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PRIME IDEALS IN GCD-DOMAINS

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1. Introduction. A GCD-*domain* is a commutative integral domain in which each pair of elements has a greatest common divisor (g.c.d.). (This is the terminology of Kaplansky [**9**]. Bourbaki uses the term "anneau pseudo-bezoutien" [**3**, p. 86], while Cohn refers to such rings as "HCF-rings" [**4**].) The concept of a GCD-domain provides a useful generalization of that of a unique factorization domain (UFD), since several of the standard results for a UFD can be proved in this more general setting (for example, integral closure, some properties of $D[X]$, etc.). Since the class of GCD-domains contains all of the Bezout domains, and in particular, the valuation rings, it is clear that some of the properties of a UFD do not hold in general in a GCD-domain. Among these are complete integral closure, ascending chain condition on principal ideals, and some of the important properties of minimal prime ideals.

The purpose of this paper is to investigate the properties of the prime ideal structure of GCD-domains in general. The investigation focuses upon the so-called PF-prime ideals (Definition 2.1), which seem to play much the same role in a GCD-domain that the principal primes play in a UFD. Moreover, in a UFD the PF-primes and the principal primes coincide (Remark 2.5), and thus many of the results obtained here concerning PF-primes generalize results which are well-known for a UFD. Bezout domains, too, are important in these considerations, since they can be characterized among GCD-domains by the fact that every proper prime is a PF-prime (Remark 2.5). Other sufficient conditions for a GCD-domain to be a Bezout domain are found, yielding, in the special case of a UFD, conditions known to be sufficient for a UFD to be a principal ideal domain (PID).

Section 2 is devoted to the definition and some important properties of PF-prime ideals. In section 3, the main result is that every proper prime ideal of a GCD-domain is a union of PF-prime ideals (Theorem 3.1); most of the results in the rest of the section follow, directly or indirectly, from this fact. In section 4 we consider the question of lengths of chains of PF-primes of D in relation to the Krull dimension of D. Finally, in section 5 we present some interpretations of these results in terms of semivaluations and the divisibility group.

Throughout this paper, the word "domain" will always mean a commutative integral domain with identity. A "prime ideal" of D cannot be equal to D

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itself, and a "proper" prime ideal is a nonzero prime ideal. In most other cases, our notation and terminology will be that of Gilmer [**6**].

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2. PF-prime ideals: definition and properties.

2.1 *Definition.* Let D be a GCD-domain. The prime ideal P of D is a *PF-prime ideal* if whenever a and b are nonzero elements of P, $\gcd(a, b)$ is in P as well. (For an explanation of the choice of the term "PF-prime", see section 5.)

Among the examples of PF-prime ideals of D are all principal primes of D. In fact, it is clear from the definition that a finitely generated prime ideal is a PF-prime if and only if it is a principal prime.

For additional examples and other important properties of PF-primes, the following characterization will be very useful:

2.2 THEOREM. *Let D be a GCD-domain and P be a prime ideal of D. Then P is a PF-prime if and only if D_P is a valuation ring.* (For convenience, we shall refer to any prime ideal P for which D_P is a valuation ring as an *essential* prime ideal.)

Proof. Assume P is a PF-prime ideal. Pick a nonzero element of the quotient field of D, and write it as a/b, where $a, b \in D$ and $\gcd(a, b) = 1$. Then since $1 \notin P$, either a or $b \notin P$, and hence either a/b or b/a is in D_P. Thus D_P is a valuation ring.

Now suppose P is not a PF-prime. Since $P \neq (0)$, we may choose $a, b \in P - \{0\}$ such that $\gcd(a, b) \notin P$. Letting d denote $\gcd(a, b)$ and a' and b' denote a/d and b/d respectively, it is clear that $a', b' \in P$ and $\gcd(a', b') = 1$. Then if the fraction a'/b' were in D_P, there would have to be an element s in $D - P$ such that $sa' \in b'D$. But $\gcd(a', b') = 1$ implies that b' must divide s, and hence that s is in P, a contradiction.

If b'/a' were in D_P, a similar contradiction is reached. Hence D_P is not a valuation ring.

