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AVERAGE DISTANCES IN COMPACT CONNECTED SPACES

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We give a simple proof of the fact that compact, connected topological spaces have the "average distance property". For a metric space (X, d) , this asserts the existence of a unique number $a = a(X)$ such that, given finitely many points $x_1, \ldots, x_n \in X$, then there is some $y \in X$ with

$$\frac{1}{n} \sum_{i=1}^{n} d(y, x_i) = a .$$

We examine the possible values of $a(X)$, for subsets of finite dimensional normed spaces. For example, if $\operatorname{diam}(X)$ denotes the diameter of some compact, convex set in a euclidean space, then $a(X) \leq \operatorname{diam}(X)/\sqrt{2}$. On the other hand, $a(X)/\operatorname{diam}(X)$ can be arbitrarily close to 1 , for non-convex sets in euclidean spaces of sufficiently large dimension.

1. The Gross-Stadje theorem

In [7], Stadje proved the following interesting result.

THEOREM 1. *Let* X *be a compact, connected topological space and* $d : X^2 \to \mathbb{R}$ *a continuous, symmetric function. Then there is a unique number* $a = a(X, d)$ *with the following property: for all* $n \in \mathbb{N}$ *, and for*

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all $x_1, \ldots, x_n \in X$, *there is a point* $y \in X$ *such that*

$$\frac{1}{n} \sum_{i=1}^{n} d(x_i, y) = a .$$

Typically, d will be a metric on X , although that assumption is not necessary. Bearing this in mind, the property characterizing $a(X, d)$ is called the "average distance property". For the special case when d is a metric, Theorem 1 had previously been proved by Gross [1]. In the general case, we will call $a(X, d)$ the Gross-Stadje number for (X, d) .

One purpose of this note is to present a simple proof of the Gross-Stadje Theorem. Our proof of the existence of the Gross-Stadje number is new and completely elementary. First, it will be helpful to introduce some notation.

Let $\mathsf{F} = \bigcup_{n=1}^{\infty} X^n$. Thus F is the set of all ordered finite-tuples, with members from X . If $x \in X$, and $F = (x_1, \ldots, x_n) \in \mathsf{F}$, put $d(x, F) = \frac{1}{n} \sum_{i=1}^{n} d(x, x_i)$. Then put $\alpha_F = \inf\{d(x, F) : x \in X\}$ and $\beta_F = \sup\{d(x, F) : x \in X\}$.

We claim that $\alpha_F \le \beta_G$ whenever $F, G \in \mathsf{F}$. Let us write $F = (x_1, \ldots, x_m)$ and $G = (y_1, \ldots, y_n)$. It suffices to show that, for some $i \le n$ and some $j \le m$, we have $d(y_i, F) \le d(x_j, G)$. Suppose that this is not true. Then $d(y_i, F) > d(x_j, G)$, for all $i \le n$, $j \le m$. Summing over i and j then yields

$$m \sum_{i=1}^{n} d(y_i, F) > n \sum_{j=1}^{m} d(x_j, G) .$$

Since d is symmetric, both sides of this inequality are equal to $\sum_{i=1}^{n} \sum_{j=1}^{m} d(y_i, x_j)$. This is a contradiction, so our claim must be correct.

Now existence of $a(X, d)$ can be easily proved. What we wish to show is that $(\exists! a \in \mathbb{R})(\forall F \in \mathsf{F})(a \in \{d(x, F) : x \in X\})$. For any $F \in \mathsf{F}$, the

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map $x \mapsto d(x, F)$ is a continuous function on X. Since X is compact and connected, $\{d(x, F) : x \in X\}$ must be the closed interval $[\alpha_F, \beta_F]$. The conclusion of Theorem 1 then becomes $(\exists! a)(\forall F \in \mathcal{F})(\alpha_F \le a \le \beta_F)$. Our previous claim tells us that $\sup\{\alpha_F : F \in \mathcal{F}\} \le \inf\{\beta_F : F \in \mathcal{F}\}$. Existence of a follows immediately.

