A theorem on cardinal numbers

By W. F. Newns.

A classical theorem of Cantor states that the class of all sub-classes of a given class has a cardinal greater than that of the given class. This theorem is here established in a sharpened form, which was suggested to me by a question set by Professor J. M. Whittaker, F.R.S., in the 1950 examination for the Honours B.Sc. Degree at Liverpool.[1]

Theorem. *Given any class A, with cardinal number a, let T be the class consisting of those subclasses of A which have at least $a - 1$ members. Then $N(T) > a$.*

Here $N(X)$ denotes the cardinal number of the class X. $a - 1$ is the cardinal number of the class $A - (x)$ where $x \epsilon A$ and $N(A) = a$. The Axiom of Choice is not employed in the proof. In the case of non-reflexive cardinals, since A and subsets of the form $A - (x)$ are the only members of T, the inequality reduces to $a + 1 > a$.

Proof (i). To each member $x \epsilon A$ corresponds a subclass $A - (x)$ of A. Such a subclass has $a - 1$ members and thus is a member of T. This establishes a (1, 1) correspondence between A and part of T. Hence

$$a \leqq N(T).$$

(ii). Suppose $N(T) = a$. Then there is a correlation σ between A and T. The class A itself, as a member of T, must have a correlate, say $z \epsilon A$.

We write $$\sigma(z) = A.$$

To each $x \epsilon A$ corresponds x' such that

$$\sigma(x') = A - (x).$$

Let A_0 be the class of all such x'. Clearly

$$z \notin A_0. \qquad (1)$$

[1] The question was as follows :—" Let A be any class and let T be the class of all subclasses of A which contain more than one member. If A has more than two members, prove that T has a greater cardinal than A."

https://doi.org/10.1017/S0950184300003074 Published online by Cambridge University Press

Moreover, consideration of the obvious correlation between x and x' shows that

$$N(A_0) = a. \qquad (2)$$

Since $A_0 \subset A - (z) \subset A$, it follows from (2) that

$$a - 1 = a.$$

Hence all classes of the form $A - (x) - (y)$ have cardinal a, and similarly their correlates form a subclass A_1 of cardinal a. Since A_0 and A_1 are disjoint, the sets $A_0 \cup X$, where $X \subset A_1$, are distinct and have cardinal a. These sets are in (1, 1) correspondence with the subclasses of A_1 and therefore of A.

Thus $N(T) = a$ implies $N(T) \geqq 2^a > a$, so that

$$N(T) \neq a.$$

THE DEPARTMENT OF PURE MATHEMATICS,
THE UNIVERSITY, LIVERPOOL.

https://doi.org/10.1017/S0950184300003074 Published online by Cambridge University Press