2.3 COROLLARY. *Every prime ideal contained in a PF-prime ideal is again a PF-prime ideal.*

2.4 COROLLARY. *The set of prime ideals contained in any PF-prime ideal is linearly ordered, and hence the set of all PF-prime ideals forms a tree, that is, a partially-ordered set in which no two unrelated elements have a common upper bound.*

Proof. Both of these corollaries follow from the fact that if P is a PF-prime and $P' \subset P$, then $D_{P'}$, an overring of the valuation ring D_P, is itself a valuation ring. Since the set of overrings of a valuation ring is linearly ordered, it follows that the set of prime ideals inside P is linearly ordered.

2.5 *Remark.* As mentioned earlier, in any GCD-domain a principal prime ideal is always a PF-prime. On the other hand, it is well-known that in a UFD the essential prime ideals are precisely the principal primes, and hence in a UFD, the set of PF-primes and the set of principal primes coincide. In the case of a Bezout domain, every prime ideal is a PF-prime; and, in fact, the Bezout domains can be characterized among the GCD-domains by this property. This is a consequence of the well-known result that a GCD-domain is a Bezout domain if and only if it is a Prüfer domain.

3. Every prime ideal is a union of PF-prime ideals. While the results of the preceding section give some of the special properties of the PF-prime ideals in a GCD-domain D, they give little indication of the relation of the sub-structure of the PF-primes to the overall prime ideal structure of D. In fact, in case D is neither a Bezout domain nor a UFD, there is no obvious reason that any of the proper prime ideals of D will be PF-primes. In this section we consider questions such as these. We begin with the statement of the main result of this section.

3.1 THEOREM. *In a GCD-domain, every prime ideal is a union of PF-prime ideals.*

Proof. We will call an ideal of the GCD-domain D a *t-ideal* if for all nonzero x, y in A, $\gcd(x, y)$ is in A. (This terminology is that used by Jaffard [**8**, pp. 18–19] and others, applied to the specific case where D is a GCD-domain.) We want to show that if S is a multiplicative system in D and A is maximal in the set of all t-ideals disjoint from S, then A is prime in D, and hence a PF-prime. Let B and C be ideals properly containing A, and define

$$\bar{B} = \{x \in D | x = 0 \text{ or } x \text{ is the gcd of a finite subset of } B - \{0\}\}.$$

(In Jaffard's terminology, $\bar{B}$ is the t-ideal generated by B.) The fact that $\bar{B}$ is an ideal follows from the fact that $d \cdot \gcd(b_1, \ldots, b_n) = \gcd(db_1, \ldots, db_n)$, and the fact that $\gcd(b_1, \ldots, b_n) + \gcd(b_1', \ldots, b_m')$ is in the ideal generated by $\gcd(b_1, \ldots, b_n, b_1', \ldots, b_m')$. We define $\bar{C}$ analogously. By the maximality of A, $\bar{B}$ and $\bar{C}$ meet S. Choosing $s, t \in S$ such that $s = \gcd(b_1, \ldots, b_n)$ and $t = \gcd(c_1, \ldots, c_m)$, with each $b_i \in B - \{0\}$ and each $c_j \in C - \{0\}$, we see that $st = \gcd(\{b_i c_j\})$. Since $st \in S$ and since A is a t-ideal, some $b_i c_j$ is not in A, and hence $BC \not\subseteq A$. Thus A is a prime ideal.

To prove the theorem, we let P be a prime ideal of D, and let d be an element of P. Then dD is a t-ideal which does not meet $D - P$, and since unions of chains of t-ideals are again t-ideals, there must be an ideal A containing d which is maximal among the t-ideals not meeting $D - P$. By the argument above A is a PF-prime, which we know contains d and is contained in P.

Since our choice of P and d was arbitrary, we conclude that every prime is a union of PF-primes.

3.2 COROLLARY. *In a GCD-domain, every prime ideal contains a PF-prime, every nonunit is in a PF-prime, and every minimal prime ideal is a PF-prime.*

In the case where D is a UFD, we obtain from the preceding theorem and corollary the well-known results that every prime ideal is a union of principal prime ideals, every nonunit is in a principal prime ideal, and every minimal prime ideal is principal. These all follow from the fact mentioned in Remark 2.5 that a UFD has the property that every PF-prime is principal. Next we prove a partial converse to this observation, showing that this property characterizes the unique factorization domains among the GCD-domains.

