dots,\sigma_s)\tau)\cdot(x_1 \otimes \cdots \otimes x_s) = (\sigma_1 x_{\tau^{-1}(1)}) \otimes \cdots \otimes (\sigma_s x_{\tau^{-1}(s)}),$$
$$((\sigma_1,\dots,\sigma_s)\tau)\cdot y = \tau y$$

for all $\sigma_1,\dots,\sigma_s \in \mathfrak{S}_r$, $\tau \in \mathfrak{S}_s$, $x_1,\dots,x_s \in f_r(V)$ and $y \in \mathrm{Lie}(s)$.

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

Bryant and Schocker proved a remarkable decomposition theorem for the Lie powers.

Theorem 2.10 (see [5, Theorem 4.4]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$, and let V be an n-dimensional vector space over $\mathbb{F}$. For each $r \in \mathbb{Z}_{\geqslant 0}$, there exists $B_{p^r k}(V) \subseteq L^{p^r k}(V)$ such that $B_{p^r k}(V)$ is a direct summand of $V^{\otimes p^r k}$ as $S_{\mathbb{F}}(n, p^r k)$-modules, and*

$$L^{p^m k}(V) = L^{p^m}(B_k(V)) \oplus L^{p^{m-1}}(B_{pk}(V)) \oplus \cdots \oplus L^1(B_{p^m k}(V))$$

for all $m \in \mathbb{Z}_{\geqslant 0}$.

We note that if $k > 1$ and $n \geqslant p^m k$, then $B_{p^i k}(V)$ is non-zero for $0 \leqslant i \leqslant m$; this is implicit in [**5**].

The $S_{\mathbb{F}}(n, p^m k)$-submodules $B_{p^m k}(V)$ of $L^{p^m k}(V)$ are further studied in [**3**] and [**4**]. In particular, they give the following description for $B_{p^m k}(V)$. As mentioned at the beginning of this subsection, $\mathfrak{S}_k$ acts on $V^{\otimes k}$ from the right by place permutation. Let $a_k = (1, 2, \ldots, k) \in \mathfrak{S}_k$. For each kth root of unity δ in $\mathbb{F}$ (which is algebraically closed, with characteristic p coprime to k), let $(V^{\otimes k})_\delta$ denote the a_k-eigenspace of $V^{\otimes k}$ with eigenvalue δ.

Theorem 2.11 (see [4, Theorem 2.6]). *Let $m, k \in \mathbb{Z}_{\geqslant 0}$, with $k > 1$ and $p \nmid k$, and let V be an n-dimensional vector space, where $n \geqslant p^m k$. Then*

$$B_{p^m k}(V) \cong \bigoplus_{(\delta_1, \ldots, \delta_{p^m}) \in \Omega} (V^{\otimes k})_{\delta_1} \otimes \cdots \otimes (V^{\otimes k})_{\delta_{p^m}}$$

for some (fixed) non-empty subset Ω of the set of p^m-tuples of kth roots of unity.

We note that $(V^{\otimes k})_\delta \cong V^{\otimes k} \otimes_{\mathbb{F}\langle a_k \rangle} \mathbb{F}_\delta$ as left $S_{\mathbb{F}}(n, k)$-modules, where $\mathbb{F}_\delta$ denotes the one-dimensional left $\mathbb{F}\langle a_k \rangle$-module, in which a_k acts via multiplication by the scalar δ.

Corollary 2.12. *Keep the notation in Theorem 2.11. Then*

$$f_{p^m k}(B_{p^m k}(V)) \cong \bigoplus_{(\delta_1, \ldots, \delta_{p^m}) \in \Omega} \operatorname{Ind}_{\langle a_k \rangle^{p^m}}^{\mathfrak{S}_{p^m k}} \left(\bigotimes_{j=1}^{p^m} \mathbb{F}_{\delta_j} \right),$$

where $\langle a_k \rangle^{p^m}$ is identified with the subgroup $\prod_{j=1}^{p^m} \langle a_k \rangle [j]$ of $\mathfrak{S}_{p^m k}$.

In particular, $f_{p^m k}(B_{p^m k}(V))$ is a non-zero $\mathbb{F}\mathfrak{S}_{p^m k}$-module induced from $\langle a_k \rangle^{p^m}$.

