

UNIQUENESS AND NON-UNIQUENESS OF SOLUTIONS OF VLASOV-McKEAN EQUATIONS

MICHAEL SCHEUTZOW

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Abstract

We study the equation $dY(t)/dt = f(Y(t), Eh(Y(t)))$ for random initial conditions, where E denotes the expected value. It turns out that in contrast to the deterministic case local Lipschitz continuity of f and h are not sufficient to ensure uniqueness of the solutions. Finally we also state some sufficient conditions for uniqueness.

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1. Introduction

Vlasov-McKean equations of the form

(1) $dX(t) = b(X(t), \mathcal{L}(X(t))) dt + \sigma(X(t), \mathcal{L}(X(t))) dW(t)$, $X(0) = \zeta$ frequently occur as the equation for the dynamics of one component in the mean-field- (or Vlasov-McKean-) limit [1, 4, 5, 6, 8]. Here \mathcal{L} denotes the law of a random vector, $(W(t))_{t \geq 0}$ is an n -dimensional Brownian motion, ζ is an \mathbf{R}^d -valued random variable and b and σ are functions having domain and range such that (1) makes sense. Equation (1) is frequently called “nonlinear stochastic differential equation” since the associated Fokker-Planck (-Vlasov) equation is a nonlinear partial (integro-) differential equation. When proving limit theorems of this type one step (besides tightness and identification of the limit) usually consists of proving pathwise uniqueness of the solution of (1) (see, for example, [6, 8]). Under certain assumptions (mostly including Lipschitz conditions) particular equations of type (1) or the corresponding Fokker-Planck equations were

shown to have a unique solution (in different senses, see [1, 2, 4, 5, 6, 7, 8]). In many applications however b and/or σ are not Lipschitz continuous, but, for example of polynomial type (for example in chemical reaction models, see [6]) and the question arises whether one can still prove pathwise uniqueness. In this paper we provide a partial answer for degenerate equations, where $\sigma \equiv 0$ (like Vlasov equation) and which are of the particular structure

$$(2) \quad \dot{Y}(t) = f(Y(t), Eh(Y(t))), \quad Y(0) = \xi.$$

We give examples showing that a local Lipschitz condition on either f or h is insufficient for uniqueness of a solution even if the other function satisfies a global Lipschitz condition. These examples are in contrast to the deterministic case (that is, (2) without the expectation in front of h) where local Lipschitz continuity of f and h are known to be sufficient for uniqueness. The results, although perhaps not too surprising due to the global effect of taking the expected value of h , that is, integrating over the whole state space, are still remarkable since the formally approximating system of equations

$$(3) \quad \dot{Y}_i(t) = f\left(Y_i(t), \frac{1}{N} \sum_{j=1}^N h(Y_j(t))\right), \quad Y_i(0) = \xi_i, \quad i = 1, \dots, N,$$

with ξ_i independent and identically distributed, has a unique (local) solution for every finite N if f and h satisfy a local Lipschitz condition. By a (strong) solution of (2) for $0 \leq t \leq T$ on a probability space (Ω, \mathcal{F}, P) on which ξ is defined, we mean a stochastic process $(Y(t))_{0 \leq t \leq T}$ on that space such that $Eh(Y(t)) < \infty$ for all $0 \leq t \leq T$, $\int_0^T E|h(Y(t))| dt < \infty$, Y is continuous with probability one and satisfies the integrated equation (2). In the first part we specify a number of conditions on f and h which are insufficient for (2) to have a unique solution by providing counterexamples. In the last part we prove three positive results on uniqueness. More precisely we verify the following table

$f \backslash h$	Lipschitz and compact support	Lipschitz	locally Lipschitz
Lipschitz and compact support	unique (P1, P2, P3)	unique (P1, P2)	unique (P2)
Lipschitz	unique (P1, P3)	unique (P1)	CE 1, CE 1'
locally Lipschitz	unique (P3)	CE 2	CE 1, CE 1', CE 2
continuous and compact support	CE 3	CE 3	CE 3

“CE” stands for “counterexample”, “P” stands for “proposition”. We omitted the column “ f continuous and compact support” since counterexamples are well known in that case. The table holds true for every dimension d ($1 \leq d < \infty$) of the state space of Y and every dimension n ($1 \leq n < \infty$) of the range of h .