3.3 THEOREM. *A GCD-domain is a UFD if and only if every PF-prime is principal.*

Proof. We need only consider the "if" part. If every PF-prime of D is principal, then by 3.2, every proper prime ideal contains a principal prime. But Kaplansky [**9**, Theorem 5, p. 4] has shown that this property characterizes a UFD, and the proof is complete.

This theorem yields the following well-known result on a UFD as an easy consequence.

3.4 COROLLARY. *Any Noetherian GCD-domain is a UFD.*

Proof. In a Noetherian GCD-domain D, every PF-prime ideal is finitely generated, and hence principal. By 3.3, D must be a UFD.

The following corollary is a special case of a result due to Griffin [**7**, Proposition 4 and Theorem 5, p. 715] concerning the class of v-multiplication rings, which includes the GCD-domains as a subclass.

3.5 COROLLARY (Griffin). *Every GCD-domain is an intersection of essential valuation overrings. In particular $D = \bigcap_\alpha D_{P_\alpha}$, where the set $\{P_\alpha\}$ may be taken to be the set of all PF-prime ideals or the set of all "maximal" PF-prime ideals, that is, "maximal" in the set of all PF-primes.* (Note that such maximal PF-primes exist since the union of a chain of PF-primes is again a PF-prime.)

Proof. It is clear that $D \subseteq \bigcap D_{P_\alpha}$. Pick a, b in D such that a/b is not in D. Then by dividing out the gcd of a and b, we may assume $\gcd(a, b) = 1$. Since $a/b \notin D$, b is a nonunit of D. Hence b is in some (maximal) PF-prime P_β, and $a \notin P_\beta$, since $\gcd(a, b)$ is not in P_β. Therefore $a/b \notin D_{P_\beta}$. Thus $D \supseteq \bigcap D_{P_\alpha}$, and equality is proved.

3.6 COROLLARY. *A GCD-domain with only a finite number of (maximal) PF-prime ideals is a Bezout domain.*

Proof. By 3.5, such a domain will necessarily be a finite intersection of valuation rings and therefore a Bezout domain, by Nagata's theorem on independence of valuations. (See [**9**, Theorem **107**, p. 78].)

It is well-known that in a Bezout domain, the partially-ordered set of prime ideals forms a tree. (In fact, we showed in general (Corollary 2.4) that the set of PF-primes in a GCD-domain forms a tree.) Next we prove a converse—that is, that this property characterizes the Bezout domains among the GCD-domains.

3.7 THEOREM. *Let D be a GCD-domain. Then D is a Bezout domain if and only if the prime ideals of D form a tree.*

Proof. We need only consider the "if" part. Let D be a GCD-domain in which the primes form a tree. Let M be a maximal ideal of D. By hypothesis, the set of primes contained in M is linearly ordered, and hence by 3.1, M must be a union of a linearly-ordered set of PF-primes. Hence M is a PF-prime. Since every maximal ideal of D is essential, D must be a Prüfer domain, and hence a Bezout domain. (See Remark 2.5.)

3.8 COROLLARY. *A GCD-domain in which the set of prime ideals is linearly ordered is a valuation ring.*

3.9 COROLLARY. *A one-dimensional GCD-domain is a Bezout domain.*

Proofs. The proofs follow immediately from the fact that either linear ordering on primes or one-dimensionality imply that the primes form a tree.

Remark. The result in 3.8 is a special case of a result by McAdam [**10**, Theorem 1, p. 239], and has also been proved independently by Vasconcelos [**13**, App., Proposition A.]. Corollary 3.9 was proved independently by Dawson and Dobbs [**5**]. In each case the method used was quite different from that used here.

4. Krull dimension and chains of PF-primes.

4.1 *Definition.* Let D be a GCD-domain. Then we say the *PF-dimension of D* (denoted PF-dim(D)) is the number of steps in the longest chain of PF-prime ideals of D, or infinity if there is no such longest chain.

Remark. In an arbitrary domain D, we could give a definition of the *essential dimension of D* by replacing the term "PF-prime ideal" with "essential prime ideal". Then for a GCD-domain the PF-dimension and essential dimension are equal, by Theorem 2.2. Some other elementary properties of the PF-dimension of a GCD-domain and its relation to the Krull-dimension (denoted K-dim(D) in this section) are listed in the following proposition, which is an immediate consequence of Theorem 3.1 and Remark 2.5.