It is a little harder to prove the uniqueness of a. First note that each $F \in \mathcal{F}$ induces, in a natural way, an atomic probability measure on X. We will use the same symbol for the probability measure and the ordered tuple; thus $d(x, F) = \int_X d(x, y) dF(y)$. Let $\mathcal{P}$ denote the set of all regular Borel probability measures on X, equipped with the vague topology. (For details of the relationship between $\mathcal{P}$ and X, we refer the reader to [4, Section 22A].) In this topology, a net P_α is convergent to P if and only if $\int_X f(x) dP_\alpha(x) \to \int_X f(x) dP(x)$, for every continuous function $f : X \to \mathbb{R}$. Then $\mathcal{P}$ is a compact, convex set, and it can be deduced from the Krein-Milman theorem [4, Section 13B] that $\mathcal{F}$ is dense in $\mathcal{P}$. If $F_\alpha \to P$ vaguely, and $x_\alpha \to x$ in X, it is a routine exercise to show that $d(x_\alpha, F_\alpha) \to d(x, P)$. From these facts it follows that

$$\underline{v} = \sup_{F \in \mathcal{F}} \min_{x \in X} d(x, F) = \max_{P \in \mathcal{P}} \min_{x \in X} d(x, P) ,$$

and

$$\overline{v} = \inf_{F \in \mathcal{F}} \max_{x \in X} d(x, F) = \min_{P \in \mathcal{P}} \max_{x \in X} d(x, P) .$$

We have already shown that $\underline{v} \le \overline{v}$. A generalization of Ville's version of the minimax theorem [6, p. 69] tells us that $\underline{v} = \overline{v}$. Thus $\sup_{F \in \mathcal{F}} \alpha_F = \inf_{F \in \mathcal{F}} \beta_F$, and so a is unique. This completes our proof of Theorem 1.

Graham Elton has pointed out (private communcation) that (X, d) will have the following strong version of the average distance property: given any regular, Borel probability measure P on X, there is a point $x \in X$ with $d(x, P) = a(X, d)$. This result follows from the last paragraph.

It also follows from the proof above that there are two probability measures, P and Q, on X such that $d(x, P) \le a(X, d) \le d(x, Q)$ for all $x \in X$. Joan Cleary and Sid Morris (private communication) have used this idea to calculate the Gross-Stadje numbers of regular polygons.

If there is a single probability measure P, on X, such that $d(x, P)$ is independent of x, then $a(X, d)$ is easy to determine. We only have to calculate $d(x, P)$ for a convenient point $x \in X$. Morris and Nickolas [5] have used this to evaluate the Gross-Stadje numbers of sufficiently symmetric metric spaces, such as spheres.

Sometimes it is difficult to find $a(X, d)$ exactly, and we must settle for some sort of estimate. In such a situation, the following result might be useful.

PROPOSITION 2. *Fix* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \le \beta$.

(i) Suppose (X, d) *has the following property: given any* $F \in \mathcal{F}$, *there is a point* $y \in X$ *with* $\alpha \le d(y, F) \le \beta$. *Then* $\alpha \le a(X, d) \le \beta$.

(ii) Suppose there is a point $y \in X$ *such that* $\alpha \le d(x, y) \le \beta$ *for all* $x \in X$. *Then* $\alpha \le a(X, d) \le \beta$.

Proof. *(i)* The hypothesis clearly implies that

$$\sup_F \min_x d(x, F) \le \beta \quad \text{and} \quad \alpha \le \inf_F \max_x d(x, F) .$$

(ii) This follows immediately from *(i)*. //

2. The range of values for metric spaces

From now on, (X, d) will be a compact, connected metric space, and $a(X, d)$ will be abbreviated to $a(X)$. We assume further that X is not a singleton, and so has strictly positive diameter, $\mathrm{diam}(X)$. Normalizing, we define the dispersion number of X by $m(X) = a(X)/\mathrm{diam}(X)$. It can then be shown [7, p. 277] that $\frac{1}{2} \le m(X) < 1$.