2. Counterexamples

COUNTEREXAMPLE 1. To construct the first counterexample let $(p_i)_{i \in \mathbb{N}_0}$ be a sequence of strictly positive numbers such that $\sum_{j=1}^\infty p_j = 1$ and $\sum_{j=0}^\infty jp_j < \infty$. Furthermore let $(g_i(\cdot))_{i \in \mathbb{N}_0}$ be a sequence of twice continuously differentiable functions from the interval $[0, \frac{1}{2}]$ to $[0, \frac{1}{2}]$ with the properties

- (i) $g_i(0) = 0$.
- (ii) $g'_i(s) > 0, 0 < s < \frac{1}{2}$ (the prime denotes derivative w.r.t. s).
- (iii) There exists some $s_i > 0$ such that $g_i(s) = s^2$ for $0 \leq s \leq s_i$.
- (iv) $g_i(\frac{1}{2}) = \frac{1}{2}$, and
- (v) $\sum_{j=0}^n p_j g_j(t)$ converges to $h(t) := 2t - 2t^2$ pointwise (and hence uniformly) on $[0, \frac{1}{2}]$ as $n \rightarrow \infty$.

To see that such a sequence of functions exists, define

$$R_0(t) := h(t) = 2t - 2t^2, \quad 0 \leq t \leq \frac{1}{2},$$

$$s_{-1} := \frac{1}{2},$$

$$s_i := \min \left(\frac{1}{2} \left(1 - \frac{p_i}{\sum_{j=i}^\infty p_j} \right) R_i(s_{i-1}), \frac{1}{2} \left(\frac{R_i(s_{i-1})}{\sum_{j=i}^\infty p_j} \right)^{1/2} \right) \quad \text{and}$$

$$R_{i+1}(t) := \begin{cases} R_i(t) - p_i t^2 \left(= 2t - \left(2 + \sum_{j=0}^i p_j \right) t^2 \right), & t \leq s_i, \\ R_i(t) \left(1 - \frac{p_i}{\sum_{j=i}^\infty p_j} \right), & t \geq s_{i-1}, \end{cases} \quad i = 0, 1, 2, \dots,$$

and let R_{i+1} be twice continuously differentiable on $[0, \frac{1}{2}]$ such that $0 < R'_{i+1}(t) < R'_i(t)$ for $0 < t < \frac{1}{2}$. The inequalities for $R'_{i+1}(t)$ are clearly satisfied for $0 < t \leq s_i$ and $s_{i-1} \leq t < \frac{1}{2}$ by induction, where $R'_{i+1}(t) > 0$ for $t \leq s_i$ follows from $s_i \leq \frac{1}{4}$ (since $R_i(t) \leq \frac{1}{2}$ by induction) which implies $R'_{i+1}(t) = 2 - 2(2 + \sum_{j=0}^i p_j)t \geq 2 - 6t > 0$ for $t \leq s_i$. Note that $s_i < s_{i-1}$ since $R'_i(t) \leq 2$ by induction and hence $R_i(s_{i-1}) \leq 2s_{i-1}$.

To show that it is possible to choose $R_{i+1}(t)$ in $[s_i, s_{i-1}]$ such that the inequalities for $R'_{i+1}(t)$ hold it is enough to ensure that

$$0 < R_{i+1}(s_{i-1}) - R_{i+1}(s_i) < R_i(s_{i-1}) - R_i(s_i).$$

The right inequality is equivalent to

$$R_i(s_{i-1}) \left(1 - \frac{p_i}{\sum_{j=i}^{\infty} p_j} \right) - R_i(s_i) + p_i s_i^2 < R_i(s_{i-1}) - R_i(s_i)$$

and hence to $s_i^2 < R_i(s_{i-1}) / \sum_{j=i}^{\infty} p_j$, the left one is equivalent to

$$2s_i - s_i^2 \left(2 + \sum_{j=0}^i p_j \right) < \left(1 - \frac{p_i}{\sum_{j=i}^{\infty} p_j} \right) R_i(s_{i-1})$$

which follows from the definition of s_i . Now $g_i(t) := (R_i(t) - R_{i+1}(t)) / p_i$ has all the properties (i)–(v). Note that $R_n(t) = h(t) - \sum_{j=0}^{n-1} p_j g_j(t)$ converges to zero at $t = \frac{1}{2}$ and hence, due to $R'_n(t) \geq 0$ and $R_n(0) = 0$, for all $0 \leq t \leq \frac{1}{2}$.