Proof. By Theorem 2.11 and [**7**, § 2.5, Lemma], we have that

$$\begin{aligned} f_{p^m k}(B_{p^m k}(V)) &\cong \bigoplus_{(\delta_1, \ldots, \delta_{p^m}) \in \Omega} \operatorname{Ind}_{(\mathfrak{S}_k)^{p^m}}^{\mathfrak{S}_{p^m k}} \left(\bigotimes_{j=1}^{p^m} f_k(V^{\otimes k} \otimes_{\mathbb{F}\langle a_k \rangle} \mathbb{F}_{\delta_j}) \right) \\ &\cong \bigoplus_{(\delta_1, \ldots, \delta_{p^m}) \in \Omega} \operatorname{Ind}_{(\mathfrak{S}_k)^{p^m}}^{\mathfrak{S}_{p^m k}} \left(\bigotimes_{j=1}^{p^m} \mathbb{F}\mathfrak{S}_k \otimes_{\mathbb{F}\langle a_k \rangle} \mathbb{F}_{\delta_j} \right) \end{aligned}$$

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

$$= \bigoplus_{(\delta_1,\dots,\delta_{p^m})\in\Omega} \mathrm{Ind}_{(\mathfrak{S}_k)^{p^m}}^{\mathfrak{S}_{p^m k}} \left(\bigotimes_{j=1}^{p^m} \mathrm{Ind}_{\langle a_k\rangle}^{\mathfrak{S}_k} \mathbb{F}_{\delta_j} \right)$$
$$= \bigoplus_{(\delta_1,\dots,\delta_{p^m})\in\Omega} \mathrm{Ind}_{\langle a_k\rangle^{p^m}}^{\mathfrak{S}_{p^m k}} \left(\bigotimes_{j=1}^{p^m} \mathbb{F}_{\delta_j} \right).$$

□

3. Main results

In this section, we prove the main results of this paper.

Lemma 3.1. *Let $n \in \mathbb{Z}^+$, and let $(r_1, r_2, \dots, r_t)$ be a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{t} p^{r_i} = p\lfloor n/p\rfloor$. For each $j = 1, \dots, t$, let $s_j = \sum_{i=1}^{j} p^{r_i}$. Let*

$$E = \prod_{j=1}^{t} \mathcal{E}_{r_j}[s_j/p^{r_j}].$$

Then

$$c_E(\mathrm{Lie}(n)) \begin{cases} = 0, & \text{if } p\nmid n, \\ \leqslant r_t, & \text{if } p\mid n. \end{cases}$$

Proof. If $p\nmid n$, then $E \subseteq \mathfrak{S}_{n-1}$. Since $\mathrm{Lie}(n)$ is free of rank 1 as an $\mathbb{F}\mathfrak{S}_{n-1}$-module by Lemma 2.8, we see that $\mathrm{Lie}(n)$ is projective as an $\mathbb{F}E$-module. Thus, $c_E(\mathrm{Lie}(n)) = 0$ by Theorem 2.1 (1).

If $p\mid n$, let

$$E' = \prod_{j=1}^{t-1} \mathcal{E}_{r_j}[s_j/p^{r_j}],$$

so $E = E' \times \mathcal{E}_{r_t}[n/p^{r_t}]$. Since $E' \subseteq \mathfrak{S}_{n-1}$, we see, as before, that $\mathrm{Lie}(n)$ is projective as an $\mathbb{F}E'$-module, so that $c_{E'}(\mathrm{Lie}(n)) = 0$. Thus, $c_E(\mathrm{Lie}(n)) \leqslant r_t$ by Lemma 2.3. □

Theorem 3.2. *We have that $c_{\mathfrak{S}_n}(\mathrm{Lie}(n)) \leqslant m$, where $p^m\mid n$ and $p^{m+1}\nmid n$.*

Proof. By Theorems 2.1 (2) and 2.4, it suffices to show that $c_E(\mathrm{Lie}(n)) \leqslant m$ for all E of the form

$$E = \prod_{j=0}^{t} \mathcal{E}_{r_j}[s_j/p^{r_j}],$$

where $(r_1, r_2, \dots, r_t)$ is a weakly decreasing sequence of positive integers such that

$$\sum_{i=1}^{t} p^{r_i} = p\lfloor n/p\rfloor \quad \text{and} \quad s_j = \sum_{i=1}^{j} p^{r_i}.$$

If $p\nmid n$ (i.e. $m = 0$), then $c_E(\mathrm{Lie}(n)) = 0 = m$ by Lemma 3.1.

