Canad. Math. Bull. Vol. 25 (3), 1982

FREE SUMMANDS OF STABLY FREE MODULES

BY
ROBERT W. SWIFT

ABSTRACT. In this note we show that the generic orthogonal stably free modules of type (2, 7) and (3, 8) have one free summand. This completes the work of other authors on free summands of orthogonal stably free modules.

Let $P^0_{m,n}(R)$ be the generic module for orthogonal stably free R-modules of type (m, n). Let $\rho(M)$ denote the maximal number of free summands of a module M. Let $\rho(n)=8c+2^d$ where $n=2^ab$, with b odd, and $a=4c+d$ with $0\le d<4$, [4].

THEOREM. *Let R be any ring which admits a ring map to the reals. Then*

$$\rho(P^0_{m,n}(R))=\begin{cases}\rho(n)-1 & \text{if} \quad m=1\\ 1 & \text{if} \quad m=n-1 \text{ or } (m,n)=(2,7) \text{ or } (3,8)\\ 0 & \text{otherwise.}\end{cases}$$

Thus any orthogonal stably free module of type (m, n) has at least the indicated number of free summands and this bound is the best possible.

Most of this theorem is known. The case $m=n-1$ is trivial and in [3] and [4] the authors independently showed $\rho(P^0_{1,n}(R))\ge\rho(n)-1$. Well-known topological results (see Adams [1] and James [5]) give the appropriate upper bounds for $\rho(P^0_{m,n}(R))$. Thus all that remains to show to complete a proof of the theorem is $\rho(P^0_{2,7}(R))\ge 1$ and $\rho(P^0_{3,8}(R))\ge 1$ and that is the purpose of this note.

We will now define $P^0_{m,n}(R)$ for a commutative ring R. Let X_{ij}, $1\le i\le m$, $1\le j\le n$, be a collection of commuting indeterminates over R. Write in matrix form as $X=(X_{ij})$. Consider the polynomial ring $R[X]=R[\cdots X_{ij}\cdots]$ and the ideal generated by the elements of the m by m matrix XX^t-1. We denote the ideal by (XX^t-1) and define

$$R^0_{m,n}=\frac{R[X]}{(XX^t-1)}.$$

Received by the editors February 27, 1980 and in revised form December 15, 1980 and March 18, 1981

AMS Classification No: 13C10.

Financial support for the author was given by the Natural Sciences and Engineering Research Council of Canada in the form of a postgraduate scholarship.

https://doi.org/10.4153/CMB-1982-051-6 Published online by Cambridge University Press

Let x denote the image X in $R^0_{m,n}$. Thinking of x as a matrix, it defines a map from $(R^0_{m,n})^n$ to $(R^0_{m,n})^m$. Define $P^0_{m,n}(R)=\ker x$.

A section to the natural map $R^0_{m,n}\to R^0_{m+k,n}$ is equivalent to being able to complete the m by n matrix x to an orthogonal $m+k$ by n matrix over $R^0_{m,n}$. Such a section guarantees that $\rho(P^0_{m,n}(R))\geq k$ since a criterion for $P^0_{m,n}(R)$ to have a free summand of rank k is that the matrix x can be completed to an $m+k$ by n matrix which is right invertible; see Gabel [2]. We will do this for $(m,n)=(2,7)$ and $(3,8)$ with $k=1$.

We consider each row of x as an element of the 8-dimensional Cayley algebra over $R^0_{3,8}$. Let $a=(x_{11},x_{12},\ldots,x_{18})$, $b=(x_{21},x_{22},\ldots,x_{28})$, $\bar{b}=(x_{21},-x_{22},\ldots,-x_{28})$ and $c=(x_{31},x_{32},\ldots,x_{38})$. Let the new row be the Cayley product $a(\bar{b}c)$. By some brute force calculations one verifies that the resulting matrix is orthogonal:

$$a\cdot a(\bar{b}c)=|a|^2(b\cdot c)=0$$

$$b\cdot a(\bar{b}c)=-|b|^2(a\cdot c)+2(a\cdot b)(b\cdot c)=0$$

$$c\cdot a(\bar{b}c)=|c|^2(a\cdot b)=0$$

$$|a(\bar{b}c)|^2=|a|^2\,|b|^2\,|c|^2=1$$

Note that in $R^0_{3,8}$ we have $a\cdot b=b\cdot c=a\cdot c=0$ and $|a|^2=|b|^2=|c|^2=1$.

An easier construction works for the case $(m,n)=(2,7)$. We define a new row for x by taking the last seven coordinates of the Cayley product of $a=(0,x_{11},\ldots,x_{17})$ and $b=(0,x_{21},\ldots,x_{27})$. The resulting 3 by 7 matrix is orthogonal over $R^0_{2,7}$. The motivation for this comes from Zvengrowski's paper [6].

References

1. J. F. Adams, *Vector Fields on Spheres*, Annals of Math. **75** (1962), 603–632.

2. M. R. Gabel, *Stably Free Projectives over Commutative Rings*, Ph.D. Dissertation, Brandeis University, 1972.

3. M. R. Gabel, *Generic Orthogonal Stable Free Projectives*, Journal of Algebra **29** (1974), 477–488.

4. A. V. Geramita and N. J. Pullman, *A Theorem of Hurwitz and Radon and Orthogonal Projective Modules*, Proc. Amer. Math. **42** Number 1 (1974), 51–56.

5. I. M. James, *The Topology of Stiefel Manifolds*, London Math. Soc. Lecture Notes 24, Cambridge Press, Cambridge, 1976.

6. P. Zvengrowski, *A 3-fold Vector Product in* R^8, Comment. Math. Helv. **40** (1966), 149–152.

Department of Mathematics
University of Western Ontario
London, Ontario N6A 5B7

https://doi.org/10.4153/CMB-1982-051-6 Published online by Cambridge University Press