Let us now construct a locally Lipschitz continuous function $\phi: \mathbf{R} \rightarrow \mathbf{R}$ such that the equation

$$(4) \quad \dot{Y}(t) = \phi(Y(t)) + EY(t) - \sum_{j=0}^{\infty} j p_j$$

with $P\{Y(0) = i\} = p_i, i \in \mathbf{N}_0$, has at least two different solutions on the interval $[0, \frac{1}{4}]$. Note that the sequence $(p_i)_{i=1,2,\dots}$ is fixed and therefore $\sum_{j=0}^{\infty} j p_j$ is a constant (which could have been incorporated into the function ϕ). Fixing a sequence of functions $(g_i)_{i \in \mathbf{N}_0}$ satisfying the assumptions above, define

$$Y_i(t) := i + g_i(t), \quad 0 \leq t \leq \frac{1}{2},$$

and $\phi: \mathbf{R} \rightarrow \mathbf{R}$ by

$$\phi(i + t) = g'_i(g_i^{-1}(t)) - (2g_i^{-1}(t) - 2(g_i^{-1}(t))^2)$$

for $i \in \mathbf{N}_0$ and $0 \leq t \leq g_i(\frac{1}{4})$.

Let $\phi(s) = 0$ for $s < 0$ and define ϕ on the intervals $[i + g_i(\frac{1}{4}), i + 1]$ ($i \in \mathbf{N}_0$) by linear interpolation. The function ϕ is well defined since the functions g_i are strictly monotonically increasing on $[0, \frac{1}{4}]$. Obviously $\phi(i) = 0$ for all $i \in \mathbf{N}_0$ and ϕ is continuous. For $0 \leq t \leq \frac{1}{4}$

$$\begin{aligned} \phi(Y_i(t)) &= \phi(i + g_i(t)) = g'_i(t) - (2t - 2t^2) = g'_i(t) - \sum_{j=0}^{\infty} p_j g_j(t) \\ &= \dot{Y}_i(t) - \sum_{j=0}^{\infty} p_j Y_j(t) + \sum_{j=0}^{\infty} p_j j \\ &= \dot{Y}_i(t) - \sum_{j=0}^{\infty} p_j Y_j(t) + \sum_{j=0}^{\infty} p_j Y_j(0). \end{aligned}$$

Define the process $(Y(t))_{t \in [0, 1/4]}$ by $P\{Y(t) = Y_i(t) \text{ for all } 0 \leq t \leq \frac{1}{4}\} = p_i$. Then it follows that Y satisfies equation (4) with the initial condition $P\{Y(0) = i\} = p_i$.

Since $\phi(i) = 0$ for $i = 0, 1, 2, \dots$ the equation is also solved by the process \tilde{Y} defined by $P\{\tilde{Y}(t) = i \text{ for all } t \geq 0\} = p_i$ which clearly has a law different from that of Y .

All that remains to show is that the function ϕ is not only continuous but in fact locally Lipschitz continuous. It suffices to check the Lipschitz continuity on every interval $[i, i + g_i(\frac{1}{4})]$. Differentiating and evaluating at $i + g_i(t)$ we get

$$\phi'(i + g_i(t)) = \frac{g_i''(t)}{g_i'(t)} - \frac{1}{g_i'(t)}(2 - 4t)$$

for all $0 < t < \frac{1}{4}$. For $0 < t \leq s_i$

$$\phi'(i + g_i(t)) = \frac{2}{2t} - \frac{1}{2t}(2 - 4t) = 2.$$

For $s_i \leq t \leq \frac{1}{4}$, $g_i'(t)$ is bounded away from zero which shows that $\phi'(t)$ is bounded on $]i, i + g_i(t)[$ and hence that ϕ is locally Lipschitz continuous.