If $p\mid n$, then $c_E(\mathrm{Lie}(n)) \leqslant r_t$ by Lemma 3.1. Since $\sum_{i=1}^{t} p^{r_i} = n$ and $(r_1, \dots, r_t)$ is weakly decreasing, we see that $p^{r_t}\mid n$. Thus, $r_t \leqslant m$. □

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

Theorem 3.3. *Let $m, k \in \mathbb{Z}^+$, with $p \nmid k$ and $k > 1$. Then*

$$c_{\mathfrak{S}_{p^m k}}(\mathrm{Lie}(p^m k)) = \max\{c_{\mathfrak{S}_{p^i}}(\mathrm{Lie}(p^i)) \mid 1 \leqslant i \leqslant m\}.$$

Proof. By applying the exact and direct-sum-preserving Schur functor $f_{p^m k}$ to Theorem 2.10, we obtain, by Theorem 2.9, that

$$\mathrm{Lie}(p^m k) \cong \bigoplus_{i=0}^{m} \mathrm{Ind}_{\mathfrak{S}_{p^{m-i}k} \wr \mathfrak{S}_{p^i}}^{\mathfrak{S}_{p^m k}}((f_{p^{m-i}k}(B_{p^{m-i}k}(V)))^{\otimes p^i} \otimes_{\mathbb{F}} \mathrm{Lie}(p^i)).$$

For each $0 \leqslant i \leqslant m$, by Proposition 2.7 with

$$M = \bigoplus_{(\delta_1,\dots,\delta_{p^{m-i}}) \in \Omega} \bigotimes_{j=1}^{p^{m-i}} \mathbb{F}_{\delta_j},$$

$G = \langle a_k \rangle^{p^{m-i}}$, $K = \mathfrak{S}_{p^{m-i}k}$, so that, by Corollary 2.12,

$$N = \mathrm{Ind}_G^K M = f_{p^{m-i}k}(B_{p^{m-i}k}(V)),$$

and, with $n = p^i$ and $S = \mathrm{Lie}(p^i)$, we have that

$$c_{\mathfrak{S}_{p^{m-i}k} \wr \mathfrak{S}_{p^i}}((f_{p^{m-i}k}(B_{p^{m-i}k}(V)))^{\otimes p^i} \otimes_{\mathbb{F}} \mathrm{Lie}(p^i)) = c_{\mathfrak{S}_{p^i}}(\mathrm{Lie}(p^i)).$$

Applying Theorem 2.1 (3) and (4) completes the proof. □

We conjecture that the inequality in Theorem 3.2 is in fact an equality. This assertion is equivalent to the following statements.

Corollary 3.4. *The following statements are equivalent.*

(1) *For all $n \in \mathbb{Z}^+$, $c_{\mathfrak{S}_n}(\mathrm{Lie}(n)) = m$ where $p^m | n$ and $p^{m+1} \nmid n$.*

(2) *For all $m \in \mathbb{Z}^+$, $c_{\mathfrak{S}_{p^m}}(\mathrm{Lie}(p^m)) = m$.*

(3) *For all $m \in \mathbb{Z}^+$, $c_{\mathcal{E}_m}(\mathrm{Lie}(p^m)) = m$.*

(4) *For all $m \in \mathbb{Z}^+$, $V_{\mathcal{E}_m}^{\#}(\mathrm{Lie}(p^m)) = \mathbb{F}^m$.*

Proof. Parts (1) and (2) of are equivalent by Theorem 3.3, while the equivalence of (2) and (3) follows from Theorem 2.1 (2) and Lemma 3.1. That (3) and (4) are equivalent is trivial. □

4. Some examples

We end the paper with the computation of the complexity of some $\mathrm{Lie}(n)$. This provides some evidence in support of our conjecture that the inequality in Theorem 3.2 is in fact an equality.

https://doi.org/10.1017/S0013091513000473 Published online by Cambridge University Press

4.1. The case $n = pk$ where p does not divide k

By [**8**], any non-projective summand of $\mathrm{Lie}(n)$ has a vertex of order p and is, therefore, periodic as a module for $\mathbb{F}\mathfrak{S}_n$. Furthermore, such a summand always exists, and hence $c_{\mathfrak{S}_n}(\mathrm{Lie}(n)) = 1$ in this case.