In this section, we will consider the range of values that $m(X)$ may take. The dispersion number gives us some information about how 'spread out' a space is, although perhaps not as much information as we would like. If $m(X)$ is close to 1, then the points of X are, generally speaking, far apart from one another. The converse is not true. It is possible that $m(X) = \frac{1}{2}$, for a space X which we might intuitively describe as fairly

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spread out.

PROPOSITION 3. *Suppose that* $m(X) > \frac{1}{2}$. *Then there is a metric space* Y , *obtained from* X *by gluing on a line segment, for which* $m(Y) = \frac{1}{2}$.

Proof. Let I^{∞} denote the space of all sequences from $[0, 1]$, equipped with the metric $d\big((\alpha_n), (\beta_n)\big) = \sup_{n=1}^{\infty} |\alpha_n - \beta_n|$. Assume without loss of generality that $\operatorname{diam}(X) = 1$. Since X is separable, we may suppose that it is embedded in I^{∞} . Let $a \in I^{\infty}$ be the constant sequence $(\frac{1}{2}, \frac{1}{2}, \ldots)$. Since X is compact, there is a point $x \in X$ such that $d(a, x)$ is a minimum; that is, $d(a, x) = d(a, X)$. Let $Y = X \cup [x, a]$, where $[x, a]$ is the line segment joining x to a . Then Y is certainly a compact, connected set. Since x is a closest point (in X) to a , $Y \backslash X$ is just the half-open segment $(x, a]$. Moreover $d(a, y) \leq \frac{1}{2}$ for all $y \in I^{\infty}$, and hence for all $y \in Y$. It follows from Proposition 2 that $a(Y) \leq \frac{1}{2}$. Clearly $\operatorname{diam}(Y) = 1$, and thus $m(Y) = \frac{1}{2}$. //

If (S, d) is any metric space, let $H(S)$ denote the family of all compact, connected, non-empty subsets of S . We can turn $H(S)$ into a metric space as follows. For any $X, Y \in H(S)$, define $\rho(X, Y) = \sup\{d(x, Y) : x \in X\}$, and $d_H(X, Y) = \max\{\rho(X, Y), \rho(Y, X)\}$. Then d_H is a metric for $H(S)$ [2, Section 28]. If S is compact, then so is $H(S)$ under this metric.

PROPOSITION 4. *For any metric space* (S, d) , *the map* $a : H(S) \to \mathbb{R}$ *is continuous. More precisely, we have*

$$|a(X)-a(Y)| \leq \rho(X, Y) + \rho(Y, X) \leq 2d_H(X, Y)$$

for all $X, Y \in H(S)$.

Proof. Fix $X, Y \in H(S)$, and put $\delta_1 = \rho(X, Y)$ and $\delta_2 = \rho(Y, X)$. Let $x_1, \ldots, x_n$ be any points in X . Then there are points $y_1, \ldots, y_n \in Y$ with $d(x_i, y_i) \leq \delta_1$ for each i . Determine $y \in Y$ so so that $a(Y) = \frac{1}{n} \sum_{i=1}^{n} d(y, y_i)$. Then there is a point $x \in X$ with

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$d(x, y) \le \delta_2$. Routine calculations then show that

$$a(Y) - \delta_1 - \delta_2 \le \frac{1}{n} \sum_{i=1}^{n} d(x, x_i) \le a(Y) + \delta_1 + \delta_2 .$$

An application of Proposition 2 yields

$$a(Y) - \delta_1 - \delta_2 \le a(X) \le a(Y) + \delta_1 + \delta_2 . \qquad //$$

THEOREM 5. *Let E be any finite dimensional normed space. Then there is a constant $k = k(E) < 1$ such that $m(X) \le k$ whenever X is a compact connected subset of E. Moreover,*

$$\{m(X) : X \text{ is a compact, connected subset of } E\}$$

is the whole interval $[\frac{1}{2}, k(E)]$.

