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# A NOTE ON POLYHEDRAL CONES

BIT-SHUN TAM

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**Abstract**

In this short note, two results on a solid, pointed, closed cone $C$ in $R^n$ will be given: first, $C$ is polyhedral iff it has a finite number of maximal faces; second, for any face $F$ of $C$, $C^* \cap F^\perp$ is a face of its dual cone $C^*$ of dimension $n - \dim F$.

## 1. Introduction

Many interesting properties of a cone in a Euclidean space are readily suggested by geometric intuition. For example, it is reasonable to guess that for a cone to be polyhedral a necessary and sufficient condition is that it has a finite number of maximal faces. Although this is true, the proof is by no means trivial. It is a purpose of this short note to give a simple proof of this characterization of polyhedral cones. Another result that will be proved is: for any face $F$ of a polyhedral cone $C$ in $R^n$, $C^* \cap F^\perp$ is a face of its dual cone $C^*$ of dimension $n - \dim F$. Although this result is known (cf. Stoer and Witzgall (1970), p. 70, (2. 13. 3)), it is included here because the proof is interesting.

## 2. Notation and preliminaries

A familiarity with elementary results on cones will be assumed (see, for instance, Barker (1973)). For convenience and clarity, we review some of the definitions and results.

A subset $C$ of the Euclidean space $R^n$ is called a *cone* if

(i) $C$ is a non-empty, closed subset of $R^n$,

(ii) $C + C \subset C$,

(iii) $\alpha C \subset C$ for all $\alpha \geqq 0$,

(iv) $C \cap (-C) = \{0\}$,

(v) $C - C = R^n$.

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In other words, the 'cone' considered here is what usually referred to as 'solid, pointed, closed cone'.

A subset $F$ of $R^n$ is called a *face* of $C$ if it is contained in $\partial C$, satisfies (i), (ii), (iii) above, together with the following condition: if $x, y \in C$ such that $x + y \in F$, then $x, y \in F$. If $F$ is not properly contained in any other face of $C$, then it is said to be maximal. By the *dimension of $F$*, we mean $\dim L(F)$, where $L(F)$ is the subspace spanned by $F$.

For any $x \in \partial C$, denote by $\Phi(x)$ the set $\{y \in C: x - \alpha y \in C$ for some $\alpha > 0\}$. It can be shown that $\Phi(x)$ is a face; in fact, for any face $F$ of $C$, any $x \in \partial C$, $x$ belongs to the relative interior of $F$ iff $\Phi(x) = F$.

A vector $x \in C$ is called extremal iff $x = y + z$ with $y, z \in C$ implies that both $y, z$ are non-negative multiples of $x$. In this case, $x \in \partial C$ and $\Phi(x) = \{\lambda x: \lambda \geqq 0\}$.

A cone $C$ is said to be *generated by a set of vectors* if each vector in $C$ can be written as a finite non-negative linear combination of these vectors. It can be shown that any cone in $R^n$ is generated by its extremal vectors. A cone is called *polyhedral* iff it has a finite number of extremal vectors.

For any non-empty subset $S$ of $R^n$, the polar $S^*$ of $S$ is the set $\{z \in R^n: (z, y) \geqq 0$ for all $y \in S\}$. The following result is well-known: for any non-empty subset $S$ which is closed under addition and multiplication by non-negative scalars, $\mathrm{cl} S = S^{**}$.

The polar $C^*$ of a cone $C$ is a cone, known as the *dual cone* of $C$.

## 3. A characterization of polyhedral cones

To begin with, we prove the following result for general cones.