REMARK. The assumption that the p_i are (strictly) positive can be relaxed by only requiring that infinitely many of the p_i are strictly positive and the others are zero. If however only finitely many are nonzero the construction does not work.

COUNTEREXAMPLE 1'. One might suspect that the construction of two different solutions of equation (4) was possible only because $EY(t)$ is not a (weakly) continuous functional of the law of $Y(t)$. The following generalization of counterexample 1 shows however that this is not true.

Let h be a continuously differentiable function from \mathbf{R} to \mathbf{R} satisfying $h'(x) > 0$ for all $x \in \mathbf{R}$ (h can be bounded or unbounded). Furthermore let $(p_i)_{i=0,1,2,\dots}$ be a sequence of nonnegative numbers which contains infinitely many nonzero elements satisfying $\sum_{j=0}^{\infty} p_j = 1$ and $\sum_{j=0}^{\infty} p_j h(j) < \infty$. Almost the same construction as before shows that

$$(5) \quad \dot{Y}(t) = \phi(Y(t)) + Eh(Y(t)) - \sum_{j=0}^{\infty} p_j h(j),$$

$$P\{Y(0) = i\} = p_i, \quad i = 0, 1, 2, \dots,$$

has more than one solution on some interval $[0, t_0]$ if ϕ is defined in the same way as before, but where the g_i satisfy

$$\sum_{j=0}^{\infty} p_j (h(j + g_j(t)) - h(j)) = 2t - 2t^2 \quad \text{for } 0 \leq t \leq t_0$$

instead of $\sum_{j=0}^{\infty} p_j g_j(t) = 2t - 2t^2$ and $g_i(t_0) \leq \frac{1}{2}$ instead of $g_i(\frac{1}{2}) = \frac{1}{2}$. It is not hard to see that such g_i exist and that one can choose any $t_0 > 0$ which is not greater than $(1 - (1 - \min(A, 1))^{1/2})/2$ where $A = \sum_{j=0}^{\infty} p_j \min_{j \leq s \leq j+1} h'(s)$ because in this case

$$\sum_{j=0}^{\infty} p_j (h(j + g_j(t_0)) - h(j)) \geq \sum_{j=0}^{\infty} p_j \min_{j \leq s \leq j+1} h'(s) \cdot g_j(t_0)$$

which can, by an appropriate choice of the g_j , assume any value between 0 and $A/2$, but $A/2$ is greater or equal to $2t_0 - 2t_0^2$ if t_0 is chosen in the way above.

COUNTEREXAMPLE 2. We give an example showing that the equation

$$(6) \quad \dot{Y}(t) = Eh(Y(t)), \quad Y(0) = \xi$$

can have more than one solution if h is locally Lipschitz continuous.

Take a sequence of nonnegative numbers $(p_i)_{i \in \mathbb{N}_0}$ which contains infinitely many nonzero members such that $\sum_{j=0}^{\infty} p_j = 1$. Now let h be any locally Lipschitz continuous function satisfying $\sum_{j=0}^{\infty} p_j h(j + u) = (3u)^{2/3}$ where the sum converges uniformly for all $0 \leq u \leq 1/3$. Define $g(t) := t^3/3$ for $0 \leq t \leq 1$ and $Y_i(t) = i + g(t)$. Then $\dot{Y}_i(t) = \dot{g}(t) = t^2 = (3g(t))^{2/3} = \sum_{j=0}^{\infty} p_j h(j + g(t)) = \sum_{j=0}^{\infty} p_j h(Y_i(t))$, that is, the process Y defined by $P\{Y(t) = Y_i(t) \text{ for } 0 \leq t \leq 1\} = p_i, i \in \mathbb{N}_0$ satisfies equation (6) provided $P\{\xi = i\} = p_i$ for all $i = 0, 1, 2, \dots$. On the other hand equation (6) is also solved by $\dot{Y}(t) = \xi$ for all $t \geq 0$ since $\sum_{j=0}^{\infty} p_j h(j) = 0$.