4.2. The case $n = 2^m$ with $m = 2, 3$ and $p = 2$

When $n = 8$, the results in [**6**], which were obtained with the help of computer calculations, can be used to find the complexity. Recall that any finite-dimensional module M is a direct sum $M = M^{\mathrm{pf}} \oplus M^{\mathrm{pr}}$ where M^{pr} is projective and M^{pf} does not have a non-zero projective summand. Clearly, $c(M) = c(M^{\mathrm{pf}})$. The projective-free part $\mathrm{Lie}(8)^{\mathrm{pf}}$ of $\mathrm{Lie}(8)$ is indecomposable, with the regular elementary abelian subgroup $\mathcal{E}_3$ of order 8 as its vertex, and a source of dimension 21.

Generally, if the projective-free part M^{pf} is indecomposable and has an elementary abelian vertex E of order p^m, then $V_E^{\#}(M) = V_E^{\#}(S)$, where S is a source of M in E. If p does not divide $\dim_{\mathbb{F}}(S)$, then, clearly, $V_E^{\#}(S) = \mathbb{F}^m$. Hence, $\mathrm{Lie}(8)$ has complexity 3.

When $n = 4$, it is easy to see that $\mathrm{Lie}(4)$ is isomorphic to $\Omega^{-1}(D)$, where D is the two-dimensional simple module of $\mathbb{F}\mathfrak{S}_4$. This has vertex the regular elementary abelian subgroup $\mathcal{E}_2$ of order 4, and its source has dimension 3. Hence, the same argument as for $n = 8$ implies that $c_{\mathfrak{S}_4}(\mathrm{Lie}(4)) = 2$.

4.3. The case $n = 9$ and $p = 3$

In this case, there are similar results by [**6**]. Again, $\mathrm{Lie}(9)^{\mathrm{pf}}$ is indecomposable, with a vertex the regular elementary abelian subgroup $\mathcal{E}_2$ of order 9, and a source of dimension 16. The above argument can still be applied to get that $c_{\mathfrak{S}_9}(\mathrm{Lie}(9)) = 2$.

Acknowledgements. Supported by the EPSRC (Grant EP/G025487/1) and the Singapore Ministry of Education Academic Research Fund (Grant R-146-000-135-112).

References

1. A. Adem and R. J. Milgram, *Cohomology of finite groups* (2nd edn), Grundlehren der Mathematischen Wissenschaften, Volume 309 (Springer, 2004).
2. D. J. Benson, *Representations and cohomology, II, Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics, Volume 31 (Cambridge University Press, 1991).
3. R. M. Bryant and M. Johnson, Lie powers and Witt vectors, *J. Algebraic Combin.* **28** (2008), 169–187.
4. R. M. Bryant and M. Johnson, A modular version of Klyachko's theorem on Lie representations of the general linear group, *Math. Proc. Camb. Phil. Soc.* **153** (2012), 79–98.
5. R. M. Bryant and M. Schocker, The decomposition of Lie powers, *Proc. Lond. Math. Soc.* **93** (2006), 175–196.
6. R. M. Bryant, K. Erdmann, S. Danz and J. Müller, Vertices of Lie modules, preprint (2013).
7. S. Donkin and K. Erdmann, Tilting modules, symmetric functions, and the module structure of the free Lie algebra, *J. Alg.* **203** (1998), 69–90.

8. K. Erdmann and M. Schocker, Modular Lie powers and the Solomon descent algebra, *Math. Z.* **253** (2006), 295–313.
9. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Volume 52 (Springer, 1977).
10. K. J. Lim and K. M. Tan, The Schur functor on tensor powers, *Arch. Math.* **96** (2012), 513–518.
11. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (2nd edn) (Dover, New York, 1976).
12. C. Reutenauer, *Free Lie algebras* (Clarendon Press, Oxford, 1993).
13. P. Selick and J. Wu, *On natural coalgebra decomposition of tensor algebras and loop suspensions*, Memoirs of the American Mathematical Society, Volume 701 (American Mathematical Society, Providence, RI, 2000).