THEOREM 1. *Let $C$ be a cone in $R^n$. For any $x \in \partial C$*

$$\dim \Phi(x) < n - 1$$

*iff every neighbourhood of $x$ meets at least two maximal faces of $C$.*

PROOF. "If Part": Suppose $\dim \Phi(x) = n - 1$. Choose any $w \in \operatorname{int} C$. Then the set $\{\alpha w + y: \alpha \geqq 0, y \in$ relative boundary of $\Phi(x)\}$ is closed, and for every $y \in \Phi(x)$, $\alpha > 0$, $\alpha w + y \in \operatorname{int} C$. To prove the former assertion, let $(\alpha_i w + y_i)_{i \in \mathbf{N}}$, with $\alpha_i \geqq 0$, $y_i \in$ relative boundary of $\Phi(x)$, be a convergent sequence. Clearly, it is sufficient to establish the convergence of $(\alpha_i)_{i \in \mathbf{N}}$. Take a non-zero $z \in \partial C^*$ such that $z \perp \Phi(x)$. Then the continuity of the inner product guarantees the convergence of the sequence $((\alpha_i w + y_i, z))_{i \in \mathbf{N}} = (\alpha_i (w, z))_{i \in \mathbf{N}}$, and hence that of $(\alpha_i)_{i \in \mathbf{N}}$ because $(w, z) \neq 0$. Now $x$ is a point outside the above closed set. If $r > 0$ is sufficiently small, then the open set $\{y \in R^n: d(x, y) < r\}$ meets only one maximal face of $C$, namely, $\Phi(x)$.

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"Only If Part": Suppose $\dim \Phi(x) < n - 1$. Let $G$ be a maximal face containing $\Phi(x)$, and let $z \neq 0 \in C^*$ be orthogonal to $G$. Consider a convex neighbourhood $U$ of $x$. Clearly, there exists $w \in (U \cap \{z\}^{\perp}) \backslash C$. Take any $y \in \operatorname{int} C \cap U$. Then for some $\lambda \in R$, $0 < \lambda < 1$, $\lambda y + (1 - \lambda)w \in \partial C \cap U$. But any maximal face which contains $\lambda y + (1 - \lambda)w$ must be different from $G$.

COROLLARY. *Let $C$ be a cone in $R^n$. If $C$ has a face of dimension less than $n - 1$ contained in only one maximal face, then $C$ has infinitely many maximal faces.*

PROOF. Let $\Phi(x)$ be a face of dimension less than $n - 1$ contained in only one maximal face, say $H$. By the theorem, every neighbourhood of $x$ meets a maximal face $C$ other than $H$. However, any such face is at positive distance from $x$, and hence the corollary follows.

We now come to our first main result on polyhedral cones.

THEOREM 2. *Let $C$ be a cone in $R^n$. $C$ is polyhedral iff $C$ has a finite number of maximal faces. If the condition is satisfied, then every maximal face of $C$ is a polyhedral cone of dimension $n - 1$.*

PROOF. Since every face is generated by its extremal vectors, the necessity part of the theorem is obvious. We now prove the sufficiency part, together with the last statement, by induction on $n$.

Let $C$ be a cone in $R^n$ with a finite number of maximal faces. From the corollary of Theorem 1, each maximal face of $C$ must be of dimension $n - 1$. The proof is complete if each maximal face is polyhedral. Assume to the contrary that $C$ has a non-polyhedral maximal face, say $H$. By induction, $H$ as a cone of dimension $n - 1$ has infinitely many maximal faces. However, each maximal face of $H$ is a face of $C$ of dimension less than $n - 1$, and so from the corollary of Theorem 1, each of them belongs to at least one maximal face of $C$ other than $H$. But two different maximal faces of $H$ cannot belong to the same maximal face of $C$ except $H$. Consequently, $C$ has infinitely many maximal faces, which is a contradiction.

## 4. A relation between the dimensions of $F$ and $C^* \cap F^{\perp}$

Our second result on polyhedral cones is:

THEOREM 3. *Let $C$ be a polyhedral cone in $R^n$. For any face $F$ of $C$ of dimension $r$, the largest face of $C^*$ orthogonal to $F$, i.e. $C^* \cap F^{\perp}$, is of dimension $n - r$.*

We need the following lemma whose proof is easy enough to be omitted (cf. McMullen and Shephard (1971), p. 70, Theorem 15).