REMARKS. (i) h can be chosen to be bounded and even to converge to zero at an arbitrary speed (in particular h is then uniformly continuous): Pick a sequence of positive numbers $(a_i)_{i \in \mathbb{N}_0}$ converging to zero monotonically. Then we construct a locally Lipschitz continuous nonnegative function $h: \mathbb{R} \rightarrow \mathbb{R}$ such that $\max_{i \leq s \leq i+1} h(s) \leq a_i$ for every $i \in \mathbb{N}_0$ and such that (6) has more than one solution. Let $t_0 := \min(\frac{1}{3}(\sum_{j=0}^{\infty} p_j a_j)^{3/2}, \frac{1}{3})$ and define $h(t) = 0$ for $t \leq 0$. For $0 \leq t \leq 1$ let h be Lipschitz continuous, nonnegative and satisfy $\max_{0 \leq t \leq 1} h(t) \leq a_0, h(1) = 0$ and $0 \leq (3t)^{2/3} - p_0 h(t) \leq \sum_{j=1}^{\infty} p_j a_j$ for all $0 \leq t \leq t_0$. Such a function exists by the definition of t_0 . Continue by defining h in the interval $[1, 2]$ such that h is nonnegative, Lipschitz continuous and satisfies $\max_{1 \leq t \leq 2} h(t) \leq a_1, h(2) = 0$ and $0 \leq (3t)^{2/3} - \sum_{j=0}^1 p_j h(j + t) \leq \sum_{j=2}^{\infty} p_j a_j$ for all $0 \leq t \leq t_0$ and so forth. It follows that $\sum_{j=0}^{\infty} p_j h(j + u) = (3u)^{2/3}$ for $0 \leq u \leq t_0$. The same reasoning as in counterexample 2 shows that there exist at least two solutions on the interval $[0, t_0]$.

(ii) As mentioned in the introduction one main interest in studying McKean equations is due to the fact that they often appear as limiting equations for the motion of one particle in a field of N interacting like particles as N tends to

infinity. This is also true for the example given above: Consider N particles with positions $(Y^{(1,N)}(t), \dots, Y^{(N,N)}(t))$ having independent $\mathcal{L}(Y(0))$ -distributed initial conditions $Y^{(i,N)}(0)$ and satisfying

$$(7) \quad \dot{Y}^{(i,N)}(t) = \frac{1}{N} \sum_{j=1}^N h(Y^{(j,N)}(t)), \quad i = 1, \dots, N,$$

with h defined like in the preceding remark. Since h is locally Lipschitz continuous the only solution of (7) is $Y^{(i,N)}(t) = Y^{(i,N)}(0)$ for all $t \geq 0$ and $i = 1, 2, \dots, N$. So the processes $Y^{(1,N)}(\cdot)$ converge in law to the solution \tilde{Y} of equation (6) but not to the first solution Y ! So the infinite particle system can exhibit a solution that cannot be approximated by the finite dimensional system (7).

(iii) The results of Shiga and Tanaka [8] show (in particular) that the equation $dX(t) = Eh(X(t)) dt + dW(t)$ does have a unique solution if h is bounded and measurable. So the noise helps to achieve uniqueness. We conjecture however that one can construct examples of nonuniqueness for the same equation if h is allowed to be unbounded but is locally bounded (or even locally Lipschitz continuous).

COUNTEREXAMPLE 3. The following counterexample is well known. It shows that the equation

$$(8) \quad \dot{Y}(t) = Eh(Y(t)), \quad Y(0) = \xi$$

can have more than one solution if h is continuous (but not Lipschitz) and has compact support: Let $P\{\xi = 0\} = 1$ and define

$$h(x) = \begin{cases} 0, & x \leq 0, x \geq 2, \\ x^{1/2}, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2. \end{cases}$$

Then (8) is solved by the two deterministic processes $Y_1(t) = t^2/4$ and $Y_2(t) = 0$ for $0 \leq t \leq 2$.

3. Some positive results

Let us now state three positive results about uniqueness of solutions of (2).

