of course there may be repetitions if replacements are allowed. The part of the probable value due to this drawing is

$$\frac{1}{N}(k_\alpha + k_\beta + \ldots . + k_\mu),$$

and the total probable value is

$$\frac{1}{N}\Sigma(k_\alpha + k_\beta + \ldots . + k_\mu),$$

where Σ denotes summation of N sets of m terms each, in all mN terms.

Now, since no coin is singled out for special favour, α will equally often be 1, 2, ..., or n; and the same is true of $\beta, \gamma, \ldots.$, and μ. Hence in the sum above the expression $k_1 + k_2 + \ldots . + k_n$ must occur a whole number of times, and this whole number must be mN/n. Thus finally, since the value of all the coins is P, the probable value in question is

$$\frac{1}{N} \cdot \frac{mN}{n} \cdot P = \frac{mP}{n}.$$

The invariant property of mathematical expectation is thus brought out. (For a rather similar result see Chrystal's *Algebra*, Part II, pp. 594-5.)

The case where the number of replacements allowed is not limited leads to an identity in combinatory analysis which is by no means obvious, namely

$$\sum_{r=1}^{m} \frac{m!}{m_1!\, m_2! \ldots . m_r!}(m_1 a_1 + m_2 a_2 + \ldots . + m_r a_r) = mn^{m-1}(a_1 + a_2 + \ldots . + a_n)$$

where $m_1 + m_2 + \ldots . + m_r = m$, and Σ includes all r-part *compositions* (*i.e.* partitions in which order of parts is relevant) of m, associated with all r-ary combinations of 1, 2, ..., n.

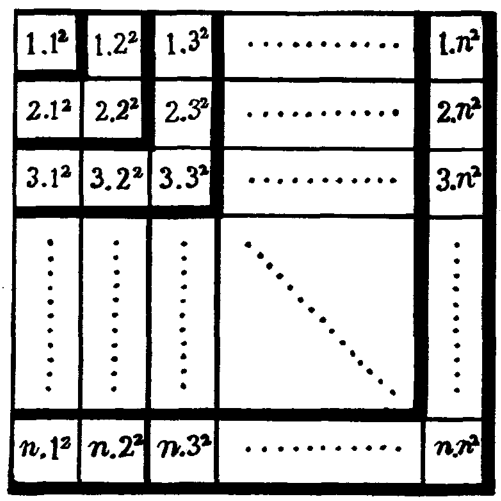
A Simple Method of Finding Sums of Powers of the Natural Numbers

By I. M. H. Etherington.

Let $1^\alpha + 2^\alpha + 3^\alpha + \ldots . + n^\alpha$ be denoted by S_α. It is well known that S_α can be expressed as a polynomial in n of degree $(\alpha + 1)$. The expressions for $S_1, S_2, S_3 \ldots .$ can be found in succession by elementary methods, which also give numerous relations such as $S_3 = S_1^2$,

$12S_2 S_3 = 7S_6 + 5S_4$. The elegant method which I am about to explain is not original. It is due in essence to the Arabian mathematician Alkarkhi* (*circa* 1000 B.C.).

The method consists of arranging numbers in a square, adding them up in two ways, and equating the results. An example will make it clear. To find S_4, assuming that we know $S_1 = \frac{1}{2}n(n+1)$ and $S_2 = \frac{1}{6}n(n+1)(2n+1)$, consider this arrangement of numbers:



Adding up by *rows* or *columns*, the sum of all the numbers is seen to be $S_1 S_2$. But we can also add by *gnomons*, as indicated by the heavy lines. The sum of the numbers comprising the n^{th} *gnomon*, *i.e.* the last row and column,

$$
\begin{aligned}
&= nS_2 + n^2 S_1 - n^3 \\
&= \tfrac{1}{6} n^2 (n+1)(2n+1) + \tfrac{1}{2} n^3 (n+1) - n^3 \\
&= \tfrac{5}{6} n^4 + \tfrac{1}{6} n^2.
\end{aligned}
$$

Thus the total is $\sum_1^n \left(\frac{5}{6}n^4 + \frac{1}{6}n^2\right)$

$$= \tfrac{5}{6}S_4 + \tfrac{1}{6}S_2.$$

Equating the results,

$$6S_1 S_2 = 5S_4 + S_2,$$

whence, substituting for S_1 and S_2, we find:

$$S_4 = \tfrac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).$$

---

* See his *Fakhri* (Woepcke, Paris, 1853), p. 61.