Proof. It suffices to show that $a(X) \le k$ whenever $\operatorname{diam}(X) = 1$. Clearly any set of diameter 1 is isometric to a subset of S, the closed unit ball of E. So let $T = \{X \in H(S) : \operatorname{diam}(X) = 1\}$. Then T is a compact, metric space, $a : T \to \mathbb{R}$ is continuous, and $a(X) < 1$ for every $X \in T$. It follows that $k(E) = \max\{a(X) : X \in T\} < 1$.

Finally, choose $X \in T$ so that $a(X) = m(X) = k(E)$, and let Y be a line segment with length 1. If $0 \le \lambda \le 1$, then

$$\lambda X + (1-\lambda)Y = \{\lambda x+(1-\lambda)y : x \in X, y \in Y\}$$

is a compact, connected subset of E, and is clearly not a singleton. Thus we can define a continuous map $[0, 1] \to \mathbb{R}$ by $\lambda \mapsto m(\lambda X+(1-\lambda)Y)$. Since $0 \mapsto \frac{1}{2}$ and $1 \mapsto k(E)$, the intermediate value theorem finishes the proof. //

Let us define $k_n = \sup\{k(E) : E \text{ is an } n\text{-dimensional normed space}\}$. It is almost obvious that $k_1 = \frac{1}{2}$. It would be interesting to know* whether $k_n < 1$ for $n = 2, 3, \ldots$. The next result shows that $k_n \to 1$ as $n \to \infty$.

THEOREM 6. *If X is a compact, convex set in some n-dimensional normed space, then $m(X) \le n/(n+1)$. This estimate is sharp.*

Proof. For each $x \in X$, let $A(x) = (1/(n+1))(x+nX)$. Then each

* See note added in proof.

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$A(x)$ is a compact, convex subset of X. If $x_1, x_2, \ldots, x_{n+1}$ are any elements of X, it is easy to verify that $(1/(n+1)) \sum_{i=1}^{n+1} x_i \in \bigcap_{i=1}^{n+1} A(x_i)$. We can then deduce from Helly's theorem [3], the existence of some $a \in \bigcap_{x \in X} A(x)$. Now fix $x \in X$. Since $a \in A(x)$, there is a $y \in X$ with $a = (1/(n+1))(x+ny)$. Then $\|x-a\| = (n/(n+1))\|x-y\| \le (n/(n+1))\operatorname{diam}(X)$ and so $a(X) \le (n/(n+1))\operatorname{diam}(X)$.

To see that this estimate is sharp, give $\mathbb{R}^{n+1}$ the l_1-norm, $\|(\alpha_0, \alpha_1, \ldots, \alpha_n)\| = \sum_{i=0}^{n} |\alpha_i|$, and let X be the convex hull of $F = \{e_0, e_1, e_2, \ldots, e_n\}$. Then X is contained in the n-dimensional affine subspace $\left\{(\alpha_0, \alpha_1, \ldots, \alpha_n) : \sum_{i=0}^{n} \alpha_i = 1\right\}$. Routine calculations show that $d(x, F) = 2n/(n+1)$, for any $x \in X$. Since $\operatorname{diam}(X) = 2$, it follows that $m(X) = n/(n+1)$. //

3. Subsets of euclidean spaces

Stadje claims [7, p. 278] that if X is a compact, convex subset of the euclidean space $\mathbb{R}^n$, then $m(X) \le \frac{1}{2}\sqrt{5-2\sqrt{3}}$. Recently, Strantzen [8] has shown that $m(X) \le \sqrt{n/(2n+2)}$ for any such X, and that this bound is sharp. This improves Stadje's estimate for $n = 2$ and $n = 3$, and disproves his claim for $n \ge 4$. However, we still have the uniform estimate $m(X) \le 1/\sqrt{2}$, whenever X is a compact, convex subset of some euclidean space. Theorem 6 shows that no such uniform bound exists for non-euclidean spaces.

It is still of interest to know whether there is such a uniform bound, for non-convex sets in euclidean spaces. There is not; we will see from Theorem 9 that $k(\mathbb{R}^n) \to 1$ as $n \to \infty$. The next two results help to identify those sets with large dispersion numbers.