PROPOSITION 1. *If f and h are Lipschitz-continuous functions and $0 < T < \infty$, then there exists at most one (strong) solution of (2) on $[0, T]$.*

PROOF. Let $(Y(t))_{0 \leq t \leq T}$ and $(\tilde{Y}(t))_{0 \leq t \leq T}$ be two solutions of (2) defined on the same space (Ω, \mathcal{F}, P) and let K be a Lipschitz constant for both f and h . Then

$$\begin{aligned} |Y(t) - \tilde{Y}(t)| &\leq \int_0^t |f(Y(s), Eh(Y(s))) - f(\tilde{Y}(s), Eh(\tilde{Y}(s)))| ds \\ &\leq K \int_0^t |Y(s) - \tilde{Y}(s)| ds + K \int_0^t |Eh(Y(s)) - Eh(\tilde{Y}(s))| ds \end{aligned}$$

almost surely. Using the Lemma of Bellman and Gronwall (see Liptser and Shiriyayev [3], page 139) we get

$$(9) \quad |Y(t) - \tilde{Y}(t)| \leq K \int_0^t e^{K(t-s)} K \int_0^s |Eh(Y(u)) - Eh(\tilde{Y}(u))| du ds + K \int_0^t |Eh(Y(s)) - Eh(\tilde{Y}(s))| ds.$$

Since the right hand side is deterministic the inequality remains true if we take the expected value on the left side. It follows that there exists a constant L such that

$$\begin{aligned} |Eh(Y(t)) - Eh(\tilde{Y}(t))| &\leq KE|Y(t) - \tilde{Y}(t)| \\ &\leq L \int_0^t |Eh(Y(u)) - Eh(\tilde{Y}(u))| du \end{aligned}$$

for all $0 \leq t \leq T$. Applying the Lemma of Bellman and Gronwall once more we get $Eh(Y(t)) = Eh(\tilde{Y}(t))$ for all $0 \leq t \leq T$ and therefore, from (9) $P\{Y(t) = \tilde{Y}(t) \text{ for all } 0 \leq t \leq T\} = 1$ which proves the assertion.

REMARKS. (i) The proposition and its proof remain true in the multidimensional case, that is, $h: \mathbf{R}^d \rightarrow \mathbf{R}^n$ and $f: \mathbf{R}^d \times \mathbf{R}^n \rightarrow \mathbf{R}^d$.

(ii) Under the assumptions of Proposition 1 the solution of (2) is also unique in law, that is, the laws of any two solutions Y and \tilde{Y} defined on (Ω, \mathcal{F}, P) and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively with $\mathcal{L}(Y(0)) = \mathcal{L}(\tilde{Y}(0))$ coincide. To see this let $m(t) := Eh(Y(t))$. Then $\dot{Y}(t) = f(Y(t), m(t))$, $Y(0) = \xi$ has a solution on (Ω, \mathcal{F}, P) in the usual sense (see Walter [9], page 82) which is measurable with respect to ξ and which solves (2) in the sense stated in the introduction. Therefore this solution can also be realized (in law) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ which, together with proposition 1, proves the assertion. This shows that strong uniqueness always implies uniqueness in law (not only under the assumptions of Proposition 1).

(iii) The assumptions of Proposition 1 do not imply the existence of a solution even if $E|h(Y(0))| < \infty$. Define for example

$$\begin{aligned} h(2^i) &= 0, & i &= 0, 1, 2, \dots, \\ h\left(\frac{3}{2} \cdot 2^i\right) &= 2^{i-1}, & i &= 0, 1, 2, \dots, \\ h(t) &= 0, & t &\leq 1, \end{aligned}$$

and interpolate h linearly otherwise. Then h is Lipschitz with constant 1. Let $P\{Y(0) = 2^i\} = p_i, i \in \mathbb{N}_0$, such that $\sum_{j=0}^\infty 2^j p_j = \infty$ and define

$$f(x, y) = x + \alpha(\arctan y + \pi/2) \quad (0 < \alpha < (4\pi)^{-1}),$$

that is, we consider the system

$$\dot{Y}(t) = Y(t) + \alpha(\arctan Eh(Y(t)) + \pi/2).$$

Solving this linear equation for Y we get

$$Y(t) = Y(0)e^t + \alpha \int_0^t e^{-s} \int_0^s \arctan Eh(Y(u)) + \pi/2 \, du \, ds,$$