In general, by taking $a^{\alpha} b^{\beta}$ as the occupant of the cell in the a^{th} row and b^{th} column, we obtain:

$$S_{\alpha} S_{\beta} = \sum_{1}^{n} (n^{\alpha} S_{\beta} + n^{\beta} S_{\alpha} - n^{\alpha+\beta}).$$

Assuming that we know the expressions for S_{α} and S_{β}, we can substitute these in the right hand side. Then:

$$S_{\alpha} S_{\beta} = A_1 S_1 + A_2 S_2 + \ldots\ldots + A_{\alpha+\beta+1} S_{\alpha+\beta+1},$$

where A_1, A_2, ..., $A_{\alpha+\beta+1}$ are numbers which depend on the coefficients in the expressions for S_{α}, S_{β}. Actually, (provided α, $\beta \neq 0$) $A_{\alpha+\beta}$ always vanishes, and the last coefficient $A_{\alpha+\beta+1}$ is $(\alpha+\beta+2)/(\alpha+1)(\beta+1)$. These follow from the fact that the polynomial for S_r always begins $\dfrac{n^{r+1}}{r+1} + \dfrac{n^r}{2} + \ldots\ldots$

Thus we can find $S_{\alpha+\beta+1}$ if we know the expressions for S_1, S_2, ..., $S_{\alpha+\beta-1}$.

An interesting case arises when $\beta = 0$. Since $S_0 = n$, we then obtain:

$$nS_{\alpha} = \sum_{1}^{n} (n^{\alpha+1} + S_{\alpha} - n^{\alpha})$$
$$= S_{\alpha+1} + \sum_{1}^{n} S_{\alpha} - S_{\alpha},$$

giving the useful formula

$$S_{\alpha+1} + \sum_{1}^{n} S_{\alpha} = (n+1) S_{\alpha}.$$

The method may be extended by generalising the square to t dimensions and filling it with numbers of the form

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots\ldots a_t^{\alpha_t}$$

where α_1, α_2, ..., α_t are fixed, while a_1, a_2, ..., a_t vary independently from 1 to n. The analogous result is:

$$S_{\alpha_1} S_{\alpha_2} \ldots\ldots S_{\alpha_t} = \sum_{1}^{n} G_n$$

where
$$G_n = [n^{\alpha_1} S_{\alpha_2} S_{\alpha_3} \ldots\ldots S_{\alpha_t} + n^{\alpha_2} S_{\alpha_1} S_{\alpha_3} \ldots\ldots S_{\alpha_t} + \ldots\ldots]$$
$$- [n^{\alpha_1+\alpha_2} S_{\alpha_3} \ldots\ldots S_{\alpha_t} + \ldots\ldots] + [n^{\alpha_1+\alpha_2+\alpha_3} S_{\alpha_4} \ldots\ldots S_{\alpha_t} + \ldots\ldots]$$
$$- \ldots\ldots \pm n^{\alpha_1+\alpha_2+\ldots\ldots+\alpha_t}.$$

Assuming that the expressions for S_{α_1}, S_{α_2}, ... S_{α_t} are known, we can substitute polynomials in n for the S's, and obtain for G_n a polynomial of degree $\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_t + t - 1$. Thus $S_{\alpha_1} S_{\alpha_2} \ldots\ldots S_{\alpha_t}$ can be expressed linearly in terms of S_1, S_2, ... S_d, where

$$d = \alpha_1 + \alpha_2 + \ldots\ldots + \alpha_t + t - 1;$$

and the expression can be calculated if the polynomial expressions for $S_{a_1}, S_{a_2}, \ldots . S_{a_t}$ are known.

As an example, let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $t = 3$.

Then $G_n = nS_2 S_3 + n^2 S_3 S_1 + n^3 S_1 S_2 - n^5 S_1 - n^4 S_2 - n^3 S_3 + n^6$

$$= n \cdot \tfrac{1}{6}n(n+1)(2n+1) \cdot \tfrac{1}{4}n^2(n+1)^2 + \text{etc.},$$

reducing to

$$G_n = \tfrac{3}{8}n^8 + \tfrac{7}{12}n^6 + \tfrac{1}{24}n^4.$$

Hence $24S_1 S_2 S_3 = 9S_8 + 14S_6 + S_4.$

A few further results, easily proved in this way, or by repeated applications of the square method, may be quoted:—

$$S_1^2 = S_3, \qquad 6S_1 S_2 = 5S_4 + S_2,$$
$$4S_1^3 = 3S_5 + S_3, \qquad 12S_1^2 S_2 = 7S_6 + 5S_4,$$
$$2S_1^4 = S_7 + S_5, \qquad 24S_1^3 S_2 = 9S_8 + 14S_6 + S_4,$$
$$16S_1^5 = 5S_9 + 10S_7 + S_5, \qquad 48S_1^4 S_2 = 11S_{10} + 30S_8 + 7S_6.$$

https://doi.org/10.1017/S1757748900002255 Published online by Cambridge University Press