THEOREM 7. *Let X be a compact, connected subset of some normed space. Let Y be a closed, connected subset of X, and suppose that the convex hull of Y contains X. Then $m(X) \le m(Y)$.*

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Proof. Clearly $\operatorname{diam}(Y) = \operatorname{diam}(X)$, so we need show only that $a(X) \le a(Y)$. Let F be any finite ordered-tuple from Y . Then F is also a finite ordered-tuple from X , so $d(x, F) = a(X)$ for some $x \in X$. However, $x = \sum_i \lambda_i y_i$, for some $y_i \in Y$, $\lambda_i \ge 0$ with $\sum_i \lambda_i = 1$. Then $a(X) = d\left(\sum_i \lambda_i y_i, F\right) \le \sum_i \lambda_i d\left(y_i, F\right)$, and so $a(X) \le d\left(y_i, F\right)$, for at least one value of i . It follows from Proposition 2 that $a(X) \le a(Y)$. //

COROLLARY 8. *Let X be a compact, connected subset of a finite dimensional normed space, whose boundary ∂X is connected. Then* $m(X) \le m(\partial X)$.

Proof. It follows from the separation theorem that X is contained in the convex hull of ∂X . //

Corollary 8 was first proved by Graham Elton for finite dimensional euclidean spaces.

Let S^n denote, as usual, the surface of the unit ball in $\mathbb{R}^{n+1}$. Graham Elton, Sid Morris and Peter Nickolas (private communication) have shown that the sequence $m\left(S^n\right)$ increases monotonically, and has limit $1/\sqrt{2}$. Given Corollary 8, it is then tempting to conjecture that $m(X) \le 1/\sqrt{2}$, whenever X is contained in a finite dimensional euclidean space. The truth is quite different.

THEOREM 9. *There exist compact, connected sets $X_n \subset \mathbb{R}^n$ such that $m\left(X_n\right) \to 1$ as $n \to \infty$. More precisely, X_n can be chosen so that $m\left(X_n\right) \ge n/(n+1)$. (Obviously X_n cannot be convex.)*

Proof. In $\mathbb{R}^{n+1}$, let F_n be the finite set $\left\{e_i/\sqrt{2} : 0 \le i \le n\right\}$. For $0 \le j < k \le n$, let $A(j, k)$ be the arc with centre at $\left(1/\left((n-1)\sqrt{2}\right)\right)\left(\sum_{i=0}^{n} e_i - e_j - e_k\right)$, which joins $e_j/\sqrt{2}$ to $e_k/\sqrt{2}$. Then $A(j, k)$ has radius $\sqrt{n/(2n-2)}$, and parameterization

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$$x_j(\theta) = \frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{2(n^2-1)}} ,$$

$$x_k(\theta) = \frac{n\sin\theta}{\sqrt{2(n^2-1)}} ,$$

$$x_i(\theta) = \frac{1-\cos\theta}{\sqrt{2}(n-1)} - \frac{\sin\theta}{\sqrt{2(n^2-1)}} , \text{ for } i \neq j, k ,$$

where $0 \leq \theta \leq \arccos(1/n)$. Then $X_n = \cup\{A(j, k) : 0 \leq j \leq k \leq n\}$ is a compact, connected subset of the n-dimensional affine subspace $\left\{(x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i = \sqrt{2}\right\}$. It is routine to show that $\|x-e_i/\sqrt{2}\| = 1$, whenever $x \in A(j, k)$ and $j \neq i \neq k$. Thus, for any $x \in X$,

$$\begin{aligned} d(x, F_n) &= \frac{1}{n+1} \sum_{i=0}^{n} \|x-e_i/\sqrt{2}\| \\ &= \frac{1}{n+1} \left(n-1+\|x-e_j/\sqrt{2}\|+\|x-e_k/\sqrt{2}\|\right) \quad \text{for suitable } j, k \\ &\geq \frac{1}{n+1} \left(n-1+\|e_j/\sqrt{2}-e_k/\sqrt{2}\|\right) \\ &= \frac{n}{n+1} . \end{aligned}$$

It follows that $a(X_n) \geq n/(n+1)$. When $n = 2$, X_n is the well-known Reuleaux triangle, and it is easy to see that $\text{diam}(X_2) = 1$. Thus $m(X_2) \geq 2/3$.