but $Eh(Y(t)) \geq \sum_{j=0}^\infty p_j h(2^j e^t) = \sum_{j=0}^\infty p_j (2^j e^t - 2^j) = (e^t - 1) \sum_{j=0}^\infty p_j 2^j = \infty$ for $0 < t \leq \ln 5/4$ because

$$0 \leq \alpha \int_0^t e^{-s} \int_0^s \arctan Eh(Y(u)) + \pi/2 \, du \, ds \leq \pi/4\pi = 1/4$$

and $2^j e^t + 1/4 \leq 2^{j5/4} + 1/4 \leq (3/2)2^j$ for $0 < t \leq \ln 5/4$ and $j = 0, 1, 2, \dots$

(iv) The proof is also valid for (multidimensional) stochastic equations of the form

$$(10) \quad dY(t) = f(Y(t), Eh(Y(t))) \, dt + dM(t)$$

where M is any fixed continuous L^2 -martingale, because the stochastic term drops out in the proof. If in addition to the assumption in Proposition 1 h is bounded and $EY^2(0) < \infty$ then uniqueness (and also existence) of the solution of equation (10) follows from a (more general) result of Oelschläger [5]. In fact, if $E|Y(0)| < \infty$, then the usual recursive approximation scheme will give existence of a solution of (10) even without the boundedness condition on h .

PROPOSITION 2. *If f is locally Lipschitz continuous and h is Lipschitz continuous and has compact support, then for every $0 < T < \infty$ there exists at most one strong solution of (2).*

PROOF. Let M_1 be chosen such that $\text{supp } h \subseteq \{x \mid |x| \leq M_1\}$ and let $M_2 > M_1$. Furthermore define $\bar{h} := \max_{x \in \text{supp } h} |h(x)|$. Since f is bounded on $\{(x, z) \mid |x| \leq M_2, |z| \leq \bar{h}\}$ there exists some $0 < t_0 \leq T$ such that for all measurable functions $(z(t))_{0 \leq t \leq t_0}$ with $\sup_{0 \leq t \leq t_0} |z(t)| \leq \bar{h}$ and for all y_0 satisfying $|y_0| \geq M_2$, the solution of $\dot{Y}(t) = f(Y(t), z(t)), Y(0) = y_0$ satisfies $\inf_{0 \leq t \leq t_0} |Y(t)| > M_1$. Now let \tilde{f} be any globally Lipschitz function which agrees with f on the set $\{(x, z) \mid |x| \leq M_2, |z| \leq \bar{h}\}$. According to Proposition 1

$$(11) \quad \dot{Y}(t) = \tilde{f}(Y(t), Eh(Y(t))), \quad Y(0) = \xi$$

has at most one solution.

Let Y_1 and Y_2 be two solutions of (2) with $Y_1(0) = Y_2(0) = \xi$. If we insert $Eh(Y_1(t))$ for $Eh(Y(t))$ into (11) we see that the corresponding unique solution $\tilde{Y}(t)$ agrees with $Y_1(t)$ on the support of h for $0 \leq t \leq t_0$. In particular $Eh(\tilde{Y}(t)) = Eh(Y_1(t))$ showing that \tilde{Y} satisfies (11). Since (11) has exactly one solution one can show $Eh(\tilde{Y}(t)) = Eh(Y_2(t))$ in the same way. So $Eh(Y_1(t)) = Eh(Y_2(t))$ for $0 \leq t \leq t_0$ which implies $P\{Y_1(t) = Y_2(t), 0 \leq t \leq t_0\} = 1$, that is, uniqueness on the interval $[0, t_0]$. Repeating the argument with 0 replaced by t_0 and t_0 replaced by $t_1 = 2t_0$ etc. we get uniqueness on $[0, T]$.

REMARK. The proposition and its proof are also valid in the multidimensional case.