4.2 PROPOSITION. *Let D be a GCD-domain that is not a field. Then*

$$1 \leqq \text{PF-dim}(D) \leqq \text{K-dim}(D),$$

and in case D is a Bezout domain, PF-dim(D) = K-dim(D).

Since we have already found several properties that characterize Bezout domains among GCD-domains, it seems reasonable to ask if equality of Krull dimension and PF-dimension is another such property. We have already seen positive results in this direction. Corollary 3.9 can be considered as stating that if D is a GCD-domain with PF-dimension one, then D is Bezout if and only if PF-dim(D) = K-dim(D). Thus we are led to make the following conjecture:

4.3 *Conjecture.* If D is a GCD-domain and PF-dim(D) = K-dim(D), then D is a Bezout domain.

The remainder of this section will be devoted to considering this conjecture for PF-dimensions other than one. For finite dimensions greater than one, we are unable to prove or disprove it, but we can show that among several large classes of examples of GCD-domains, no counterexamples exist. In the case of infinite dimension, the conjecture is false, as the following example illustrates:

4.4 *Example.* Let D be a GCD-domain that is not a Bezout domain. (A two-dimensional UFD—for example, the ring of polynomials in two indeterminates over a field—will suffice). Let K be its quotient field. Let V be a valuation ring of infinite rank that can be written in the form $K + M$, where M is its maximal ideal. Then the subring $D + M$ of V is the example we want. In particular it has the following properties:

4.4.1 $D + M$ is a GCD-domain,

4.4.2 $D + M$ is not a Bezout domain,

4.4.3 PF-dim$(D + M) = \infty$,

4.4.4 K-dim$(D + M) = \infty$.

Property 4.4.1 follows from the fact that D is a GCD-domain with quotient field K [**2**, Theorem 3.13], and 4.4.2 follows from the fact that D is not a Bezout domain [**2**, Theorem 2.1 (l')]. Now let $\{P_\alpha\}$ be an infinite chain of nonzero prime ideals of V. Then each P_α is again a prime ideal of $D + M$, and $(D + M)_{P_\alpha} = V_{P_\alpha}$, which is a valuation overring of $D + M$. In other words, each P_α is a PF-prime, and hence the PF-dimension of $D + M$ (and of course the Krull dimension as well) is infinity. Thus the example is complete.

We return now to the search for a finite-dimensional counterexample. In particular we need a GCD-domain which is neither a Bezout domain nor a UFD. (In fact it must have PF-dimension $\geqq 2$.) One method of constructing such domains is the $D + M$ construction used in the preceding example.

Unfortunately, in the finite-dimensional case no new counterexamples to the conjecture can be obtained by this method, as the following theorem shows:

4.5 THEOREM. *If a domain of the form $D + M$ is a finite-dimensional counterexample to Conjecture* 4.3, *then so is D.*

Proof. Let $D + M$ be a subring of the valuation ring $K + M$ which is a GCD-domain but not a Bezout domain and such that PF-dim$(D + M)$ = K-dim$(D + M) < \infty$. Then D is a GCD-domain with quotient field K [**2**, Theorem 3.13], and D is not a Bezout domain [**2**, Theorem 2.1 (l')]. Moreover,

$$\text{K-dim}(D + M) = \text{K-dim}(D) + \text{K-dim}(V)$$

[**2**, Theorem 2.1 (f)]. If we can show the same formula holds for PF-dimension, then we are done, since we can subtract K-dim(V) (which equals PF-dim(V)) and conclude PF-dim(D) = K-dim(D). To verify the formula, we consider a PF-prime P^* of $D + M$. If $P^* \subseteq M$, it is a prime ideal (necessarily a PF-prime) of V. If $P^* \nsubseteq M$, then P^* is of the form $P + M$ where P is a prime ideal of D. Moreover P is a PF-prime in D since $(D + M)_{P+M} = D_P + M$ and $D_P + M$ can be a valuation ring only when D_P is already a valuation ring. Hence a chain of primes of $D + M$ is a chain of PF-primes if and only if it is of the form

$$Q_1 \subset Q_2 \subset \ldots \subset Q_m \subset P_1 + M \subset P_2 + M \subset \ldots \subset P_m + M,$$

where each Q_i is a PF-prime of V, and each P_i is a PF-prime of D. Thus the longest chain possible has (PF-dim(V)) + (PF-dim(D)) terms, and the formula is verified. This completes the proof of the theorem.

