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A NOTE ON A MATRIX RESULT OF RYSER

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The purpose of this note is to give a short proof of a generalisation of a theorem of Ryser, Theorem 10.2.3 of [1], concerning matrices that occur in the theory of symmetric block designs.

The two main results of matrix theory required in the proof given below are:

(1) If B, C are square matrices such that $BC=zI$ where z is a non-zero complex number, then $CB=zI$.

(2) A matrix S which is both symmetric (i.e. $S'=S$) and skew-symmetric (i.e. $S'=-S$) is zero.

THEOREM. *Let A be a real v by v matrix. Let E be a real symmetric idempotent matrix (i.e. $E'=E$ and $E^2=E$). Let α, β, γ be non-zero real numbers. Consider the following four conditions:*

$$AA' = \alpha I+\beta E. \tag{3}$$

$$A'A = \alpha I+\beta E. \tag{4}$$

$$AE = \gamma E. \tag{5}$$

$$EA = \gamma E. \tag{6}$$

Then, if A satisfies either (3) *or* (4) *and either* (5) *or* (6), *it follows that* (3), (4), (5) *and* (6) *all hold.*

Proof. *Case* 1. Suppose (3) and (5) hold. Then

$$\begin{aligned}(A-\beta^{1/2}E)(A'+\beta^{1/2}E) &= AA'+\beta^{1/2}(AE-EA')-\beta E\\ &= \alpha I+\beta E+\beta^{1/2}(\gamma E-\gamma E)-\beta E\\ &= \alpha I.\end{aligned}$$

So, from (1),

$$(A'+\beta^{1/2}E)(A-\beta^{1/2}E) = \alpha I.$$

i.e.

$$A'A+\beta^{1/2}(EA-A'E)-\beta E = \alpha I.$$

i.e.

$$A'A-\alpha I-\beta E = \beta^{1/2}(A'E-EA).$$

The left side of this last equation is symmetric, while the right side is skew-symmetric. It follows from (2), that each side is the zero matrix. Hence

$$A'A = \alpha I+\beta E, \quad \text{which is (4).}$$

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To obtain (6), note that

$$AA'A = \alpha A+\beta EA, \quad \text{from (3)}$$

and

$$AA'A = \alpha A+\beta AE, \quad \text{from (4)}.$$

Subtract these last two equations to deduce that $AE=EA$, which proves (6).

Case 2. Suppose (3) and (6) hold. Premultiply (3) by E and use (6). Hence

$$\gamma EA' = (\alpha+\beta)E.$$

Transpose to deduce

$$AE = (\alpha+\beta)\gamma^{-1}E. \tag{5a}$$

We can now use (5a) to replace (5) in the argument in Case 1.

Case 3. If (4) is given together with either (5) or (6), the problem is reduced to Case 1 or Case 2 by considering A' instead of A.

REMARK. In Ryser's result the role of the matrix E is taken by the matrix $v^{-1}J$, where J is the v by v matrix, each of whose entries is 1. Also, in that case, $\alpha=k-\lambda$, $\beta=\lambda v$, $\gamma=k$. Then, it is worth noting that $(k-\lambda)^{-1/2}(A-tJ)$ is a real orthogonal v by v matrix, where t is either root of the quadratic equation $vt^2-2kt+\lambda=0$.

REFERENCE

1. M. Hall, *Combinatorial Theory* (Blaisdell), 1967.

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