

ON A CONJECTURE OF J.S. FRAME

NOBORU ITO*

To Professor Kiyoshi Noshiro on His Sixtieth Birthday

Let \mathcal{G} be a transitive group of degree n , and let \mathcal{G}_1 be the stabilizer of a symbol in \mathcal{G} . Then we owe to J.S. Frame the following remarkable relations between the lengths n_i of the orbits of \mathcal{G}_1 and the degrees f_i of the absolutely irreducible components of the permutation matrix representation \mathcal{G}^* of \mathcal{G} :

(A) If the irreducible constituents of \mathcal{G}^* are all different, then the rational number

$$F = n^{k-2} \prod_{i=1}^k n_i / f_i$$

is an integer, where k is the number of the orbits of \mathcal{G}_1 .

(C) If the irreducible constituents of \mathcal{G}^* all have rational characters, then F is a square.

Further J.S. Frame made the following conjecture ([1]):

(B) If the k numbers n_i are all different, then F is a square.

(B) is true for $k \leq 3$ ([3], §30).

Now the purpose of this short note is to show that (B) is not true in general for $k=4$.

Let $LF_r(q)$ be the r -dimensional projective special linear group over the field of q elements such that $p = \frac{q^r - 1}{q - 1}$ is a prime and r is odd. Let $V_r(q)$ and $W_r(q)$ be the r -dimensional spaces of column and row vectors over the field of q elements, respectively. Let V and W be the set of one-dimensional subspaces of $V_r(q)$ and $W_r(q)$, respectively. $\langle \begin{smallmatrix} x_1 \\ \vdots \\ x_r \end{smallmatrix} \rangle \in V$ and $\langle y_1, \dots, y_r \rangle \in W$ denote the one-dimensional subspaces generated by $\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in V_r(q)$ and (y_1, \dots, y_r)

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$\in W_r(q)$, respectively.

$LF_r(q)$ can be considered in a natural manner as permutation groups on V and also on W . The number of elements in V and W are equal to $p = (q^r - 1)/(q - 1)$, and two conjugacy classes of subgroups of $LF_r(q)$ of index p correspond to the stabilizers of the symbols of V and W respectively. Let \mathfrak{A} and \mathfrak{B} be the stabilizers of $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $\langle 1, 0, \dots, 0 \rangle$ in $LF_r(q)$ respectively. Let \mathfrak{B}

be a Sylow p -subgroup of $LF_r(q)$ and let P be a generating element of \mathfrak{B} . Then $P^{-i} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ ($i = 0, \dots, p - 1$) and $\langle 1, 0, \dots, 0 \rangle P^j$ ($j = 0, \dots, p - 1$) are all

different. Making $P^{-i} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $\langle 1, 0, \dots, 0 \rangle P^j$ correspond to $\mathfrak{A}P^i$ and $P^{-j}\mathfrak{B}$,

respectively, let us denote the permutation representations of $LF_r(q)$ over V and W by A and B respectively.

\mathfrak{A} consists of the matrices of the form $\begin{pmatrix} 0 \\ * \\ \vdots \\ 0 \\ \rho \end{pmatrix}$, $\rho \in GF(q)$, whose determinants

are equal to 1. $B(\mathfrak{A})$ has two orbits, namely, $D = \{ \langle \dots, 0 \rangle \}$ and $W - D$, whose lengths are equal to $k = \frac{q^{r-1} - 1}{q - 1}$ and $p - k$, respectively. Thus the conjugacy class of B in G is divided into two A -classes, each of which contain k and $p - k$ subgroups, respectively.

$LF_r(q)$ admits an involutory automorphism τ such that $X^\tau = (X^t)^{-1}$ for every element X of $LF_r(q)$, where t denotes the transpose operation.

It is easy to see that \mathfrak{A}^τ is conjugate to \mathfrak{B} . In fact, \mathfrak{A}^τ is the stabilizer of $\langle 0, \dots, 0, 1 \rangle$ in $LF_r(q)$. Let \mathfrak{G} be the split extension of $LF_r(q)$ by τ . We notice that $A\tau X = AX^\tau\tau$ for every element X of $LF_r(q)$.

Let us consider the following permutation representation of \mathfrak{G} by the subgroup \mathfrak{A} :

$$X \rightarrow \begin{pmatrix} \mathfrak{A}, & \mathfrak{A}P, & \dots, & \mathfrak{A}\tau, & \mathfrak{A}\tau P, & \dots \\ \mathfrak{A}X, & \mathfrak{A}PX, & \dots, & \mathfrak{A}\tau X, & \mathfrak{A}\tau PX, & \dots \end{pmatrix}$$

Then it is easy to see that the lengths of the orbits of \mathfrak{A} are equal to 1, $p - 1$, k and $p - k$. Thus the permutation representation decomposes into four absolutely irreducible components. Then it is quite easy to see that their

degrees are equal to 1, 1, $p-1$ and $p-1$. Now $F = (2p)^4(p-1)k(p-k)/(p-1)^2 = (2p)^4q^{r-2}$ is not a square, since q is a non-square and r is odd.

REFERENCES

- [1] J.S. Frame, The degrees of the irreducible components of simply transitive permutation groups, *Duke Math. J.* 3, 8–17 (1937).
- [2] J.S. Frame, The double cosets of a finite group, *Bull. Amer. Math. Soc.* 47, 458–467 (1941).
- [3] H. Wielandt, *Finite Permutation Groups*, Academic Press (1964).

Nagoya University and University of Illinois Chicago, Illinois