Another way to construct a GCD-domain that is neither a Bezout domain nor a UFD is to adjoin a polynomial indeterminate to a GCD-domain that is not a UFD. This method not only fails to yield any new counterexamples, as the $D + M$ method did, but even fails to yield any counterexamples at all.

4.6 THEOREM. *Let D be a GCD-domain that is not a field. Then*

$$\text{PF-dim}(D[X]) = \text{PF-dim}(D).$$

Consequently, if D has finite Krull dimension, then

$$\text{PF-dim}(D[X]) \neq \text{K-dim}(D[X]).$$

Proof. Arnold and Brewer [**1**, Lemma 1, p. 483] have shown for an arbitrary domain D and a prime ideal Q of $D[X]$, that if $D[X]_Q$ is a valuation ring, then either $Q = (Q \cap D)D[X]$, or $Q \cap D = (0)$. Ideals of the first type are just those of the form $P[X]$, for some essential prime ideal P of D. An ideal of the second type must extend to a proper prime ideal of $K[X]$; that is, it is the center of a $p(X)$-adic valuation on $K[X]$. Thus for the case of a GCD-domain D, the PF-primes of $D[X]$ are those of the type $P[X]$ where P is a PF-prime of D, together with the centers of the $p(X)$-adic valuations of $K[X]$ on $D[X]$. Let Q be a prime of the latter type. Since $Q \cap D = (0)$, we can factor the gcd of the coefficients out of an element q_1 of Q and get a polynomial $q_2 \in Q$ for which the gcd of the coefficients is 1. Then for any PF-prime P of D, the set of coefficients of g_2 cannot lie inside of P, since if they did, 1 would be in P as well. Hence $q_2 \notin P[X]$. In other words, there are no containment relations between the centers of $p(X)$-adic valuations on $D[X]$ and PF-primes of the type $P[X]$. Moreover, the $p(X)$-adic valuations are all rank-one discrete, so their centers are height one primes. Thus any chain of PF-primes of length greater than 1 will be of the form $\{P_\alpha[X]\}$ and will correspond to a chain $\{P_\alpha\}$ of PF-primes of D. Hence PF-dim$(D[X])$ = PF-dim(D). The second assertion of the theorem follows at once, since for finite Krull dimension,

$$\text{K-dim}(D[X]) \geqq \text{K-dim}(D) + 1 \geqq \text{PF-dim}(D) + 1 > \text{PF-dim}\, D[X].$$

Remark. For a proof that $D[X]$ is a GCD-domain, see Jaffard [**8**, Chapter IV, § 2, Proposition 4, p. 100].

5. A group-theoretic interpretation. The *divisibility group* $G(D)$ of an integral domain D with quotient field K is the partially ordered abelian group of nonzero principal fractional ideals of D in K, with the ordering defined as follows: $k_1D \geqq k_2D$ if and only if $k_1D \subseteq k_2D$. Thus the zero element of $G(D)$ is the fractional ideal D itself, and the positive elements in the group are precisely the ordinary principal ideals of D. The natural mapping $\omega : K - \{0\} \to G(D)$ taking k to kD is called the *semivaluation of K associated with D*. (For further details about semivaluations and the divisibility group, see Gilmer [**6**, App. 4] or Mott [**11**].)

It is straightforward to verify that D is a GCD-domain if and only if the ordering on $G(D)$ is a lattice ordering. Thus we can utilize the specialized structure of lattice-ordered abelian groups to gain insight into the properties of GCD-domains. Of particular interest in considering the prime ideal structure of D are the prime filters of $G(D)$ defined below.

5.1 *Definition*. Let L be a lattice-ordered abelian group with positive cone L_+. A proper subset F of L_+ is a *prime filter of* L_+ if it satisfies these three properties:

5.1.1 if $x \in F$ and $y \geqq x$, then $y \in F$;

5.1.2 if $x, y \in F$, then $\inf\{x, y\} \in F$;

5.1.3 if $z, w \in L_+ - F$, then $z + w \in L_+ - F$.