PROPOSITION 3. *If f is Lipschitz and has compact support w.r.t. the first argument, that is, there exists some $M > 0$ such that $f(y, z) = 0$ for all $z \in \mathbf{R}$ and $|y| \geq M$ and if h is locally Lipschitz continuous then for every $0 < T < \infty$ there exists at most one strong solution of (2).*

PROOF. If $E|h(Y(0))| = \infty$ there is nothing to prove. Otherwise let \tilde{h} be a Lipschitz function which agrees with h inside a ball of radius M and satisfies $E\tilde{h}(Y(0)) = Eh(Y(0))$. Obviously any solution of (2) satisfies

$$(12) \quad \dot{Y}(t) = f(Y(t), E\tilde{h}(Y(t))), \quad Y(0) = \xi$$

since $\dot{Y}(t) = 0$ if $|Y(t)| \geq M$ and

$$\begin{aligned} E\tilde{h}(Y(t)) &= E(\tilde{h}(Y(t)) | |Y(0)| < M) P\{|Y(0)| < M\} \\ &\quad + E(\tilde{h}(Y(t)) | |Y(0)| \geq M) P\{|Y(0)| \geq M\} \\ &= E(h(Y(t)) | |Y(0)| < M) P\{|Y(0)| < M\} \\ &\quad + E(\tilde{h}(Y(0)) | |Y(0)| \geq M) P\{|Y(0)| \geq M\} \\ &= Eh(Y(t)). \end{aligned}$$

The assertion follows from the fact that equation (12) has at most one solution due to Proposition 1.

REMARK. Again the proposition and its proof are also valid in the multidimensional case.

FINAL REMARK. Instead of (2) some authors treat the (nonlinear) equation

$$(13) \quad \dot{X}(t) = \int_{\mathbf{R}^d} f(X(t), y) du_t(y), \quad u_t = \mathcal{L}(X(t))$$

with $f: \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ (mostly plus a noise term). If we agree to call $(X(t))_{t \geq 0}$ a (strong) solution of (13) if its law $u_t = \mathcal{L}(X(t))$ satisfies $\sup_{0 \leq t \leq T} \int_{\mathbf{R}^d} |y| du_t(y) < \infty$ for every $T > 0$, $(X(t))_{t \geq 0}$ has continuous paths and (13) holds in its integrated form for almost all paths, then the following table holds true for the function f where uniform (locally) Lipschitz means that the (local) Lipschitz constants can be chosen independently of the other component. Here the uniqueness result is a special case of [2], page 342.

1st comp. \ 2nd comp.	unif. Lipschitz	unif. loc. Lipschitz
unif. Lipschitz	unique	CE 2
unif. loc. Lipschitz	CE 1	CE 1, 2

References

- [1] D. A. Dawson, 'Critical dynamics and fluctuations for a mean-field model of cooperative behavior', *J. Statist. Phys.* **31** (1983), 29–85
- [2] T. Funaki, 'A certain class of diffusion processes associated with nonlinear parabolic equations', *Z. Wahrsch. Verw. Gebiete* **67** (1984), 331–348.
- [3] R. S. Liptser and A. N. Shiriyayev, *Statistics of random processes I* (Springer Verlag, Berlin, 1977)
- [4] H. P. McKean, Jr., 'Propagation of chaos for a class of nonlinear parabolic equations', *Lecture series in differential equations*, Vol. 2, pp. 177–194 (Van Nostrand Reinhold, New York, 1969).
- [5] K. Oelschläger, 'A martingale approach to the law of large numbers for weakly interacting stochastic processes', *Ann. Probab.* **12** (1984), 458–479.
- [6] M. Scheutzow, 'Periodic behavior of the stochastic Brusselator in the mean-field limit', *Probab. Theory Rel. Fields* **72** (1986), 425–462
- [7] T. Shiga and H. Tanaka, 'Central limit theorem for a system of Markovian particles with mean field interactions', *Z. Wahrsch. Verw. Gebiete* **69** (1985), 439–459.
- [8] A. S. Sznitman, 'An example of nonlinear diffusion process with normal reflecting boundary conditions and some related limit theorems', preprint, Laboratoire de Probabilités, Paris, February 1983
- [9] W. Walter, *Gewöhnliche Differentialgleichungen* (Springer Verlag, Berlin, 1976).

Fachbereich Mathematik
 Universität Kaiserslautern
 D-6750 Kaiserslautern
 West Germany