Unfortunately, it is not true that $\text{diam}(X_n) = 1$ for $n \geq 3$. One can show that $\|x-y\|$ is a maximum (over $x, y \in X_n$) when x and y are the midpoints of two arcs which do not share a common vertex. It follows that $\text{diam}(X_n) = \left(\sqrt{n(n+1)}-\sqrt{2}\right)/(\ -1) \to 1$ as $n \to \infty$. Thus $m(X_n) \to 1$ as $n \to \infty$.

If Y_n is any closed, connected subset of X_n which contains F_n , the same reasoning shows that $a(Y_n) \geq n/(n+1)$ and $\text{diam}(Y_n) \to 1$. For

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example, we could choose

(i) $Y_n = \cup\{A(i-1, i) : 1 \le i \le n\}$, or

(ii) $Y_n = A(0, n) \cup \cup\{A(i-1, i) : 1 \le i \le n\}$, or

(iii) $Y_n = \cup\{A(0, i) : 1 \le i \le n\}$.

In case (i), Y_n will be homeomorphic to a line segment, and in case (ii), Y_n will be homeomorphic to a circle. For the last choice, it is possible to show that $\operatorname{diam}(Y_n) = 1$ and so $m(Y_n) \ge n/(n+1)$. //

Some numerical calculations show that the dispersion number for the Reuleaux triangle is 0.668 (to three significant figures). It follows that $k(\mathbb{R}^2) \ge 0.668$. Graham Elton (private communication) has shown that $k(\mathbb{R}^2) \le 0.775$. It would be interesting to narrow this gap.

ADDED IN PROOF (25 June 1982). We have recently shown that $k_n < 1$. A sketch of the proof follows.

If $\|\cdot\|$ is any norm on $\mathbb{R}^n$, let ν denote its restriction to $I^n = [-1, 1]^n$, and also the derived metric. If E is any n-dimensional normed space, Auerbach's Lemma [J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1977, Proposition 1.c.3] asserts the existence of norm-one vectors $x_1, \ldots, x_n \in E$ and norm-one functionals $f_1, \ldots, f_n \in E^*$ with $f_i(x_j) = \delta_{ij}$. Easy calculations show that

$$\max_{i=1}^{n} |\alpha_i| \le \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \le \sum_{i=1}^{n} |\alpha_i| \quad \text{for all } \alpha_1, \ldots, \alpha_n \in \mathbb{R} .$$

Identifying E with $\mathbb{R}^n$, we then have $\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1$. Thus if X is a compact, connected subset of some n-dimensional normed space, with diameter one, then X is isometric to a metric space of the form (Y, ν) , where $Y \in H(I^n)$ and $\nu_\infty \le \nu \le \nu_1$.

Now $N = \{\nu : \nu_\infty \le \nu \le \nu_1\}$ is a compact subset of $C(I^n)$; by the

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Ascoli-Arzela theorem. It is routine to show that the maps $a : H(I^n) \times N \to \mathbb{R}$ and $\operatorname{diam} : H(I^n) \times N \to \mathbb{R}$ are continuous. It follows from compactness that

$$k_n = \sup\{a(X, \nu) : (X, \nu) \in H(I^n) \times N \text{ and } \operatorname{diam}(X, \nu) = 1\}$$

is strictly less than one.

We also note that $k_n \geq 1 - 2^{-n}$. To see this, give $\mathbb{R}^n$ the norm $\|\cdot\|_\infty$, and consider the subset

$$X = \{(\alpha_1, \ldots, \alpha_n) : 0 \leq \alpha_i \leq 1 \text{ for all } i, \text{ and } 0 < \alpha_i < 1 \text{ for at most one value of } i\} .$$

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