In an earlier paper by this author it was shown that if D is a Bezout domain, then the semivaluation ω induces a one-to-one inclusion-preserving correspondence between the proper prime ideals of D and the prime filters of $G(D)_+$ [**12**, Theorem 2.2]. In a GCD-domain in general the inverse image under ω of a prime filter in $G(D)_+$ is always a proper prime ideal, but it is not true that the image under ω of every proper prime ideal of D is a prime filter. To be precise, the image of P is a prime filter if and only if P is a PF-prime ideal, and this will hold true if and only if P is the inverse image of a prime filter. Thus, in general, ω induces a one-to-one inclusion-preserving correspondence between the proper PF-prime ideals of D and the prime filters of $G(D)_+$. The terminology "PF-prime" was chosen to refer to this fact. (Note that the use of the terms "inverse image" and "image" in the preceding paragraph presumes the inclusion or deletion of the zero element of D wherever it is necessary to make sense, since 0_D is not in the domain of ω.)

Several of the ideas in this paper can be translated directly into statements about prime filters in the divisibility group. For example, the results that the union of a chain of PF-primes is again a PF-prime and that the set of all PF-primes forms a tree are equivalent to the facts that unions of prime filters are again prime filters and that the set of prime filters in the positive cone L_+ of any lattice-ordered group forms a tree. The technique used in the proof of Theorem 3.1 could be used to show that a filter (that is, a subset of L_+ which satisfies 5.1.1 and 5.1.2) which is maximal with respect to missing a subsemigroup of L_+ is a prime filter of L_+.

A related point of view that yields further insight into these subjects comes from the Krull-Jaffard-Ohm Theorem, which states that every lattice-ordered abelian group is order-isomorphic to the divisibility group of a Bezout domain. Thus each GCD-domain D shares its divisibility group G with a Bezout domain D', and the special properties of D' can be used to prove properties of D. For example, this approach yields a proof of Theorem 3.1 along the following lines: Let P be a prime ideal of D, and let $S = D - P$. Then there is a saturated multiplicative system S' of D' which corresponds to S, in the sense that their images under the respective semivaluations are equal. Since D' is a Bezout domain, S' is the complement of a union of a family $\{P_\alpha'\}$ of PF-prime ideals of D'. But each of the prime ideals P_α' corresponds, in the same way as described above, to a PF-prime ideal P_α of D. Since $S' = D' - \bigcup P_\alpha'$, the special properties of this correspondence guarantee that $S = D - \bigcup P_\alpha$, which proves that P is the union of the family $\{P_\alpha\}$ of PF-primes of D.

References

1. J. Arnold and J. Brewer, *Kronecker function rings and flat $D[X]$-modules*, Proc. Amer. Math. Soc. *27* (1971), 483–485.
2. E. Bastida and R. Gilmer, *Overrings and divisorial ideals of rings of the form $D + M$*, Michigan J. Math. *20*(1973), 79–95.
3. N. Bourbaki, *Algèbre commutative*, Chapter 7 (Diviseurs) (Hermann, Paris, 1965).
4. P. M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc. *64* (1968), 251–264.
5. J. Dawson and D. Dobbs, *On going down in polynomial rings* (to appear in Can. J. Math.).
6. R. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure and Applied Math., no. *12* (Queen's University, Kingston, Ontario, 1968).
7. M. Griffin, *Some results on v-multiplication rings*, Can. J. Math. *19* (1967), 710–722.
8. P. Jaffard, *Les systèmes d'idéaux* (Dunod, Paris, 1960).

9. I. Kaplansky, *Commutative rings* (Allyn and Bacon, Boston, 1970).
10. S. McAdam, *Two conductor theorems*, J. Algebra *23* (1972), 239–240.
11. J. Mott, *The group of divisibility and its applications*, Conference on Commutative Algebra, Lawrence, Kansas, 1972; Lecture Notes in Mathematics, No. *311* (Springer–Verlag, New York, 1973).
12. P. Sheldon, *Two counterexamples involving complete integral closure in finite-dimensional Prüfer domains* (to appear in J. Algebra).
13. W. Vasconcelos, *The local rings of global dimension two*, Proc. Amer. Math. Soc. *35* (1972), 381–386.

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