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LEMMA. *Let $C$ be a cone in $R^n$. For any subspace $H$ of $R^n$, $C^* \cap H = p[C]^\#$ where $p$ is the orthogonal projection of $R^n$ onto $H$, and $p[C]^\#$ is the polar of $p[C]$ in $H$.*

PROOF OF THEOREM 3. We proceed by induction on $n$. It can be assumed that $\dim F \leqq n-2$. Let $H$ be a maximal face of $C$ containing $F$. Then $H$ is a polyhedral cone of the $n-1$ dimensional subspace $L(H)$ and $F$ is a face of $H$. Denote by $p$ the orthogonal projection of $R^n$ onto $L(H)$, and by $H^\#$ the dual cone of $H$ (which is considered as a cone in $L(H)$). From the lemma, $H = C \cap L(H) = p[C^*]^\#$, and hence $H^\# = p[C^*]^{\#\#} = \mathrm{cl}\, p[C^*] = p[C^*]$. By induction, the largest face of $H^\#$ orthogonal to $F$ is of dimension $n-1-r$. It follows that there exist $z_1, \cdots, z_{n-r-1} \in C^* \backslash H^\perp$ orthogonal to $F$ such that their projections on $L(H)$ form an independent family. But we can also find a non-zero $z_{n-r} \in C^* \cap H^\perp$. Therefore, the dimension of $C^* \cap F^\perp$ is $n-r$.

We omit the detailed proof of the following simple corollaries:

COROLLARIES.

**1.** Every polyhedral cone in $R^n$ has at least $n$ maximal faces.

**2.** For each face $F$ of a polyhedral cone $C$, there exists $z \in C^*$ such that $F = C \cap \{z\}^\perp$.

**3.** Let $C$ be a polyhedral cone in $R^n$, and let $y \neq 0 \in \partial C$. Then $y$ is an extremal vector of $C$ iff $C^* \cap \{y\}^\perp$ is a maximal face of $C^*$.

**4.** Let $C$ be a polyhedral cone in $R^n$. If $F$ is a face of $C$ of dimension $r$, then there exist at least $n-r$ maximal faces of $C$ containing $F$.

## 5. Remarks

**1.** Theorem 2 implies the following theorem on convex sets: A closed and bounded convex set in $R^n$ is a polytope iff it has a finite number of maximal faces.

(Our definition of 'face' is different from that of McMullen and Shephard (1971), p. 39. A subset $F$ of a convex set $K$ is called a face of $K$ if $F$ is closed, convex and satisfies: $\lambda x + (1-\lambda)y \in F$, $x, y \in K$, $0 \leqq \lambda \leqq 1$ implies $x, y \in F$. For polytopes the two definitions coincide.)

The proof depends on the following easily-proved result:

Let $K$ be a closed and bounded convex set with non-empty interior in $R^n$. Let $C = \{\alpha(y, 1) \in R^{n+1}: \alpha \geqq 0,\ y \in K\}$. Then $C$ is a cone in $R^{n+1}$. Furthermore, we have,

(i) $(y, 1) \in \partial C$ iff $y \in \partial K$.

(ii) $(y, 1)$ is an extremal vector of $C$ iff $y$ is an extreme point of $K$.

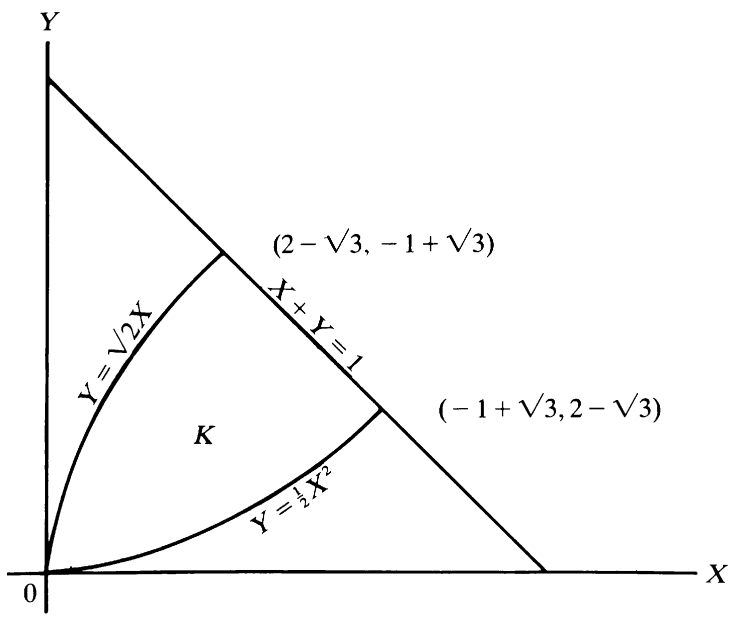
(iii) Each face of $C$ is of the form $\{\alpha(y,1) \in R^{n+1}: \alpha \geqq 0, y \in F\}$ for some face $F$ of $K$.

**2.** Theorem 2 also suggests a simple proof of the following well-known result (cf. Stoer and Witzgall (1970), p. 56).

THEOREM OF WEYL. *The dual cone $C^*$ of a polyhedral cone $C$ is polyhedral.*

Suppose $C^*$ is not polyhedral. Then by Theorem 2, it has an infinite number of maximal faces. But for any two distinct maximal faces $G$, $H$ of $C^*$, $C \cap G^\perp$, $C \cap H^\perp$ are distinct faces of $C$ because $(C \cap G^\perp) \cap (C \cap H^\perp) = 0$. It follows that $C$ has infinitely many faces, and hence is non-polyhedral.

**3.** It is difficult, if not impossible, to prove Theorem 2 by the Theorem of Weyl. The difficulties are, maybe for some maximal face of $C$, there are infinitely many extremal vectors of $C^*$ orthogonal to it; or it may happen that there exist extremal vectors of $C^*$ not orthogonal to any maximal faces of $C$. The example below shows that, unlike polyhedral cones, in general, a cone $C$ does not possess the following properties: if $F$ is a maximal face of $C$, then $F^\perp \cap C^*$ is an extreme ray of $C^*$; if $y$ is an extremal vector of $C$, then $\Phi(y)^\perp \cap C^*$ is a maximal face of $C^*$.

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EXAMPLE. Let $C$ be the cone in $R^3$ defined by: $C = \{\alpha(x, y, 1) \in R^3: \alpha \geqq 0, (x, y) \in K\}$ where $K$ is the compact convex set in $R^2$ bounded by the curves $y = \frac{1}{2}x^2$, $y = \sqrt{2x}$ and $x + y = 1$. Now, $\Phi(0,0,1)$ is a maximal face of $C$, but $\Phi(0,0,1)^\perp \cap C^* = \Phi(1,0,0) \vee \Phi(0,1,0)$, not an extreme ray of $C^*$. Also, in $C^*$ there are two extremal vectors orthogonal to the vector $(2-\sqrt{3}, -1+\sqrt{3}, 1)$ of $\partial C$: one of them is normal to the subspace spanned by $(2-\sqrt{3}, -1+\sqrt{3}, 1)$ and $(-1+\sqrt{3}, 2-\sqrt{3}, 1)$; the other is the vector $((-1+\sqrt{3})^{-1}, -1, (-1+\sqrt{3})/2)$, normal to the surface $x = uv$, $y = v\sqrt{2u}$, $z = v$ at $(2-\sqrt{3}, -1+\sqrt{3}, 1)$. Here, even though $((-1+\sqrt{3})^{-1}, -1, (-1+\sqrt{3})/2)$ is an extremal vector of $C^*$, $\Phi((-1+\sqrt{3})^{-1}, -1, (-1+\sqrt{3})/2)^\perp \cap C = \Phi(2-\sqrt{3}, -1+\sqrt{3}, 1)$ is not a maximal face of $C$.

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Department of Mathematics,
University of Hong Kong,
Hong Kong.